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On the Distribution of Zero Sets of Holomorphic Functions: III. Converse Theorems

B. N. Khabibullin and F. B. Khabibullin

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Abstract. Let M be a subharmonic function in a domain $D \subset \mathbb{C}^n$ with Riesz measure ν_M, and let $\mathsf{Z} \subset D$. As was shown in the first of the preceding papers, if there exists a holomorphic function $f \neq 0$ in D such that $f(\mathsf{Z}) = 0$ and $|f| \leqslant \exp M$ on D, then one has a *scale* of integral uniform upper bounds for the distribution of the set Z via ν_M. The present paper shows that for $n = 1$ this result "almost has a converse." Namely, it follows from such a *scale* of estimates for the distribution of points of the sequence $\mathsf{Z} := \{\mathsf{z}_k\}_{k=1,2,\dots} \subset D \subset \mathbb{C}$ via ν_M that there exists a nonzero holomorphic function f in D such that $f(\mathsf{Z}) = 0$ and $|f| \leqslant \exp M^{\uparrow r}$ on D, where the function $M^{\uparrow r} \geqslant M$ on D is constructed from the averages of M over circles rapidly narrowing when approaching the boundary of D with a possible additive logarithmic term associated with the rate of narrowing of these circles.

Key words: holomorphic function, sequence of zeros, subharmonic function, Jensen measure, test function, balayage.

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1. Introduction

The present paper uses the notation, definitions and conventions in [1] and [2] with their natural adaptations for the *complex plane* $\mathbb{C}$ and its *Aleksandrov one-point compactification* $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The main goal is to give the converse of the results in the original paper [1] for domains $D \subset \mathbb{C}_\infty$ in the form of a criterion in the subharmonic version and also in a form close to a criterion when holomorphic functions in D are considered.

1.1. Notation, definitions, conventions. Throughout the paper, $\mathbb{N} := \{1, 2, \dots\}$ stands for positive integers, $\mathbb{R} \subset \mathbb{C}$ is the *real line*, $\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geqslant 0\}$ is the *positive semiaxis*, and

$$\mathbb{R}^+_* := \mathbb{R}^+ \setminus \{0\}, \quad \mathbb{R}_{\pm\infty} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}, \quad \mathbb{R}^+_{+\infty} := \mathbb{R}^+ \cup \{+\infty\}, \tag{1.1}$$

where the order on $\mathbb{R}$ is supplemented with the natural inequalities $-\infty \leqslant x \leqslant +\infty$ for any $x \in \mathbb{R}_{\pm\infty}$. For $r \in \mathbb{R}^+_*$ and $z \in \mathbb{C}$, let $D(z, r) := \{z' \in \mathbb{C} \colon |z' - z| < r\}$ be the *open disk with center z and radius r*, let $D(r) := D(0, r)$, let $\mathbb{D} := D(1)$, and let $D(z, +\infty) := \mathbb{C}$. For $z = \infty$, it is convenient for us to set $D(\infty, r) := \{z \in \mathbb{C}_\infty \colon |z| > 1/r\}$, $|\infty| := +\infty$, and $D(\infty, +\infty) := \mathbb{C}_\infty \setminus \{0\}$. The open disks $D(z, r)$, $r \in \mathbb{R}^+_*$, form a base of open neighborhoods of a point $z \in \mathbb{C}_\infty$. For $S \subset \mathbb{C}_\infty$, by $\operatorname{int} S$, $\operatorname{clos} S$, and ∂S we denote the *interior*, *closure*, and *boundary* of S in $\mathbb{C}_\infty$. For $S \subset S' \subset \mathbb{C}_\infty$, we write $S \Subset S'$ if S is a relatively compact subset of S'. A (*sub*)*domain* in $\mathbb{C}_\infty$ is an open connected subset of $\mathbb{C}_\infty$. Throughout the following,

$$D \neq \varnothing \ \textit{ is a proper subdomain of } \mathbb{C}_\infty \neq D. \tag{1.2}$$

Just as in [1], by $\operatorname{har}(S)$, $\operatorname{sbh}(S)$, δ-$\operatorname{sbh}(S)$, $\operatorname{Hol}(S)$, and $C^k(S)$, $k \in \mathbb{N} \cup \{\infty\}$, we denote the classes of *harmonic*, *subharmonic* ([3], [4]), *δ-subharmonic* [1, 3.1], *holomorphic*, and *k times continuously differentiable* functions, respectively, on open subsets of $\mathbb{C}_\infty$ containing $S \subset \mathbb{C}_\infty$; however, $C(S)$ is the class of *continuous* functions on S. By $\boldsymbol{-\infty}$ and $\boldsymbol{+\infty}$ we denote the functions identically equal to $-\infty$ and $+\infty$, respectively. In this notation,

$$\begin{gathered} \operatorname{sbh}_*(S) := \operatorname{sbh}(S) \setminus \{\boldsymbol{-\infty}\}, \quad \delta\text{-}\operatorname{sbh}_*(S) := \delta\text{-}\operatorname{sbh}(S) \setminus \{\boldsymbol{\pm\infty}\}, \\ \operatorname{Hol}_*(S) := \operatorname{Hol}(S) \setminus \{0\}. \end{gathered} \tag{1.3}$$

The symbol 0 stands for the zero vector or the origin in a vector or an affine space. Positivity in an ordered vector space X is everywhere understood as $\geqslant 0$; $+\infty \geqslant 0$ in* $\mathbb{R}^+_{+\infty} \overset{(1.1)}{\subset} \mathbb{R}_{\pm\infty}$. For $A \subset X$, by A^+ we denote the set of positive elements in A. The class of all functions $f\colon X \to Y$ is denoted by Y^X. If $F(S) \subset \mathbb{R}^S_{\pm\infty} := (\mathbb{R}_{\pm\infty})^S$ is any class of *extended real functions*, then $F^+(S) \subset (\mathbb{R}^+_{+\infty})^S$ is the subclass of all positive functions in $F(S)$.

Further, $\mathrm{Meas}(S)$ is the class** of *real Borel measures*, also called *charges* [4], on Borel subsets of a set $S \subset \mathbb{C}_\infty$; $\mathrm{Meas}_{\mathrm{c}}(S)$ is the subclass of measures $\nu \in \mathrm{Meas}(S)$ with *compact support* $\operatorname{supp}\nu \Subset S$; $\mathrm{Meas}^+(S)$ is the subset of positive charges, i.e., just *measures*; λ is the *Lebesgue measure* in $\mathbb{C}$; and δ_z is the *Dirac measure* at a point $z \in \mathbb{C}_\infty$.

Let $f \overset{(1.3)}{\in} \mathrm{Hol}_*(D)$. We say that a function f *vanishes on* a sequence $\mathsf{Z} = \{\mathsf{z}_k\}_{k=1,2,\dots}$ of points lying in D (we write $\mathsf{Z} \subset D$) if the multiplicity of zero, or root, of f at each point $z \in D$ is not less than the number of occurrences of this point in the sequence Z (we write $f(\mathsf{Z}) = 0$). To a sequence $\mathsf{Z} = \{\mathsf{z}_k\}_{k=1,2,\dots} \subset D$ *without limit points in* D, we assign

[div] The *divisor of the sequence* Z on D, that is, the function (denoted by the same symbol) $\mathsf{Z}\colon \mathsf{D} \to \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ that takes each point $z \in D$ to the number of occurrences of z in Z; namely,

$$\mathsf{Z}(z) := \sum_{\mathsf{z}_k = z} 1 = \sum_k \delta_{\mathsf{z}_k}(\{z\}), \qquad z \in D. \tag{1.4}$$

[cm] The *counting measure*

$$n_{\mathsf{Z}}(S) := \sum_{\mathsf{z}_k \in S} 1 = \sum_k \delta_{\mathsf{z}_k}(S), \qquad S \subset D, \tag{1.5}$$

i.e., the number of points of Z that lie in S. It is obvious that $\mathsf{Z}(z) \overset{(1.4)}{\equiv} n_{\mathsf{Z}}(\{z\})$, $z \in D$.

Departing from the traditional interpretation of a sequence as a function of a positive integer or integer argument, we say that two sequences are equal if their divisors (or, equivalently, their counting measures) coincide. See [5, 1.1] and [6, 0.1.2] for more detail.

The *sequence of zeros*, or roots, of a function $f \in \mathrm{Hol}_*(D)$, renumbered in some way counting multiplicities, is denoted by Zero_f. Here $\ln|f| \in \mathrm{sbh}_*(D)$, and for $f \neq 0$ the relationship between the Riesz measure $\nu_{\ln|f|}$ of the function $\ln|f|$ and the counting measure n_{Zero_f} of the sequence of zeros of f is given by the formula ([3, Theorem 3.7.8], [1, 1.2.4])

$$\nu_{\ln|f|} = \frac{1}{2\pi}\Delta \ln|f| \overset{(1.5)}{=} n_{\mathrm{Zero}_f} \in \mathrm{Meas}^+(D), \qquad \text{where } \Delta \text{ is the Laplace operator.} \tag{1.6}$$

Obviously, $f(\mathsf{Z}) = 0$ if and only if $n_{\mathsf{Z}} \leqslant n_{\mathrm{Zero}_f}$ for $n_{\mathsf{Z}}, n_{\mathrm{Zero}_f} \in \mathrm{Meas}^+(D)$ on D.

1.2. Main Results.

Definition 1 (a version of the notion of *balayage* ([3], [4], [7])). Let $S \Subset D$, and let $F \overset{(1.1)}{\subset} \mathbb{R}^{D\setminus S}_{\pm\infty}$ be some class of extended real functions on $D \setminus S$. A charge $\mu \in \mathrm{Meas}(D)$ is called an *affine balayage* of a charge $\nu \in \mathrm{Meas}(D)$ for D outside $S \Subset D$ relative to F (and we write $\nu \preccurlyeq_{S,F} \mu$) if there exists a number $C \in \mathbb{R}$ such that

$$\int_{D\setminus S} v \,\mathrm{d}\nu \leqslant \int_{D\setminus S} v \,\mathrm{d}\mu + C \quad \text{for all } v \in F, \tag{1.7}$$

where the integrals in (1.7) are, generally speaking, upper integrals [8]. In particular, for a sequence $\mathsf{Z} = \{\mathsf{z}_k\}_{k=1,2,\dots}$ with counting measure n_{Z} defined in (1.5), a charge $\mu \in \mathrm{Meas}(D)$ is called an

*A reference over a relation sign ((in)equality, inclusion, etc.) means that this relation is somehow connected with the object being referenced.

**The notation $\mathscr{M}(S)$ was used in [1].

affine balayage of the sequence Z for D outside $S \Subset D$ with respect to the class F (and we write $\mathsf{Z} \preccurlyeq_{S,F} \mu$) if $n_{\mathsf{Z}} \preccurlyeq_{S,F} \mu$, i.e., if there exists a number $C \in \mathbb{R}$ such that

$$\sum_{\mathsf{z}_k \in D \setminus S} v(\mathsf{z}_k) \overset{(1.5)}{=:} \int_{D\setminus S} v \, \mathrm{d} n_{\mathsf{Z}} \overset{(1.7)}{\leqslant} \int_{D \setminus S} v \, \mathrm{d}\mu + C \quad \text{for all } v \in F. \tag{1.8}$$

Obviously, the *preorder* relation $\preccurlyeq_{S,F}$ on $\mathrm{Meas}(D)$ with $F = F^+$ is weaker than the standard order relation $\nu \leqslant \mu$ on $\mathrm{Meas}(D)$. For a function $v \colon D \setminus S \to \mathbb{R}_{\pm\infty}$ with $S \Subset D$, set

$$\lim_{\partial D} v := \lim_{D \ni z' \to z} v(z') \in \mathbb{R}, \qquad z \in \partial D, \tag{1.9}$$

if the limit on the right-hand side exists and if it is the same for any point $z \in \partial D$.

To avoid some purely technical complications associated with the need to apply inversion of the complex plane and the Kelvin transform of functions [1, 1.2.2], for the time being we only consider domains $D \subset \mathbb{C}$; i.e., $\infty \notin D$.

Theorem 1 (a criterion for subharmonic functions)**.** *Let $D \subset \mathbb{C}$ be a domain with a nonpolar boundary $\partial D \subset \mathbb{C}_\infty$, let M be a function in $\mathrm{sbh}_*(D) \cap C(D)$ with Riesz measure $\mu \in \mathrm{Meas}^+(D)$, let $\nu \in \mathrm{Meas}^+(D)$, and let $b \in \mathbb{R}^+_*$. Then the following three assertions are equivalent.*

s1. *There exists a function $u \in \mathrm{sbh}_*(D)$ with Riesz measure $\nu_u \geqslant \nu$ such that $u \leqslant M$ on D.*

s2. *For any subset $S \Subset D$ satisfying the conditions*

$$\varnothing \neq \operatorname{int} S \subset S = \operatorname{clos} S \Subset D, \tag{1.10}$$

the measure μ is an affine balayage of the measure ν for D outside S with respect to the class of test subharmonic positive functions*

$$\mathrm{sbh}_0^+(D \setminus S; \leqslant b) := \Big\{ v \in \mathrm{sbh}^+(D \setminus S) \colon \lim_{\partial D} v \overset{(1.9)}{=} 0, \ \sup_{D \setminus S} v \leqslant b \Big\}. \tag{1.11}$$

s3. *There exists a subset $S \Subset D$ satisfying condition* (1.10) *for which the measure μ is an affine balayage of ν for D outside S with respect to the class $\mathrm{sbh}_{00}^+(D \setminus S; \leqslant b) \cap C^\infty(D \setminus S)$, where*

$$\mathrm{sbh}_{00}^+(D \setminus S; \leqslant b) := \big\{ v \in \mathrm{sbh}_0^+(D \setminus S; \leqslant b) \colon \exists D_v \Subset D, \ v \equiv 0 \ \text{on } D \setminus D_v \big\} \tag{1.12}$$

is the class of test subharmonic positive compactly supported functions.

Remark 1. The implication s2 ⇒ s3 is obvious for *any* domain $D \subset \mathbb{C}_\infty$ and *any* measure $\mu \in \mathrm{Meas}^+(D)$ without the condition of continuity of the function M on D. The same, as shown in Sec. 2.1, is true for the implication s1 ⇒ s2. Only the proof of the implication s3 ⇒ s1 uses both the continuity of the function M and the fact that the boundary $\partial D \subset \mathbb{C}_\infty$ is a nonpolar set. Recall that the latter is equivalent to the existence of a *Green function* g_D for the domain D ([3, 4.4], [12, 3.7, 5.7.4]). The boundary ∂D is nonpolar, say, if at least one of its connected components contains more than one point [3, Corollary 3.6.4] or if the Hausdorff dimension of ∂D is greater than zero [12, 5.4.1].

The implication s3 ⇒ s1 is a special case of Theorem 3 in Sec. 2.2.

Let us proceed to the holomorphic version of Theorem 1, in which there arises a gap between necessary and sufficient conditions. This gap, which is often insignificant, can hardly be bridged even for the disk $D = \mathbb{D}$ in the general situation considered here. We denote by $\mathrm{dist}(\cdot\,,\cdot)$ the *Euclidean distance* between two points, between a point and a set, and between two sets in $\mathbb{C}$. By definition, we set $\mathrm{dist}(\cdot\,,\varnothing) := \mathrm{dist}(\varnothing,\cdot) := \inf\varnothing := +\infty =: \mathrm{dist}(z,\infty) =: \mathrm{dist}(\infty,z)$ for $z \in \mathbb{C}$.

1.2.1. *The choice of lift of the function M.* In what follows, $r \colon D \to \mathbb{R}^+$ is an arbitrary continuous function satisfying the condition

$$0 < r(z) < \min\{\mathrm{dist}(z, \partial D), 1\} \quad \text{for all } z \in D. \tag{1.13}$$

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*Similar classes of test subharmonic functions were studied and used in [1], [2], and [9]–[11].

To a function $M \in L^1_{\mathrm{loc}}(D)$, where $L^1_{\mathrm{loc}}(D) \subset \mathbb{R}^D_{\pm\infty}$ is the class of functions *locally integrable with respect to the Lebesgue measure* λ, we assign its *variable means over disks*,

$$M^{*r}(z) := \frac{1}{\lambda(D(z,r(z)))} \int_{D(z,r(z))} M \,\mathrm{d}\lambda$$
$$= \frac{1}{\pi r^2(z)} \int_0^{2\pi} \int_0^{r(z)} M(z+te^{i\theta})\, t \,\mathrm{d}t\,\mathrm{d}\theta, \qquad D(z,r(z)) \subset D, \tag{1.14}$$

and also, following [13] and [14], its *lift* $M^{\uparrow r}$, defined as follows:

(i) In the general case of $D \subset \mathbb{C}$, we set

$$M^{\uparrow r}(z) := M^{*r}(z) + \ln\frac{1}{r(z)} + (1+\varepsilon)\ln(1+|z|) \quad \text{for all } z \in D, \tag{1.15}$$

where the number $\varepsilon \in \mathbb{R}^+_*$ can be chosen to be arbitrarily small.

(ii) If $\mathbb{C}_\infty \setminus \operatorname{clos} D \neq \varnothing$ or the domain $D \subset \mathbb{C}$ is simply connected in $\mathbb{C}_\infty$, then

$$M^{\uparrow r}(z) := M^{*r}(z) + \ln\frac{1}{r(z)} \quad \text{for all } z \in D. \tag{1.16}$$

(iii) If $D = \mathbb{C}$, then for an arbitrarily large number $P > 0$ we can set

$$M^{\uparrow r}(z) := M^{*r}(z), \quad r(z) := \frac{1}{(1+|z|)^P} \quad \text{for all } z \in \mathbb{C} = D. \tag{1.17}$$

Theorem 2 (necessary/sufficient conditions for holomorphic functions)**.** *Let D be a domain in $\mathbb{C}$, let M be a function in* $\mathrm{sbh}_*(D)$ *with Riesz measure* $\mu \in \mathrm{Meas}^+(D)$, *let* $\mathsf{Z} = \{\mathsf{z}_k\}_{k=1,2,\dots} \subset D$, *and let* $b \in \mathbb{R}^+_*$. *Each of the following three assertions* h1–h3 *follows from the preceding one.*

h1. *There exists a function* $f \in \mathrm{Hol}_*(D)$ *such that* $f(\mathsf{Z}) = 0$ *and* $|f| \leqslant \exp M$ *on* D.

h2. *For any set S satisfying condition* (1.10), *the measure μ is an affine balayage of the sequence* Z *for D outside S with respect to the class of test subharmonic functions* (1.11).

h3. *There exists a set S satisfying condition* (1.10) *such that μ is an affine balayage of the sequence* Z *for D outside S with respect to the class* $\mathrm{sbh}^+_{00}(D \setminus S; \leqslant b) \cap C^\infty(D \setminus S)$.

Conversely, if, in addition, the boundary ∂D is a nonpolar set in $\mathbb{C}_\infty$ and $M \in C(D)$, then assertion h3 *implies the existence of a function* $f \in \mathrm{Hol}_*(D)$ *vanishing on* Z *and satisfying the inequality* $|f| \leqslant \exp M^{\uparrow r}$ *on D with lifts $M^{\uparrow r}$ in* (i)–(iii) *defined by formulas* (1.15)–(1.17).

Remark 2. The implication h2 ⇒ h3 is obvious. Sections 3 and 4 contain the converse Theorems 4 and 5 exclusively in terms of affine balayage with respect to the classes of Green functions and certain logarithmic potentials of analytic disks, respectively. Corollaries 1 and 2 of Theorems 4 and 5 give other versions of the implication h3 ⇒ h1.

2. Proofs of Theorems 1 and 2

2.1. Proof of the implications s1⇒s2 and h1⇒h2. This section does not assume that the function M is continuous. The nonempty domain $D \subset \mathbb{C}$ is arbitrary.

Take a $z_0 \in D$ such that $u(z_0) \neq -\infty$ and $M(z_0) \neq -\infty$. The choice of the domain $\widetilde{D}$, $S \Subset \widetilde{D} \Subset D$, regular for the Dirichlet problem in the statement of the main theorem in [1] is arbitrary. Then, by condition s1 of the main theorem in [1], there exist numbers $C, \overline{C}_M \in \mathbb{R}^+$ such that [1, (3.3)]

$$Cu(x_0) + \int_{D\setminus S} v \,\mathrm{d}\nu_u \leqslant \int_{D\setminus S} v\,\mathrm{d}\mu + C\,\overline{C}_M \quad \text{for all } v \in \mathrm{sbh}^+_0(D\setminus S; \leqslant b).$$

Since $\nu \leqslant \nu_u$, this shows by Definition 1 that $\nu \preccurlyeq_{S,F} \nu_u \preccurlyeq_{S,F} \mu$ for the affine balayage operation $\preccurlyeq_{S,F}$ for D outside S with respect to the class $F \overset{(1.11)}{=} \mathrm{sbh}^+_0(D \setminus S; \leqslant b)$.

The implication h1 ⇒ h2 is a special case of the implication s1 ⇒ s2 for $u := \ln|f|$ with the Riesz measure $n_{\mathrm{Zero}_f} \geqslant n_{\mathsf{Z}}$ in the framework of Definition (1.8) of the affine balayage μ of the sequence Z.

2.2. Proof of the implication s3⇒s1. Let us prove a more general assertion.

Theorem 3. *Let $D \subset \mathbb{C}_\infty$ be a domain with nonpolar boundary ∂D, let $S \Subset D$ be a subset satisfying condition* (1.10), *let $M \overset{(1.3)}{\in} \delta\text{-sbh}_*(D)$ be a δ-subharmonic function with Riesz charge $\nu_M \in \operatorname{Meas}(D)$, let $\nu \in \operatorname{Meas}^+(D)$, and let $b \in \mathbb{R}^+_*$. If the charge ν_M is an affine balayage of the measure ν for D outside S with respect to the class $\operatorname{sbh}^+_{00}(D \setminus S; \leqslant b) \cap C^\infty(D \setminus S)$ defined according to formula* (1.12), *i.e., if there exists a number $C \in \mathbb{R}$ such that*

$$\int_{D\setminus S} v\,\mathrm{d}\nu \leqslant \int_{D\setminus S} v\,\mathrm{d}\nu_M + C \quad \text{for all } v \overset{(1.12)}{\in} \operatorname{sbh}^+_{00}(D \setminus S; \leqslant b) \cap C^\infty(D \setminus S), \tag{2.1}$$

then for each continuous function $r\colon D \to \mathbb{R}^+$ satisfying condition (1.13) *there exists a function $u \in \operatorname{sbh}_*(D)$ with Riesz measure $\nu_u \geqslant \nu$ such that*

$$u(z) \overset{(1.14)}{\leqslant} M^{*r}(z) \quad \text{for all } z \in D. \tag{2.2}$$

If $M \in \delta\text{-sbh}_*(D) \cap C(D)$, i.e., if the function M is also continuous, then under condition (2.1) *there exists a function $u \in \operatorname{sbh}*(D)$ with Riesz measure $\nu_u \geqslant \nu$ such that $u \leqslant M$ on D.*

Proof. First, assume that, instead of condition (2.1), in addition to condition (1.10), for some nonempty subdomain

$$D_0 \Subset \operatorname{int} S \tag{2.3}$$

and some number $C \in \mathbb{R}$ one has the inequality

$$\int_{D\setminus D_0} v\,\mathrm{d}\nu \leqslant \int_{D\setminus D_0} v\,\mathrm{d}\nu_M + C \quad \text{for all } v \overset{(1.12)}{\in} \operatorname{sbh}^+_{00}(D \setminus D_0; \leqslant b); \tag{2.4}$$

i.e., the compactly supported test functions v are not necessarily differentiable, and $S \Subset D$ is somewhat narrowed to a subdomain $D_0 \Subset D$. For the measure ν on $D \supset D_0$, there always exists a point $z_0 \in D_0$ such that the value $M(z_0) \neq \infty$ is well defined; i.e., $z_0 \in D_0 \cap \operatorname{dom} M$ in the notation of [1, 3.1], and the equivalent conditions

$$\left(\int_0^{r_0} \frac{\nu(z_0,t)}{t}\,\mathrm{d}t < +\infty\right) \iff \left(\int_{D(z_0,r_0)} \ln|z'-z_0|\,\mathrm{d}\nu(z') > -\infty\right), \qquad D(z_0,3r_0) \Subset D_0, \tag{2.5}$$

hold for some number $r_0 > 0$. Conditions (2.5), in particular, ensure the existence of a function $u_0 \in \operatorname{sbh}_*(D)$ with Riesz measure $\nu_{u_0} = \nu$ and with the property $u_0(z_0) \neq -\infty$ [1, 3.1].

In what follows, we temporarily need the boundedness of the function M in a neighborhood of the point z_0. To this end, we so far transform it locally while preserving condition (2.4). Using (2.5) and the representation $M = u_+ - u_-$ of M as a difference of subharmonic functions $u_+, u_- \in \operatorname{sbh}_*(D)$, one can locally change the values of M in $D(z_0,2r_0) \Subset D_0$, namely, continue the functions u_+ and u_- into $D(z_0,2r_0)$ harmonically by the Poisson integral. We denote them by u_+° and u_-°, respectively. Then $M^\circ := u_+^\circ - u_-^\circ \in \delta\text{-sbh}_*(D)$ is a bounded function in a neighborhood of the closed disk $\operatorname{clos} D(z_0,r_0)$, and (2.4) is still true for all $v \in \operatorname{sbh}^+_{00}(D \setminus D_0; \leqslant b)$. For now, we denote the function M° by the same symbol M. By $J_{z_0}(D)$, as in [1], we denote the class of all Jensen measures $\mu \in \operatorname{Meas}^+_{\mathrm{c}}(D)$ satisfying the condition $u(z_0) \leqslant \int u\,\mathrm{d}\mu$ for all $u \in \operatorname{sbh}(D)$. We need the following theorem.

Theorem A (a special case of Theorem 6 in [5])**.** *Assume that $M \in L^1_{\mathrm{loc}}(D)$, $z_0 \in D$, $u_0 \in \operatorname{sbh}(D)$, and $u_0(z_0) \neq -\infty$. If the function M is bounded in an open neighborhood of the closure $\operatorname{clos} D_1$ of some subdomain $D_1 \Subset D$ containing z_0 and there exists a number $C_0 \in \mathbb{R}$ such that*

$$\int_D u_0\,\mathrm{d}\mu \leqslant \int_D M\,\mathrm{d}\mu + C_0 \quad \text{for any Jensen measure } \mu \in J_{z_0}(D), \tag{2.6}$$

then for each continuous function $r\colon D \to \mathbb{R}^+$ satisfying condition (1.13) *there exists a function $w \in \operatorname{sbh}_*(D)$ such that*

$$u_0 + w \overset{(1.14)}{\leqslant} M^{*r} \quad \text{on } D. \tag{2.7}$$

In our case, the role of the domain D_1 will be played by the disk $D(z_0, r_0)$. In addition, the following notion is required.

2.2.1. *Jensen potentials.* A function $V \in \operatorname{sbh}^+(\mathbb{C}_\infty \setminus \{z_0\})$ is called a *Jensen potential inside D with pole at $z_0 \in D$* [1, Definition 3] if the following two conditions are satisfied:

(1) There exists a domain $D_V \Subset D$ containing $z_0 \in D_V$ such that $V(z) \equiv 0$ for $z \in \mathbb{C}_\infty \setminus D_V$.

(2) One has a *logarithmic seminormalization at z_0*; namely,

$$\limsup_{z_0 \neq z \to z_0} \frac{V(z)}{l_{z_0}(z)} \leqslant 1, \tag{2.8o}$$

$$\text{where } l_{z_0}(z) := \begin{cases} \ln \frac{1}{|z - z_0|} & \text{for } z_0 \neq \infty, \\ \ln |z| & \text{for } z_0 = \infty. \end{cases} \tag{2.8l}$$

The class of all such Jensen potentials will be denoted by $PJ_{z_0}(D)$.

The *logarithmic potential of genus* 0 *of a probability measure $\mu \in \operatorname{Meas}_c^+(\mathbb{C}_\infty)$ with a pole at* $z_0 \in \mathbb{C}_\infty$ is defined for all $w \in \mathbb{C}_\infty \setminus \{z_0\}$ as the function

$$V_\mu(w) := \int_D \ln\Bigl|\frac{w - z}{w - z_0}\Bigr| \, \mathrm{d}\mu(z) = \int_D \ln\Bigl|1 - \frac{z - z_0}{w - z_0}\Bigr| \, \mathrm{d}\mu(z) \quad \textit{for } z_0 \neq \infty, \tag{2.9o}$$

where for $w = \infty$ the integrands are defined to be 0;

$$V_\mu(w) := \int_D \ln\Bigl|\frac{w - z}{z}\Bigr| \, \mathrm{d}\mu(z) = \int_D \ln\Bigl|1 - \frac{w}{z}\Bigr| \, \mathrm{d}\mu(z) \quad \textit{for } z_0 = \infty, \tag{2.9∞}$$

where for $z = \infty$ the integrands are defined to be 0.

Recall the main relationships between the classes $J_{z_0}(D)$ and $PJ_{z_0}(D)$. The first is the following duality statement.

Proposition 1 [15, Proposition 1.4, duality theorem]**.** *The mapping*

$$\mathscr{P}\colon J_{z_0}(D) \to PJ_{z_0}(D), \quad \mathscr{P}(\mu) \overset{(2.9)}{:=} V_\mu, \quad \mu \in J_{z_0}(D),$$

is a bijection for which $\mathscr{P}\bigl(t\mu_1 + (1-t)\mu_2\bigr) = t\mathscr{P}(\mu_1) + (1-t)\mathscr{P}(\mu_2)$ for all $t \in [0,1]$ (affinity), *and the inverse bijection $\mathscr{P}^{-1}$ is defined by the formula*

$$\mathscr{P}^{-1}(V) \overset{(2.8l)}{=} \frac{1}{2\pi} \Delta V \Bigr|_{D \setminus \{z_0\}} + \Bigl(1 - \limsup_{z_0 \neq z \to z_0} \frac{V(z)}{l_{z_0}(z)}\Bigr) \cdot \delta_{z_0}, \qquad V \in PJ_{z_0}(D). \tag{2.10}$$

The second is the extended *Poisson–Jensen formula* (2.11).

Proposition 2 [15, Proposition 1.2]**.** *Let $\mu \in J_{z_0}(D)$. Then for any function $u \in \operatorname{sbh}(D)$ with Riesz measure ν_u and with $u(z_0) \neq -\infty$ one has*

$$u(z_0) + \int_{D \setminus \{z_0\}} V_\mu \, \mathrm{d}\nu_u = \int_D u \, \mathrm{d}\mu. \tag{2.11}$$

Lemma 1. *Assume that $M \in \delta\text{-}\operatorname{sbh}_*(D)$ with Riesz charge ν_M, $z_0 \in \operatorname{dom} M$, $u_0 \in \operatorname{sbh}(D)$ with Riesz measure ν, $u_0(z_0) \neq -\infty$, $V \in PJ_{z_0}(D)$ is a Jensen potential, and $C_1 \in \mathbb{R}$. If*

$$\int_{D \setminus \{z_0\}} V \, \mathrm{d}\nu \leqslant \int_{D \setminus \{z_0\}} V \, \mathrm{d}\nu_M + C_1, \tag{2.12}$$

then for the Jensen measure $\mu \overset{(2.10)}{=} \mathscr{P}^{-1}(V) \in J_{z_0}(D)$ one has the inequality

$$\int u_0 \, \mathrm{d}\mu \leqslant \int M \, \mathrm{d}\mu + C_0, \quad \textit{where } C_0 = C_1 - M(z_0) + u_0(z_0). \tag{2.13}$$

Proof of Lemma 1. Under the condition $z_0 \in \operatorname{dom} M$, the function M can be represented by the difference $M = u_+ - u_-$ of functions $u_\pm \in \operatorname{sbh}_*(D)$ with respective Riesz measures $\nu_M^\pm \in \operatorname{Meas}^+(D)$ such that $u_\pm(z_0) \neq -\infty$. The extended Poisson–Jensen formula in Proposition 2 applies to each of the functions $u_\pm$ and hence to the function M. Thus, for the Jensen measure $\mu \overset{(2.10)}{:=} \mathscr{P}^{-1}(V)$ we obtain

$$\int_D u_0 \, \mathrm{d}\mu \overset{(2.11)}{=} \int_{D\setminus\{z_0\}} V \, \mathrm{d}\nu + u_0(z_0) \overset{(2.12)}{\leqslant} \int_{D\setminus\{z_0\}} V \, \mathrm{d}\nu_M + C_1 + u_0(z_0)$$
$$\overset{(2.11)}{=} \int M \, \mathrm{d}\mu - M(z_0) + C_1 + u_0(z_0),$$

which proves the desired inequality (2.13). □

Let us return directly to the proof of Theorem 3. For a domain D with nonpolar boundary $\partial D \subset \mathbb{C}_\infty$, there always exists a Green function $g_D(\,\cdot\,, z_0)$ *with a pole at* z_0. Throughout the subsequent proof, for brevity we write

$$g := g_D(\,\cdot\,, z_0) \textit{ is the Green function for } D \textit{ with a pole at } z_0 \in D_0.$$

Here only the following properties (see [3, 4.4] and [12, 3.7, 5.7]) of the function g are important.

g1. The normalization condition $\lim\limits_{z_0 \neq z \to z_0} \dfrac{g(z)}{l_{z_0}(z)} \overset{(2.8\mathrm{l})}{=} 1$ at the point z_0, which is stronger than (2.8o).

g2. $g \in \operatorname{har}^+(D \setminus \{z_0\})$: the harmonicity and positivity in $D \setminus \{z_0\}$.

In particular, it follows from the maximin principle owing to inclusion $z_0 \in D_0 \Subset D$ that

$$0 < \operatorname{const}_{z_0, D_0, D} := B_0 := \sup_{z \in \partial D_0} g(z) < +\infty. \tag{2.14}$$

Let $V \in PJ_{z_0}(D)$ be an *arbitrary Jensen potential.* Then, in view of properties g1–g2,

$$\limsup_{D \ni z \to z_0} \frac{(V-g)(z)}{l_{z_0}(z)} \overset{\mathrm{g1}}{\leqslant} 0, \qquad V - g \overset{\mathrm{g2}}{\in} \operatorname{sbh}_*(D \setminus \{z_0\}).$$

It follows that the point z_0 is a removable singularity of the function $V - g \in \operatorname{sbh}_*(D \setminus \{z_0\})$, and since

$$\limsup_{D \ni z' \to z} (V-g)(z') \leqslant \limsup_{D \ni z' \to z} V(z') = 0, \qquad z \in \partial D,$$

it follows by the maximum principle that the inequality $V - g \leqslant 0$ is satisfied on D for the function $V - g \in \operatorname{sbh}_*(D)$; i.e.

$$V \leqslant g \quad \text{on } D, \qquad V \overset{(2.14)}{\leqslant} B_0 \quad \text{on } \partial D_0. \tag{2.15}$$

Consequently, inequality (2.4) holds for the function

$$v := \frac{b}{B_0} V \in \operatorname{sbh}_{00}^+\big(D \setminus D_0; \leqslant b\big)$$

considered in the open neighborhood $D \setminus D_0$. Multiplying both sides by B_0/b, we obtain

$$\int_{D\setminus D_0} V \, \mathrm{d}\nu \leqslant \int_{D\setminus D_0} V \, \mathrm{d}\nu_M + \frac{B_0}{b} C, \qquad V \in PJ_{z_0}(D).$$

This inequality can be rewritten as

$$\int_{D\setminus\{z_0\}} V \, \mathrm{d}\nu \leqslant \int_{D\setminus\{z_0\}} V \, \mathrm{d}\nu_M + \frac{B_0}{b} C + \int_{D_0\setminus\{z_0\}} V \, \mathrm{d}\nu + \int_{D_0\setminus\{z_0\}} V \, \mathrm{d}\nu_M^-$$
$$\overset{(2.15)}{\leqslant} \int_{D\setminus\{z_0\}} V \, \mathrm{d}\nu_M + \bigg(\frac{B_0}{b} C + \int_{D_0\setminus\{z_0\}} (g \, \mathrm{d}\nu + g \, \mathrm{d}\nu_M^-)\bigg), \qquad V \in PJ_{z_0}(D). \tag{2.16}$$

Here the last integral is finite in view of (2.5) and the fact that $z_0 \in D_0 \cap \operatorname{dom} M$ and also does not depend on $V \in PJ_{z_0}(D)$. Thus, inequality (2.12) holds with a constant C_1 equal to the value of the "big" bracket on the right-hand side in formula (2.16) for any potential $V \in PJ_{z_0}(D)$. Then, by Lemma 1, (2.13) holds for any Jensen measure $\mu \in J_{z_0}(D)$. Therefore, for any potential $V \in PJ_{z_0}(D)$, condition (2.6) of Theorem A is satisfied and there exists a function $w \in \operatorname{sbh}_*(D)$ for which inequality (2.7) holds. Further, the inequality $\nu + \nu_w \geqslant \nu$ on D obviously holds with the Riesz measure ν_w of the function w. Therefore, the function $u^\circ := u_0 + w$ with the Riesz measure $\nu_{u^\circ} = \nu + \nu_w$ is the function needed in (2.2), so far for the function M°, which is different from M in the disk $D(z_0, 2r_0)$. Consider $M \leqslant M^\circ$. For a *continuous* function r, the functions M^{*r} and $(M^\circ)^{*r}$ are continuous in D as well, since both lie in the class $L^1_{\text{loc}}(D)$. At the same time, the subharmonic function $u^\circ \neq -\infty$ is bounded above in $D(z_0, 3r_0) \Subset D$. Therefore, there exists a sufficiently large constant $C_2 \geqslant 0$ such that $u_0 := u^\circ - C_2 \leqslant (M^\circ)^{*r}$ on D with Riesz measure $\nu_{u_0} = \nu_{u^\circ} \geqslant \nu$. Under the conditions on the function r, there exists a subdomain $D_2 \Subset D$ that includes $D(z_0, r_0)$ and for which, by the construction of the function M° and the definition of averaging on $D \setminus D_2$, one has $(M^\circ)^{*r} = M^{*r}$ and hence the inequality $u_0 \leqslant M^{*r}$ on $D \setminus D_2$. Since the function M^{*r} is continuous on D and u_0 is bounded above on D_1, it follows that there exists a sufficiently large number $C_3 \geqslant 0$ such that $u := u_0 - C_3 \leqslant M^{*r}$ on D with Riesz measure $\nu_u = \nu_{u_0} \geqslant \nu$, which gives (2.2).

If the function $M \in \delta\text{-}\operatorname{sbh}_*(D)$ is continuous, then it is necessarily locally bounded below, which allows avoiding the intermediate use of the function M° in the proof. In addition, the continuous function M is locally uniformly continuous, which allows choosing a continuous function r satisfying condition (1.13) for which $M^{*r} \leqslant M + 1$ on D. This permits one to replace the right-hand side M^{*r} in (2.2) with M.

Now assume that condition (2.1) of Theorem 3 is satisfied. From this condition, we derive condition (2.4), under which all the conclusions of Theorem 3 have already been proved.

Let $v \in \operatorname{sbh}^+_{00}(D \setminus D_0; \leqslant b)$ be an *arbitrary compactly supported* test function, where the subdomain $D_0 \Subset \operatorname{int} S$ is chosen as in (2.3). Since the function v is compactly supported, it follows that there exists a subdomain $D_v \Subset D$ such that $D_0 \Subset D_v$ and the function v is subharmonic on $D \setminus D_0$ and identically zero on $D \setminus D_v$. In this case, we set

$$\varepsilon_0 := \frac{1}{2} \min\{\operatorname{dist}(D_v, \partial D), \operatorname{dist}(D_0, D \setminus S)\} > 0. \tag{2.17}$$

Consider an infinitely differentiable function $a\colon\ RR^+ \to \mathbb{R}^+$

$$\text{with } \textit{support} \ \operatorname{supp} a \subset (0,1) \ \text{ and } \textit{normalization} \ 2\pi \int_0^{+\infty} a(x)x \,\mathrm{d}x = 1, \tag{2.18}$$

and also the measures $\alpha_\varepsilon \in \operatorname{Meas}^+(\mathbb{C})$ determined by the densities

$$\mathrm{d}\alpha_\varepsilon(z) \overset{(2.18)}{:=} \frac{1}{\varepsilon^2} a\left(\frac{|z|}{\varepsilon}\right) \mathrm{d}\lambda(z), \qquad 0 < \varepsilon < \varepsilon_0,\ z \in \mathbb{C}. \tag{2.19}$$

It is well known ([3, 2.7], [12, 3.4.1]) that, for a decreasing sequence of numbers $0 < \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$, $\varepsilon_n \overset{(2.17)}{\leqslant} \varepsilon_0$, the sequence of subharmonic infinitely differentiable convolution functions $v_n := v * \alpha_{\varepsilon_n}$ decreases in $n \in \mathbb{N}$ and is pointwise convergent to the function v on $D \setminus S$. In particular, according to (2.17), $v \leqslant v_n$ on $D \setminus S$ and $v_n \leqslant b$ on $D \setminus S$ as the averages over the measures (2.19), which are *probability measures* in view of (2.18). By construction, all the functions v_n belong to $\operatorname{sbh}^+_{00}(D \setminus S; \leqslant b) \cap C^\infty(D \setminus S)$. By condition (2.1), there exists a number C such that

$$\int_{D\setminus S} v_n \,\mathrm{d}\nu \leqslant \int_{D\setminus S} v_n \,\mathrm{d}\nu_M + C$$

for all functions v_n constructed above for all $n \in \mathbb{N}$. Hence, by the Hahn–Jordan decomposition for the Riesz charge $\nu_M = \nu_M^+ - \nu_M^-$, $\nu_M^\pm \in \operatorname{Meas}^+(D)$, we have

$$\int_{D\setminus S} v_n \, \mathrm{d}(\nu + \nu_M^-) \leqslant \int_{D\setminus S} v_n \, \mathrm{d}\nu_M^+ + C', \qquad n \in \mathbb{N},$$

which, since $v \leqslant v_n$ on $D \setminus S$, gives

$$\int_{D\setminus S} v \, \mathrm{d}\nu \leqslant \int_{D\setminus S} v_n \, \mathrm{d}\nu_M^+ - \int_{D\setminus S} v \, \mathrm{d}\nu_M^- + C, \qquad n \in \mathbb{N}.$$

By letting $n \to +\infty$ in the first integral on the right-hand side, since the sequence of compactly supported infinitely differentiable test functions v_n decreases to $v \in \operatorname{sbh}_{00}^+(D \setminus D_0; \leqslant b)$, we obtain

$$\int_{D\setminus S} v \, \mathrm{d}\nu \leqslant \int_{D\setminus S} v \, \mathrm{d}\nu_M^+ - \int_{D\setminus S} v \, \mathrm{d}\nu_M^- + C = \int_{D\setminus S} v \, \mathrm{d}\nu_M + C. \tag{2.20}$$

Let us define constants $C_4, C_5 \in \mathbb{R}^+$ independent of v by the formulas

$$0 \leqslant \int_{D_0\setminus S} v \, \mathrm{d}\nu \leqslant b\nu(D_0 \setminus S) =: C_4 < +\infty,$$
$$0 \leqslant \int_{D_0\setminus S} v \, \mathrm{d}\nu_M^- \leqslant b\nu_M^-(D_0 \setminus S) =: C_5 < +\infty.$$

Then inequality (2.20) without the intermediate difference of integrals remains valid if the integration over $D \setminus S$ is replaced with the integration over $D \setminus D_0$ and the constant C is replaced with the constant $C + C_4 + C_5$. By virtue of the arbitrariness in the choice of the compactly supported test function $v \in \operatorname{sbh}_{00}^+(D \setminus D_0; \leqslant b)$, inequality (2.4) with the new constant $C + C_4 + C_5$ instead of C is satisfied for all such v. This completes the proof of Theorem 3. □

Remark 3. In the case of a function $M \in \operatorname{sbh}_*(D)$, it suffices to require in Theorem 3 that the function r with property (1.13) be only locally separated from zero below in the sense that for each $z \in D$ there exists a number $t_z > 0$ such that

$$D(z, t_z) \Subset D, \qquad \sup_{z' \in D(z,t_z)} r(z') > 0.$$

Indeed, elementary geometric considerations using compactness (for example, exhaustion of the domain D by a sequence of relatively compact subdomains) permit one to prove the following assertion.

Lemma 2. *For the function r separated below from zero on D and satisfying condition* (1.13), *there exists a continuous and even infinitely differentiable function $\hat{r} \leqslant r$ that still satisfies condition* (1.13).

By applying Theorem 3 with $\hat{r} \in C(D)$ instead of r, we construct the desired function $u \leqslant M^{*\hat{r}} \leqslant M^{*r}$, where (2.2) and the fact that the means (1.14) increase with respect to r for $M \in \operatorname{sbh}_*(D)$ has been used.

2.3. Proof of the implication h3 ⇒ h1 with the lift $M^{\uparrow r}$. Under the assumptions of the "*Conversely,...*" part of Theorem 2, claim h3 and the implication s3 ⇒ s1 in Theorem 1 imply the existence of a subharmonic function $u \in \operatorname{sbh}_*(D)$ with Riesz measure $\nu_u \geqslant n_{\mathsf{Z}}$ satisfying the inequality $u \leqslant M$ on D. Since there always exists a holomorphic function $f_{\mathsf{Z}} \in \operatorname{Hol}_*(D)$ with the sequence of zeros $\operatorname{Zero}_{f_{\mathsf{Z}}} = \mathsf{Z}$ by the Weierstrass theorem, we see that the latter means that there exists a function $s \in \operatorname{sbh}_*(D)$ with Riesz measure $\nu_s := \nu_u - n_{\mathsf{Z}} \in \operatorname{Meas}^+(D)$ such that

$$u = \ln|f_{\mathsf{Z}}| + s \leqslant M \quad \text{on } D. \tag{2.21}$$

The following Lemma 3 does not assume that the boundary ∂D is a nonpolar set.

Lemma 3. *Assume that* $D \subset \mathbb{C}_\infty$ *is an arbitrary proper subdomain,* $u_0, s, M \in \mathrm{sbh}_*(D)$*, and*

$$u_0 + s \leqslant M \quad \text{on } D. \tag{2.22}$$

Then there exists a function $g \in \mathrm{Hol}_*(D)$ *such that*

$$u_0 + \ln|g| \leqslant M^{\uparrow r} \quad \text{on } D, \tag{2.23}$$

where the lift $M^{\uparrow r}$ *is defined in Sec.* 1.2.1 *depending on the type of domain* D *in* (i)-(iii) *based on the arbitrary choice of a continuous function* $r\colon D \to \mathbb{R}^+$ *satisfying condition* (1.13) *in* (1.15) *and* (1.16)*, the number* $\varepsilon > 0$ *in* (1.15)*, and the number* $P > 0$ *in* (1.17).

Proof. *Case* (i): Eq. (1.15). Since the function u_0 is subharmonic, we obtain, by averaging both sides of inequality (2.22) over the circles $D(z,r)$ with the Lebesgue measure λ,

$$u_0 + s^{*r} \leqslant u_0^{*r} + s^{*r} \leqslant M^{*r} \quad \text{on } D. \tag{2.24}$$

By [16, Theorem 3], there exists a function $g \in \mathrm{Hol}_*(D)$ such that

$$\ln|g(z)| \leqslant s^{*r}(z) + \ln\frac{1}{r(z)} + (1+\varepsilon)\ln(1+|z|), \qquad z \in D, \tag{2.25}$$

whence, according to (2.24) and definition (1.15), we obtain (2.23).

Case (ii): Eq. (1.16). The case of $\mathbb{C}_\infty \setminus \operatorname{clos} D \neq \varnothing$ is covered by [14, Theorem 1] and partly by [13, Theorem 1]. For a domain $D \subset \mathbb{C}$ that is simply connected in $\mathbb{C}_\infty$, the proof follows the same scheme as in the previous case but with the use of the inequality

$$\ln|g(z)| \leqslant s^{*r}(z) + \ln\frac{1}{r(z)}, \qquad z \in D,$$

based on [16, Corollary 3(iii)], instead of (2.25).

Case (iii): Eq. (1.17). The case of $D = \mathbb{C}$ was analyzed in [14, Theorem 1] and partly in [13, Theorem 1]. $\square$

By Lemma 3, inequality (2.21) written in the form (2.22) with $u_0 := \ln|f|$ implies the conclusion (2.23), which means that $\ln|f_{\mathsf{Z}}g| = \ln|f_{\mathsf{Z}}| + \ln|g| \leqslant M^{\uparrow r}$. Thus, the function $f := f_{\mathsf{Z}}g \in \mathrm{Hol}_*(D)$, which vanishes on Z, is the desired one.

3. Converse Theorem with Green Functions

The converse theorem in this section only uses the Green function [12], extended by zero, of a special system of relatively compact subdomains regular for the Dirichlet problem in D and containing a given subdomain $D_0 \Subset D$ with a fixed pole $z_0 \in D_0$. Note that *each such Green function is a subharmonic compactly supported test function for the region* D *outside the subdomain* D_0. Here, unlike Theorems 1–3, the proper *subdomain* $D \subset \mathbb{C}_\infty$ *is arbitrary*. Throughout Sec. 3, in addition to (1.2), we assume that $z_0 \in D_0 \Subset D \subset \mathbb{C}_\infty \neq D$, where D_0 is a domain.

Definition 2 (see [5, Definition 1], [17, Definition 11])**.** A system of domains $\mathscr{U}_{D_0}(D) \subset \{D' \Subset D\colon D_0 \subset D'\}$ *regular* for the Dirichlet problem is called a *regular optimally exhausting system of domains in* D *with center* D_0 if $\bigcup\{D'\colon D' \in \mathscr{U}_{D_0}(D)\} = D$ and the following two conditions hold for any domains D_1 and D_2 satisfying the inclusions $D_0 \subset D_1 \Subset D_2 \subset D$:

(1) There exists a domain $D' \in \mathscr{U}_{D'}(D)$ such that $D_1 \Subset D' \Subset D_2$ and each nonempty bounded connected component of the set $\mathbb{C}_\infty \setminus D'$ has a nonempty intersection with $\mathbb{C}_\infty \setminus D_2$.

(2) For any domain $D \in \mathscr{U}_{D_0}(D)$, there exists a domain $D'' \in \mathscr{U}_{D_0}(D)$ such that $D_1 \Subset D'' \Subset D_2$ and the union $D'' \cup D'$ lies in $\mathscr{U}_{D_0}(D)$ as well.

Finally, the system $\mathscr{U}_{D_0}(D)$ is assumed to be *conditionally invariant with respect to the shift in* D; i.e., the conditions $D' \in \mathscr{U}_{D_0}(D)$, $z \in \mathbb{C}$, and $D_0 \subset D' + z \Subset D$ imply that $D' + z \in \mathscr{U}_{D_0}(D)$.

Example 1. A simple example of a regular optimally exhausting system of domains is given by the *special system of all possible connected unions* $D' \supset D_0$ *of finitely many disks* $D(z,t) \Subset D$ *excluding those domains* D' *whose complements* $\mathbb{C}_\infty \setminus D$ *have isolated points*. With the same exceptions, the disks in this example can be replaced with all possible n-gons relatively compact in D or, more generally, with simply connected subdomains ([3, Theorems 4.2.1 and 4.2.2], [12, 2.6.3]) of some special kind.

Theorem 4. *Let* $M = M_+ - M_- \overset{(1.3)}{\in} \delta\text{-sbh}_*(D)$ *with Riesz charge* $\nu_M \in \mathrm{Meas}(D)$, *where* $M_+ \in \mathrm{sbh}_*(D) \cap C(D)$ *and* $M_- \in \mathrm{sbh}_*(D)$, *and let* $z_0 \in D_0 \cap \mathrm{dom}\, M \Subset D$. *Suppose that* (2.5) *holds for the measure* $\nu \in \mathrm{Meas}^+(D)$ *for some* $r_0 \in \mathbb{R}^+_*$. *Let* $\mathscr{U}_{D_0}(D)$ *be a regular optimally exhausting system of domains in* D *with center* D_0 *for which the inequalities*

$$\int_{D\setminus\{z_0\}} g_{D'}(\,\cdot\,, z_0)\,\mathrm{d}\nu \leqslant \int_{D\setminus\{z_0\}} g_{D'}(\,\cdot\,, z_0)\,\mathrm{d}\nu_M + C, \qquad D' \in \mathscr{U}_{D_0}(D), \tag{3.1}$$

hold with some constant $C \in \mathbb{R}$; *i.e., the charge* ν_M *is an affine balayage of the measure* ν *for* D *outside* z_0 *with respect to the class of Green functions* $g_{D'}(\,\cdot\,, z_0)$ *with* $D' \in \mathscr{U}_{D_0}(D)$. *Then there exists a function* $u \in \mathrm{sbh}_*(D)$ *with Riesz measure* $\nu_u \geqslant \nu$ *satisfying the inequality* $u \leqslant M$ *on* D.

Proof. Let ν_{M_+} and ν_{M_-} be the Riesz measures of the functions M_+ and M_-, respectively. Then a series of inequalities (3.1) uniform in the constant C can be written, setting $\nu_1 := \nu + \nu_{M_-}$, in the form

$$\int_{D\setminus\{z_0\}} g_{D'}(\,\cdot\,, z_0)\,\mathrm{d}\nu_1 \leqslant \int_{D\setminus\{z_0\}} g_{D'}(\,\cdot\,, z_0)\,\mathrm{d}\nu_{M_+} + C, \qquad D' \in \mathscr{U}_{D_0}(D), \tag{3.2}$$

where $\nu_1, \nu_{M_+} \in \mathrm{Meas}^+(D)$ are already *positive measures*. Take some subharmonic function $u_1 \in \mathrm{sbh}_*(D)$ in D with Riesz measure ν_1. In view of the condition $z_0 \in \mathrm{dom}\, M$ and also the equivalent conditions (2.5) on z_0, the Riesz measure ν_1 satisfies conditions (2.5) with ν replaced with ν_1. Therefore, $M_-(z_0) \neq -\infty$ and necessarily $u_1(z_0) \neq -\infty$. Further, we need the following variations of the assertions in [5, Main Theorem, Theorem 6]:

Theorem B (a special case of [17, Theorem (main)])**.** *Let a function* $M \in \mathrm{sbh}_*(D)$ *with Riesz measure* ν_M *be bounded below in some open neighborhood of the closure* $\mathrm{clos}\, D_0$, *let* $u \in \mathrm{sbh}_*(D)$ *be a function with Riesz measure* ν *on* D, *let* $u(z_0) \neq -\infty$, *and let* $\mathscr{U}_{D_0}(D)$ *be a regular optimally exhausting system of domains for* D *with center* $D_0 \ni z_0$. *If**

$$-\infty < \inf_{D' \in \mathscr{U}_{D_0}(D)} \biggl(-\int_{D\setminus\{z_0\}} g_{D'}(\,\cdot\,, z_0)\,\mathrm{d}\nu_u + \int_{D\setminus\{z_0\}} g_{D'}(\,\cdot\,, z_0)\,\mathrm{d}\nu_M \biggr), \tag{3.3}$$

then for any continuous function $r\colon D \to \mathbb{R}^+$ *satisfying condition* (1.13) *there exists a function* $v \in \mathrm{sbh}_*(D)$ *harmonic in an open neighborhood of the point* z_0 *such that* $u + v \leqslant M^{*r}$ *on* D *with the averaging in* (1.14). *Moreover, if, in addition,* $M \in C(D)$, *then the variable averaging* M^{*r} *on the right-hand side in the last inequality can be replaced with* M.

By Theorem B applied to the function u_1 and the *continuous* function M_+ instead of u and M, respectively, owing to inequality (3.2) corresponding to condition (3.3), there exists a function $v \in \mathrm{sbh}_*(D)$ harmonic in a neighborhood of z_0 such that $u_1 + v \leqslant M_+$ on D. By construction, $u_1 \in \mathrm{sbh}_*(D)$ with Riesz measure $\nu_1 := \nu + \nu_{M_-}$. Consequently, the Riesz measure of the function $u_0 := u_1 - M_-$ is the measure ν; i.e., there exists a function $u := u_0 + v \in \mathrm{sbh}_*(D)$ with Riesz measure $\nu_u \geqslant \nu$ such that $u \leqslant M_+ - M_- = M$ on D, which completes the proof of Theorem 4. □

Corollary 1. *Suppose that, under the assumptions of Theorem* 4 $D \subset \mathbb{C}$, $M \in \mathrm{sbh}_*(D) \cap C(D)$ *and the measure* ν_M *is the affine balayage of the sequence* $\mathsf{Z} = \{\mathsf{z}_k\}_{k=1,2,\dots} \subset D$, $z_0 \notin \mathsf{Z}$, *for* D *outside the singleton* $S := \{z_0\}$ *with respect to the class of Green functions* $g_{D'}(\,\cdot\,, z_0)$ *with* $D' \in \mathscr{U}_{D_0}(D)$; *i.e., for some number* $C \in \mathbb{R}$, *according to* (1.6)–(1.8), *condition* (3.1) *is satisfied*

*Unfortunately, in the statement of the main theorem from our work [5], on the intermediate stage of whose proof [17, Theorem (Main)] is based, an annoying typo in the ± signs crept in. Thus, the ratio used in its statement [5, item (h1), (2.11)] must look exactly like (3.3). Further comment can be found in the footnote to [17, Main Theorem].

in the following form:

$$\sum_{\mathsf{z}_k\in D'} g_{D'}(\mathsf{z}_k, z_0) \leqslant \int_{D\setminus\{z_0\}} g_{D'}(\,\cdot\,, z_0)\,\mathrm{d}\nu_M + C, \qquad D' \in \mathscr{U}_{D_0}(D).$$

Then there exists a function $f \in \mathrm{Hol}_*(D)$ *such that* $f(\mathsf{Z}) = 0$ *and the inequality* $|f| \leqslant \exp M^{\uparrow r}$ *holds on* D, *where the lift* $M^{\uparrow r}$ *is defined in Sec.* 1.2.1 *depending on the type of domain* D *in* (i)-(iii), *based on an arbitrary choice of the continuous function* $r\colon D \to \mathbb{R}^+$ *satisfying condition* (1.13) *in* (1.15), (1.16), *the number* $\varepsilon > 0$ *in* (1.15) *and the numbers* $P > 0$ *in* (1.17).

This corollary can be derived from Theorem 4 in the same way as the proof of the implication h3 $\Rightarrow$ h1 with lift $M^{\uparrow r}$ can be derived from Theorem 3 in section 2.3.

Remark 4. Based on the analysis of the subtle results of Hansen and Netuka [18] about the approximation of Jensen measures by harmonic measures, the regular optimally exhausting system of domains $\mathscr{U}_{D_0}(D)$ with center $D_0 \subset D$ in Theorem 4 and Corollary 1 can be replaced by a system of domains $D' \Subset D$ that include the domain $D_0 \Subset D$ and are obtained from a sequence, exhausting D, of domains $D_n \Subset D$, $n \in \mathbb{N}$, regular for the Dirichlet problem and having analytic, or piecewise linear, or another "good" boundary by removing various finite sets of pairwise disjoint closed disks from the domains D_n. In addition, for the resulting system of domains it is nevertheless necessary to require conditional invariance with respect to the shift in D in Definition 2.

4. Converse Theorem with Analytic and Polynomial Disks

An important subclass of Jensen measures in the class $J_{z_0}(D)$ is generated by analytic disks in D with center z_0. An *analytic closed disk in the domain* D *with center* $z_0 \in D$ is a function $g\colon \operatorname{clos}\mathbb{D} \to D$ continuous on $\operatorname{clos}\mathbb{D}$ whose restriction to $\mathbb{D}$ is holomorphic and for which $g(0) = z_0$ ([19, Ch. 3], [20]–[23]). In particular, $g(\operatorname{clos}\mathbb{D}) \Subset D$. For any such analytic closed disk g, one can readily show that the function $w \in \mathbb{C}_\infty \setminus \{z_0\}$ given by

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\frac{w - g(e^{i\theta})}{w - z_0}\right| \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|1 - \frac{g(e^{i\theta}) - z_0}{w - z_0}\right| \mathrm{d}\theta \quad \textit{for } z_0 \neq \infty, \tag{4.1o}$$

where, by analogy with (2.9o), the integrands are defined to be 0 at $w = \infty$, and

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\frac{w - g(e^{i\theta})}{g(e^{i\theta})}\right| \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|1 - \frac{w}{g(e^{i\theta})}\right| \mathrm{d}\theta \quad \textit{for } z_0 = \infty, \tag{4.1∞}$$

where, by analogy with (2.9∞), the integrands are defined to be 0 for $g(e^{i\theta}) = \infty$, is a Jensen potential inside D with a pole at z_0. In particular, (4.1) defines a subharmonic positive compactly supported test function for D outside $\{z_0\}$. In Remark 2, we dubbed the functions (4.1) the logarithmic potentials of analytic disks. If an analytic closed disk g in a domain D with center $z_0 \in D$ is a *polynomial* in the complex variable, then it is natural to call it a *polynomial disk in* D *centered at* $z_0 \in D$.

Theorem 5. *Let the function* $M \overset{(1.3)}{\in} \delta\text{-}\mathrm{sbh}_*(D)$ *with a Riesz charge* ν_M, $z_0 \neq \infty$ *and the measure* $\nu \in \mathrm{Meas}^+(D)$ *be the same as in Theorem* 4. *If the charge* ν_M *is an affine balayage of the measure* ν *for* D *outside* $S := \{z_0\}$ *with respect to the function class* (4.1o), *i.e.*, *if there exists a constant* $C \in \mathbb{R}$ *such that*

$$\begin{aligned}&\int_D \frac{1}{2\pi}\int_0^{2\pi} \ln\left|1 - \frac{g(e^{i\theta}) - z_0}{z - z_0}\right| \mathrm{d}\theta\,\mathrm{d}\nu(z)\\ &\qquad\leqslant \int_D \frac{1}{2\pi}\int_0^{2\pi} \ln\left|1 - \frac{g(e^{i\theta}) - z_0}{z - z_0}\right| \mathrm{d}\theta\,\mathrm{d}\nu_M(z) + C\end{aligned}$$

for all analytic closed or only polynomial disks g *in* D *with center* z_0, *then there exists a function* $u \in \mathrm{sbh}_*(D)$ *with Riesz measure* $\nu_u \geqslant \nu$ *such that* $u \leqslant M$ *on* D.

In the case of a subharmonic function M, a discussion of the scheme of proof of Theorem 5 is contained in [6, 1.2.1–2, Suppl. 1.2.3, 1.2.4]. This is one of the reasons why we omit the proof of Theorem 5 here. Another reason is that the multidimensional version of Theorem 5 in $\mathbb{C}^n$ is more natural and will be considered together with applications elsewhere.

Arguing in almost the same way as in the proof of the implication h3 $\Rightarrow$ h1 with the lift $M^{\uparrow r}$ in Sec. 2.3, by analogy with Corollary 1 of Theorem 5, one can derive the following assertion.

Corollary 2. *Under the conditions of Theorem* 5, *consider a sequence of points* $\mathsf{Z} = \{\mathsf{z}_k\}_{k=1,2,\dots} \subset D \subset \mathbb{C}$, $z_0 \in D \setminus \mathsf{Z}$, *instead of the measure* ν *and assume that* $M \in \operatorname{sbh}_*(D) \cap C(D)$. *If the measure* ν_M *is an affine balayage of the sequence* Z *for* D *outside* $S := \{z_0\}$ *with respect to the function class* (4.1o), *i.e.*, *there exists a constant* $C \in \mathbb{R}$ *such that the inequality*

$$\sum_{\mathsf{z}_k \in D} \int_0^{2\pi} \log \left| 1 - \frac{g(e^{i\theta}) - z_0}{\mathsf{z}_k - z_0} \right| \mathrm{d}\theta \leqslant \int_D \int_0^{2\pi} \log \left| 1 - \frac{g(e^{i\theta}) - z_0}{z - z_0} \right| \mathrm{d}\theta \, \mathrm{d}\nu_M(z) + C$$

holds for all analytic closed or only polynomial disks g *in* D *with center* z_0, *then there exists a function* $f \in \operatorname{Hol}_*(D)$ *for which* $f(\mathsf{Z}) = 0$ *and* $|f| \leqslant \exp M^{\uparrow r}$ *on* D, *where the lift* $M^{\uparrow r}$ *is defined in Sec.* 1.2.1 *with the same refinements as in the conclusion of Corollary* 1.

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Bashkir State University, Ufa, Russia
e-mail: khabib-bulat@mail.ru

Bashkir State University, Ufa, Russia
e-mail: khabibullinfb@mail.ru

Translated by V. E. Nazaikinskii