

Tangential Shear Stress in Oscillatory Flow of a Viscoelastic Incompressible Fluid in a Plane Channel

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Abstract—The problems of oscillatory flow of a viscoelastic incompressible fluid in a plane channel are solved for a given harmonic oscillation of the fluid flow rate. The transfer function (amplitude–phase frequency response) is determined. Using this function, the effect of the acceleration oscillation frequency and the relaxation properties of fluid on the ratio of the tangential shear stress on channel wall to the velocity averaged over the channel cross-section (cross-sectional velocity) is determined. It is shown that the viscoelastic properties of fluid, as well as its acceleration, are the limiting factors for using the quasi-stationary approach. The found formulas for determining the transfer function for viscoelastic fluid flow in the case of non-stationary stream make it possible to determine the dissipations of mechanical energy in a non-stationary flow of the medium which are of importance for calculation of the control of hydraulic and pneumatic systems.

Keywords: viscoelastic fluid, non-stationary flow, transfer function, oscillatory flow, amplitude, phase

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The study of oscillatory flow of viscous and viscoelastic fluids in a plane and rectangular channel under the action of harmonic oscillation of the fluid flow rate can be carried out in biological mechanics, in particular, for the operation of a microchip system [1]. These systems are designed to diagnose the work of various human organs, as well as targeted delivery of drugs to them. In addition, pneumatic micro-pumps with periodic displacement of fluid from the free volumes are often used to ensure constant fluid flow rate in biomedical installations [2]. In such systems, an apparatus with the pulsating flow rate may be economically advantageous. When transporting highly viscous and heavy oil and petroleum products over long distances and circulating drilling mud in the well, one of the important tasks is to develop an effective method for reducing the hydraulic resistance of flows [3–5]. The fluids used in all the above-mentioned industries, both medicinal products and petroleum products or drilling muds treated with high molecular weight polymers can be classified as viscoelastic fluids [3–5]. As the authors know, at present there are practically no studies on the effect of flow rate pulsations on fluctuations in the hydraulic resistance and friction resistance coefficients. However, these studies are of much importance for calculating the pressure gradient and other hydrodynamic characteristics that occupy a special place in carrying out certain biomedical and other technological investigations [1, 2]. Thus, it is necessary to note an important role of studies of the tangential shear stress on the wall, together with other flow parameters, in the vibrational flow of viscous and viscoelastic fluid.

The most simplified approach to the theoretical study of vibrational flow of a viscous fluid is based on the assumption that the viscous fluid is incompressible and its flow in infinitely long cylindrical pipe of circular cross-section is laminar under the action of the pressure gradient which varies harmonically with time. The pulsating currents of viscous incompressible fluids in rigid and elastic pipes were studied in works by Gromeka [6, 7]. In these studies Gromeka determined the velocity of propagation of pulse pressure waves and their attenuation. Then, the problems of vibrational flow of viscous fluid in a pipe were investigated in Crandall's study [8]. Crandall, solving the problems of vibrating flow of viscous fluid in a circular infinite pipe, derived formulas for the velocity profile, the fluid flow rate and the impedance during the propagation of a sinusoidal pressure wave. A few years later, Lambossy [9] has published his conclusions on the same velocity profile and, in addition, has calculated the viscous drag. Womersley [10] re-derived Lambossy's solution. His distinctive qualitative results consisted in the fact that, firstly, there

exists a phase shift between the pressure and cross-sectional-average velocity fluctuations, and, secondly, the formation of a non-monotonic distribution of the velocity profiles was revealed.

For the first time, the influence of superimposed oscillations of the cross-sectional-average velocity in laminar pipe flow was studied in experimental work [11]. At relatively high oscillation frequencies, the so-called “annular” Richardson effect was obtained, namely, there is a maximum that appears on the profile of the oscillating component of the longitudinal velocity in a narrow near-wall layer, whose thickness decreases with increase in the oscillation frequency. In the rest of the pipe, the fluid oscillates as a whole in accordance with the fluctuation of the cross-sectional-average velocity. Experiments were also carried out on pipes with the inner diameter of 40 mm, in which the piston initiates harmonic variations in the fluid flow rate about zero [12]. The points obtained from oscillograms on which the local velocities were recorded at various positions in the pipe cross-section by means of an electrothermal anemometer were plotted on the graph. From the graphs it can be seen that the maximum local velocities are observed near the wall. These experimental results are in good agreement with the results of the above study. Theoretically, the problem of pulsating laminar flow in a pipe was solved in [12]. In [13], this problem was solved similarly to [12], but under the condition that not the harmonic oscillations of the cross-sectional velocity, but the oscillations of the pressure gradient were specified. From the analytical solution of the equation of motion for pulsating flow it follows that in the case of relatively high frequencies and oscillation amplitudes, at certain Reynolds numbers of the time-average flow there is a zone of reverse (reversible) flows in the neighborhood of the wall, when the local velocity is directed counterstreaming to the mean flow. In [14] the occurrence of these zones was confirmed experimentally with very good agreement between the theory and the experiment. In [15], a similar solution of the problem of pulsating flow in a plane channel and in a cylindrical pipe was carried out. It was noted that the patterns of fluctuations of hydrodynamic quantities qualitatively coincide for the flows in the plane channel and in the circular cylindrical pipe.

In [16] unsteady pulsating flows of a viscous fluid in a circular cylindrical pipe of infinite length under the action of a harmonic variable pressure gradient were studied. By solving the problem, calculation formulas for the velocity and fluid flow rate distributions were obtained. Numerical calculations showed that the velocity, the fluid flow rate, and other hydrodynamic parameters are established slowly in the pulsating flow at the lower dimensionless oscillation frequencies starting from zero initial state and relatively rapidly at the higher oscillation frequencies, being close to the parameters of a non-pulsating flow. These parameters are established almost instantly in oscillating flows at high oscillation frequencies.

Pulsating flows of a viscous incompressible fluid in a rectangular channel were studied in [17, 18]. The problem was solved using the finite-difference method. The optimal parameters of the difference scheme were determined, and the data on the oscillation amplitude and phase of the longitudinal velocity, the hydraulic drag coefficient, and other flow parameters were obtained. At the low vibration frequencies, it was shown that all hydrodynamic parameters oscillate in accordance with the laws of the cross-sectional-average velocity. For rectangular channels with various cross-sectional shapes (plane, rectangular, and circular cylindrical) the dependences of the hydrodynamic quantities on the dimensionless oscillation frequency are of the same nature in the case of high-frequency oscillations. The influence of the rectangular channel aspect ratio on pulsating flow hydrodynamics was also analyzed. In [19] oscillatory flow in a rectangular channel in which two opposite walls were permeable was considered. As was noted, the solution of this problem can be useful in describing the blood flow in fiber membranes used for artificial kidneys. The authors also obtained an analytical solution for developed oscillating flow in triangular [20] and toroidal [21] channels. In [22], the evolution of the pattern of perturbations superimposed on plane-parallel time-periodic Newtonian viscous fluid flow in a layer, one of whose boundaries performs longitudinal harmonic oscillations along itself and slippage of the material with zero friction is possible on the other boundary, was studied. On the basis of the integral relation method based on the variational inequalities for quadratic functionals and developed for unsteady flows, sufficient integral estimates for the exponential damping of initial perturbations were derived.

Of practical interest is the study of pulsating flow of a viscoelastic fluid in a plane channel and in a cylindrical pipe under the influence of harmonic oscillations of the pressure gradient or when harmonic oscillations of the fluid flow rate are superimposed on the flow. The motion of a viscoelastic fluid along a long pipe under the action of an oscillatory pressure gradient was studied in [23]. The distinctive features of this motion in comparison with the corresponding motion of the Newtonian fluid were shown. Inertia-free oscillatory flow of a viscoelastic fluid in a circular infinite pipe under the action of an oscillatory pressure gradient was studied in [24]. It was shown that in the oscillating flow the longitudinal velocity profiles are symmetric and there is a significant phase shift between the pressure gradient and the velocity. In pulsating flows, the phase shift was absent in fact, and the axial velocity varied asymmetrically with respect to its average value over the oscillation period. Laminar oscillatory flows of the Maxwell and Oldroyd-B

viscoelastic fluids were studied in [25], where many interesting features, which absent in Newtonian fluid flows, were demonstrated. The results obtained in [25] showed that in the inertialess regime, $Re \ll 1$, the properties of the flow depend on three characteristic lengths, in particular, on the wavelength λ_0 and the length of damped viscoelastic shear waves $x_0 = \left(\frac{2\nu}{\omega_0}\right)^{1/2}$, where ν is the kinematic viscosity, ω_0 is the oscillation frequency, as well as on the characteristic transverse dimension of the system a . In this connection, they were divided in accordance with their length into three scales and three independent dimensionless groups, namely, $\frac{t_0}{\lambda}$ (viscosity up to relaxation time), De (relaxation time up to oscillation period) and X (viscosity coefficient). At the same time, the oscillatory flow regions were divided into two systems corresponding to the “wide” $\left(\frac{a}{x_0} > 1\right)$ and “narrow” $\left(\frac{a}{x_0} < 1\right)$ systems. In the wide systems, the oscillations are restricted to near-wall flows, being inviscid in the central core. In the narrow systems, transverse waves also cross the entire system and cross its center. This ultimately leads to constructive resonances which result in sharp increase in the amplitude of the velocity profile. Unsteady flows of a viscoelastic fluid are analyzed using the Oldroyd-B model in a circular infinite cylindrical pipe under the action of a time-dependent pressure gradient in the following cases: a) the pressure gradient changes with time in accordance with exponential laws; b) the pressure gradient changes according to the harmonic laws; c) the pressure gradient is constant [26]. In all the cases, formulas for the distributions of the velocity, the fluid flow rate, and other hydrodynamic quantities in pulsating flow were obtained.

In [27] the problem of unsteady oscillatory flow of a viscoelastic fluid in a circular cylindrical pipe was considered based on the Maxwell model. Formulas for determining the dynamic and frequency characteristics were obtained. With the help of numerical experiments, the influence of the oscillation frequency and the relaxation properties of the fluid on the tangential shear stress on the wall was studied. It was shown that the viscoelastic properties of the fluid, as well as its acceleration, are the limiting factors for using the quasi-stationary approach.

In recent decades, the problem of electrokinetic phenomena, including electroosmosis, flow potential, electrophoresis, and sedimentation potential, has attracted much attention and provided many applications to micro- and nanochannels. In this connection, in [28] electrokinetic flow of viscoelastic fluids in a plane channel under the action of an oscillatory pressure gradient was studied. It was assumed that the fluid flow is laminar and unidirectional; in this regard, the motion of fluid occurs in the linear regime. The surface potentials are considered to be small; therefore, the Poisson–Boltzmann equation can be linearized. A resonant behavior in which the elastic property of the Maxwell fluid prevails develops in the flow. The resonant phenomenon enhances the electrokinetic effect and, at the same time, the efficiency of electrokinetic energy conversion increases.

In the above works, the fluid velocity field is mainly studied at various regimes of variation in the pressure gradient. Variation in the tangential and normal stress that develops in the oscillatory flow has been studied relatively little. In the most cases, in the hydrodynamic models of unsteady flows, fluids were replaced by a sequence of flows with a quasi-stationary distribution of hydrodynamic quantities. However, the structures of unsteady flows differ from the structures of stationary flows, and in such cases such a substitution must be justified in each particular case. At present, the question of the legitimacy of investigation of the quasi-stationary characteristics for determining the shear stress field in non-stationary flows of viscous and viscoelastic fluids has not been practically resolved. Naturally, under such conditions, it becomes necessary to use hydrodynamic models of non-stationary processes that take into account the variation in the hydrodynamic characteristics of the flow as a function of time.

It should be noted that in the generic case the hydrodynamic characteristic in pipeline transport cannot be determined from the characteristics that correspond to the stationary flow conditions.

In the present study, the oscillatory flow of a viscoelastic fluid is investigated using the Maxwell model in a plane channel when harmonic oscillations of the fluid flow rate are superimposed on the flow. The transfer function of the amplitude–phase frequency characteristics (APhFC) is determined. This function is used to study the dependence of the non-stationary tangential shear stress on the wall on the dimensionless oscillation frequency, the acceleration, and the relaxation properties of fluid.

1. FORMULATION OF THE PROBLEM AND METHOD OF SOLVING

We will consider the problems of slow oscillatory flow of a viscoelastic incompressible fluid between two fixed parallel planes extending in both directions to infinity. We will denote the distance between the walls by $2h$. The Ox axis passes horizontally in the middle of the channel along the flow. The Oy axis is directed perpendicular to the Ox axis. Viscoelastic fluid flow occurs symmetrically along the channel axis. The differential equation of motion of a viscoelastic incompressible fluid written in stress has the following form [29, 31–33]:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial \tau}{\partial y}, \quad (1.1)$$

where u is the longitudinal velocity, p is the pressure, ρ is the density, μ is dynamic viscosity, τ is the tangential stress, and t is time.

The rheological equation of state of fluid is taken in the form of the Maxwell equation [30]:

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \tau(y, t) = -\mu \frac{\partial u}{\partial y}. \quad (1.2)$$

Here, λ is the relaxation time. For $\lambda = 0$ in (1.2) we obtain the Newton viscous friction law. Substituting (1.2) in the equation of motion (1.1) for the fluid velocity, we obtain

$$\rho \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = -\left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}. \quad (1.3)$$

We consider that the oscillatory flow of a viscoelastic fluid takes place due to a given harmonic oscillation of the fluid flow rate or the longitudinal velocity averaged over the channel cross-section

$$Q = a_Q \cos \omega t = \operatorname{Re} a_Q e^{i\omega t}, \quad \langle u \rangle = a_u \cos \omega t = \operatorname{Re} a_u e^{i\omega t},$$

where a_Q and a_u are the amplitudes of the fluid flow rate and the longitudinal velocity averaged over the channel cross-section, respectively. In the case under consideration it is assumed that the no-slip conditions are fulfilled on the channel walls, i.e., the longitudinal velocity is equal to zero on the channel walls. Then the boundary conditions are as follows:

$$\begin{aligned} u &= 0 & \text{at } y &= h, \\ u &= 0 & \text{at } y &= -h. \end{aligned} \quad (1.4)$$

Due to linearity, the equation (1.1) for the longitudinal velocity, the pressure, and the shear stress on the wall can be written as follows:

$$u(y, t) = \operatorname{Re} u_1(y) e^{i\omega t}, \quad p(x, t) = \operatorname{Re} p_1(x) e^{i\omega t}, \quad \tau(t) = \operatorname{Re} \tau_1 e^{i\omega t}. \quad (1.5)$$

Substituting expressions (1.5) in Eq. (1.3), we obtain

$$\frac{\partial^2 u_1(y)}{\partial y^2} - \frac{\rho i \omega \eta^2(i\omega)}{\mu} u_1(y) = \frac{\eta^2(i\omega)}{\mu} \frac{\partial p_1(x)}{\partial x}. \quad (1.6)$$

Here, $\eta^2(i\omega) = (1 + i\omega\lambda)$.

The fundamental solutions of Eq. (1.6) with zero right-hand side are the functions

$$\cos\left(\frac{i^{3/2}\alpha_0}{h}\eta(i\omega)y\right) \quad \text{and} \quad \sin\left(\frac{i^{3/2}\alpha_0}{h}\eta(i\omega)y\right),$$

and the solutions of the inhomogeneous equation have the constants

$$\frac{1}{\rho i \omega} \left(-\frac{\partial p_1(x)}{\partial x}\right).$$

Thus, the general solution of Eq. (1.6) takes the form:

$$u_1(y) = C_1 \cos\left(i^{3/2}\alpha_0\eta(i\omega)\frac{y}{h}\right) + C_2 \sin\left(i^{3/2}\alpha_0\eta(i\omega)\frac{y}{h}\right) + \frac{1}{\rho i \omega} \left(-\frac{\partial p_1(x)}{\partial x}\right). \quad (1.7)$$

To determine the constant coefficients C_1 and C_2 in (1.7), we use the boundary conditions (1.4)

$$\begin{aligned} u_1(h) &= C_1 \cos(i^{3/2} \alpha_0 \eta(i\omega)) + C_2 \sin(i^{3/2} \alpha_0 \eta(i\omega)) + \frac{1}{\rho i \omega} \left(-\frac{\partial p_1(x)}{\partial x} \right), \\ u_1(-h) &= C_1 \cos(i^{3/2} \alpha_0 \eta(i\omega)) - C_2 \sin(i^{3/2} \alpha_0 \eta(i\omega)) + \frac{1}{\rho i \omega} \left(-\frac{\partial p_1(x)}{\partial x} \right). \end{aligned} \quad (1.8)$$

From (1.8) we can readily find

$$\begin{aligned} C_2 &= 0, \\ C_1 &= -\frac{1}{\rho i \omega} \left(-\frac{\partial p_1(x)}{\partial x} \right) \frac{1}{\cos(i^{3/2} \alpha_0 \eta(i\omega))}. \end{aligned}$$

As a result, for determination of the velocity we have

$$u_1(y) = \frac{1}{\rho i \omega} \left(-\frac{\partial p_1(x)}{\partial x} \right) \left(1 - \frac{\cos\left(i^{3/2} \alpha_0 \eta(i\omega) \frac{y}{h}\right)}{\cos(i^{3/2} \alpha_0 \eta(i\omega))} \right), \quad (1.9)$$

where $\alpha_0 = \sqrt{\frac{\omega}{\nu}} h$ is the vibrational Womersly number (dimensionless oscillation frequency).

Using the equation

$$\tau_1 = -\frac{\mu}{\eta^2(i\omega)} \frac{\partial u_1(y)}{\partial y} \Big|_{y=h} \quad (1.10)$$

we can find the tangential shear stress on the wall

$$\tau_1 = -h \left(-\frac{\partial P}{\partial x} \right) \frac{1}{i \alpha_0^2} \left(\frac{i^{3/2} \alpha_0 \sin(i^{3/2} \alpha_0 \eta(i\omega))}{\eta(i\omega) \cos(i^{3/2} \alpha_0 \eta(i\omega))} \right). \quad (1.11)$$

Now, integrating both sides of formula (1.9) with respect to the variable y over the limits from $-h$ to h , we obtain the following formulas for the fluid flow:

$$Q_1 = 2h \left[\frac{1}{\rho i \omega} \left(-\frac{\partial p_1(x)}{\partial x} \right) \left(1 - \frac{\sin(i^{3/2} \alpha_0 \eta(i\omega))}{(i^{3/2} \alpha_0 \eta(i\omega)) \cos(i^{3/2} \alpha_0 \eta(i\omega))} \right) \right]. \quad (1.12)$$

Taking into account formula (1.12) $Q_1 = 2h \langle u_1 \rangle$, we find the longitudinal velocity averaged over the channel cross-section

$$\langle u_1 \rangle = \frac{h^2}{\mu i \alpha_0^2} \left(-\frac{\partial p_1(x)}{\partial x} \right) \left(1 - \frac{\sin(i^{3/2} \alpha_0 \eta(i\omega))}{(i^{3/2} \alpha_0 \eta(i\omega)) \cos(i^{3/2} \alpha_0 \eta(i\omega))} \right). \quad (1.13)$$

Here, $\rho i \omega$ can be written in the form:

$$\rho i \omega = i \frac{\omega}{\nu} h^2 \frac{\mu}{h^2} = i \alpha_0^2 \frac{\mu}{h^2}.$$

Then formula (1.13) with account for (1.11) takes the form:

$$\langle u_1 \rangle = -\frac{h}{3\mu} \tau_1 \frac{3 \left(i^{3/2} \alpha_0 \eta(i\omega) \cos\left(i^{3/2} \alpha_0 \eta(i\omega)\right) - \sin\left(i^{3/2} \alpha_0 \eta(i\omega)\right) \right)}{\left(i^{3/2} \alpha_0 \right)^2 \sin\left(i^{3/2} \alpha_0 \eta(i\omega)\right)}. \quad (1.14)$$

Using formula (1.14), we determine the transfer function $W_{\tau,u}(i\omega)$ for the shear stress on the wall as follows

$$W_{\tau,u_1}(i\omega) = \frac{h \tau_1(i\omega)}{3\mu u_1(i\omega)}. \quad (1.15)$$

Taking into account (1.15), from Eq. (1.14) we obtain

$$W_{\tau,u_1}(i\omega) = \frac{h \tau_1(i\omega)}{3\mu \langle u_1(i\omega) \rangle} = - \frac{\left(i^{\frac{3}{2}}\alpha_0\right)^2 \sin\left(i^{\frac{3}{2}}\alpha_0\eta(i\omega)\right)}{3 \left(i^{\frac{3}{2}}\alpha_0\eta(i\omega) \cos\left(i^{\frac{3}{2}}\alpha_0\eta(i\omega)\right) - \sin\left(i^{\frac{3}{2}}\alpha_0\eta(i\omega)\right)\right)}. \quad (1.16)$$

The transfer function (1.16) is sometimes called the amplitude–phase frequency response (APFC). This function makes it possible to determine the time dependence of the shear stress on the channel wall for a given law of variation in the longitudinal velocity averaged over the channel cross-section. As known, in most cases, when solving non-stationary problems, the shear stress on the wall obtained in the quasi-stationary fluid flow regime is used. In real cases, such assumptions are valid when the local velocity distribution over the flow cross-section has a parabolic distribution law. In this case, the tangential shear stress on the channel wall fluctuates in one phase with the fluctuation of the average longitudinal velocity over the channel cross-section.

In this case, the quantity $\tau_{0,kc}$ can be calculated from the formula $\tau_{0,kc} = \frac{3\mu}{h} \langle u_1 \rangle$ and instead of the quasi-stationary flow for the shear stress on the wall $\tau_{0,kc}$ we can take

$$\tau_{0,kc} = \tau_{hc}. \quad (1.17)$$

Thus, relation (1.17) makes it possible to change the quantity τ_{hc} to $\tau_{0,kc}$, only under the condition that the actual distribution of local velocities over the flow cross-section differs insignificantly from the quasi-stationary one. However, in many cases, in a non-stationary flow, the law of local velocity distribution differs significantly from the quasi-stationary one. In the majority of studies [9–12, 17, 18, 24, 25] it was shown that in the case of oscillatory laminar flow in a cylindrical pipe, the change in local velocities in the near-wall layers is by some time ahead of the change in local velocities in the central layers. In the oscillatory flow, due to a change in the law of local velocity distribution over the channel cross-section, the quantity τ_{hc} actually differs significantly from $\tau_{0,kc}$. In the linear model of unsteady flow, the most complete idea of the dependence of τ_{hc} on $\langle u_1 \rangle$ can be obtained using the transfer function (1.16).

2. RESULTS OF CALCULATIONS AND ANALYSIS

To determine the dependence of the shear stress on the channel wall on the longitudinal velocity averaged over the channel cross-section in unsteady flow, we use the transfer function (1.16). In this regard, we take into account the law of change in the longitudinal velocity averaged over the channel

$$\langle u_1 \rangle = a_{u_1} \cos \omega t, \quad (2.1)$$

where a_{u_1} is the amplitude of the longitudinal velocity averaged over the channel section. Using formulas (2.1), it is possible to determine the dependence of the shear stress on the wall between the longitudinal velocity averaged over the channel section. Due to the linearity of Eqs. (2.1), used to find the shear stress on the channel wall, its value will also be harmonic, but, in the general case, shifted in phase with respect to $\langle u_1 \rangle$.

Thus, change in the shear stress on the wall is determined as follows:

$$\tau_1 = a_{\tau_1} \cos(\omega t + \varphi_{\tau_1}), \quad (2.2)$$

where a_{τ_1} is the amplitude of the shear stress on the channel wall and φ_{τ_1} is the phase shift between the value τ_1 and $\langle u_1 \rangle$.

Using the relation

$$\cos(\omega t + \varphi_{\tau_1}) = \cos \omega t \cos \varphi_{\tau_1} - \sin \omega t \sin \varphi_{\tau_1}$$

and taking into account that

$$\frac{\partial \langle u_1 \rangle}{\partial t} = -a_{u_1} \omega \sin \omega t,$$

we can reduce Eq. (2.2) to the form:

$$\tau_1 = \left(\frac{a_{\tau_1}}{a_{u_1}} \cos \varphi_{\tau_1} \right) \langle u_1 \rangle + \left(\frac{a_{\tau_1}}{a_{u_1}} \sin \varphi_{\tau_1} \right) \frac{1}{\omega} \frac{\partial \langle u_1 \rangle}{\partial t}. \quad (2.3)$$

The quantities $\left(\frac{a_{\tau_1}}{a_{u_1}} \cos \varphi_{\tau_1} \right)$ and $\frac{a_{\tau_1}}{a_{u_1}} \sin \varphi_{\tau_1}$ correspond to the real and imaginary parts of the transfer function (1.16); therefore, from (1.16) we obtain

$$W_{\tau_1, u_1} = -\frac{1}{3} \left(\frac{-i \alpha_0^2 \sin(i^{3/2} \alpha_0 (1 + i De \alpha_0^2)^{1/2})}{i^{3/2} \alpha_0 (1 + i De \alpha_0^2)^{1/2} \cos(i^{3/2} \alpha_0 (1 + i De \alpha_0^2)^{1/2}) - \sin(i^{3/2} \alpha_0 (1 + i De \alpha_0^2)^{1/2})} \right) = \chi + \beta i. \quad (2.4)$$

Here, $De = \frac{v \lambda}{h^2}$ is the elastic Deborah number that characterizes the elastic properties of fluid,

$$\chi = \left(\frac{a_{\tau_1}}{a_{u_1}} \cos \varphi_{\tau_1} \right) \text{ and } \beta = \frac{a_{\tau_1}}{a_{u_1}} \sin \varphi_{\tau_1}.$$

Then formula (2.3) takes the form:

$$\frac{h}{3\mu} \frac{\tau_1}{\langle u_1 \rangle} = W_{\tau_1, u_1} = \chi + \beta \frac{1}{\omega} K_H. \quad (2.5)$$

Here, $K_H = \frac{\partial \langle u_1 \rangle}{\langle u_1 \rangle \partial t}$ is the parameter that characterizes the fluid accelerations; χ and β are dimensionless quantities, t is a dimensional quantity, so it should be converted to dimensionless form, using the transformation

$$t = \frac{h^2 \rho}{3\mu} t^*. \quad (2.6)$$

Taking into account (1.17), (2.4), and (2.6) from (2.5) we obtain the computational formulas

$$\frac{\tau_{hc}}{\tau_{0kc}} = \chi + \frac{3\beta}{\alpha_0^2} K_H. \quad (2.7)$$

Here, $\tau_{0kc} = \frac{3\mu}{h} \langle u_1 \rangle$ and $\tau_1 = \tau_{hc}$.

Using formula (2.7), in Fig. 1 we have plotted the graphs which show variation in the relative tangential stress on the wall in non-stationary flow as a function of the dimensionless oscillation frequency when the Deborah number is equal to zero. The graphs plotted in Fig. 1 show that at $K_H = 0$ the ratio $\frac{\tau_{hc}}{\tau_{0kc}}$ is close to unity, as long as α_0 is less than unity. If α_0 takes values greater than unity, then even at $K_H = 0$ the ratio $\frac{\tau_{hc}}{\tau_{0kc}}$ becomes greater than unity and increases with the dimensionless oscillation frequency. This suggests that the shear stresses on the channel wall can exceed their quasi-stationary values in the case of unsteady fluid flow even at those times when the fluid acceleration is equal to zero. The ratio $\frac{\tau_{hc}}{\tau_{0kc}}$ grows with increase in the parameter K_H , which can be explained by variation in the shear stress on the wall, which occurs with advance in phase as compared with the average cross-sectional velocity.

When a viscoelastic fluid flows in a plain channel, there is a significant variation in the shear stress on the wall at the low vibration frequencies depending on the elastic Deborah number. In [25] oscillatory flows of a viscoelastic fluid in a plane channel and in a cylindrical pipe were studied; the flow area was divided into two classes, one of which at $\alpha_0 > 1$ belongs to the “wide” class, and the other at $\alpha_0 < 1$ to the “narrow” one. In the “wide” classes, the oscillatory fluid flow is restricted to the near-wall flow, and flow

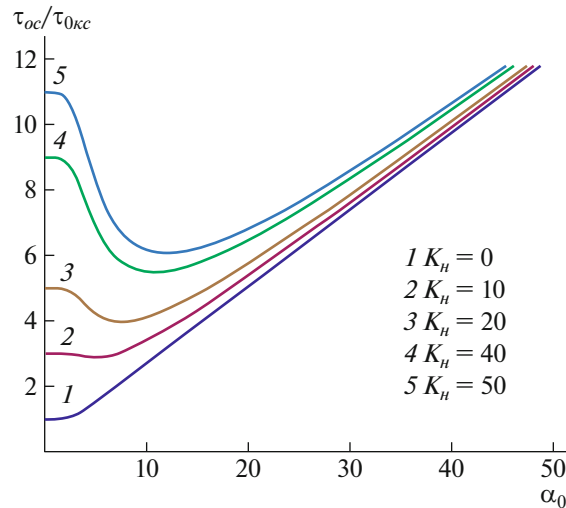


Fig. 1. Variation in the ratio of the non-stationary shear stress on the wall to the quasi-stationary shear stress depending on the dimensionless oscillation frequency for various values of the fluid acceleration parameter K_H .

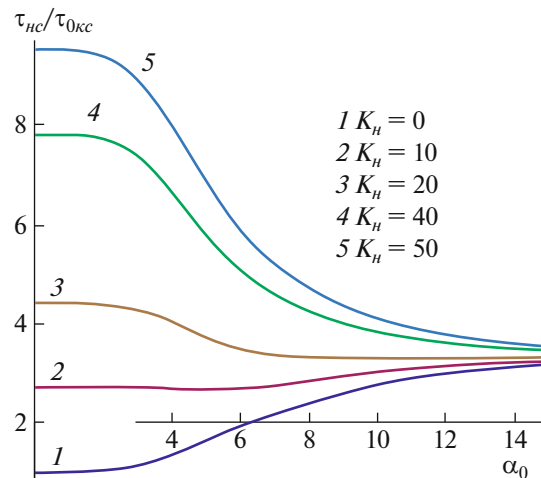


Fig. 2. Variation in the ratio of the non-stationary shear stress on the wall to the quasi-stationary shear stress depending on the dimensionless oscillation frequency for various values of the fluid acceleration parameter K_H and the elastic Deborah number $De = 0.01$.

is inviscid in the central part. In the “narrow” systems, shear waves cross the entire flow area, which ultimately leads to sharp increase in the amplitude of the velocity profile and other hydrodynamic parameters, such as the tangential shear stress on the wall and the fluid flow rate depending on the elastic Deborah number. Based on formula (2.7), in Figs. 2, 3, and 4 we have constructed the graphs that show variation in the shear stress in oscillatory flow of viscoelastic fluid in a plane channel depending on the oscillation frequency at the Deborah numbers $De = 0.01$; 0.05 ; 0.1 , respectively.

It should be noted that all graphs for the flow of viscoelastic fluid in the plane channel are oscillatory. In Fig. 2 we have shown variation in the ratio of the non-stationary shear stress on the channel wall to the quasi-stationary shear stress as a functions of the dimensionless oscillation frequency in the case $De = 0.01$. It should be also noted that in this case, in contrast to the Newtonian flow, an increase in the shear stress can be observed in the region of the near-zero oscillation frequency, depending on the fluid acceleration. Then there is a gradual decrease for $K_H = 20$; 40 ; 50 , and for $K_H = 0$ and 10 we can see an

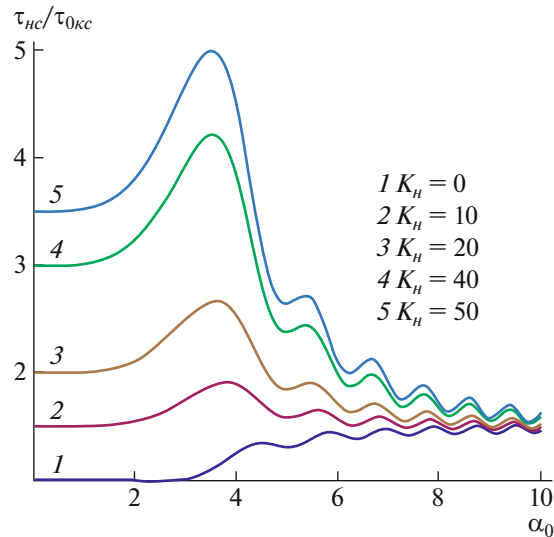


Fig. 3. Variation in the ratio of the non-stationary shear stress on the wall to the quasi-stationary shear stress depending on the dimensionless oscillation frequency for various values of the fluid acceleration parameter K_H and the elastic Deborah number $De = 0.05$.

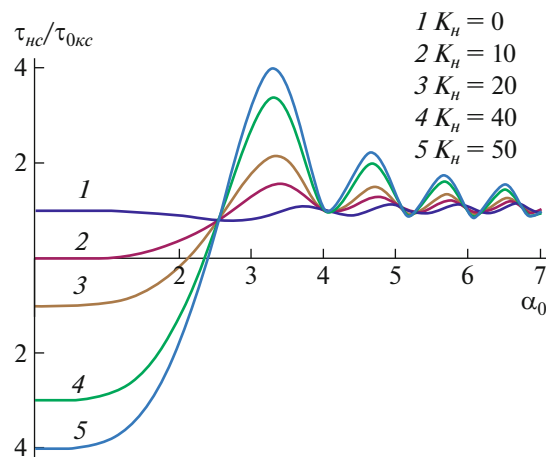


Fig. 4. Variation in the ratio of the non-stationary shear stress on the wall to the quasi-stationary shear stress depending on the dimensionless oscillation frequency for various values of the fluid acceleration parameter K_H and the elastic Deborah number $De = 0.1$.

increase to $\frac{\tau_{nc}}{\tau_{0kc}} = 3$ at the higher oscillation frequencies. In Fig. 3 we have plotted the graphs of the ratio of the non-stationary shear stress on the channel wall to the quasi-stationary shear stress as a function of the dimensionless oscillation frequency in the case of $De = 0.05$. In the case of a near-zero oscillation frequency, we can observe a decrease in the shear stress depending on the fluid acceleration, and then we can see increase to a maximum in the interval $2 < \alpha_0 < 4$ and then gradual asymptotic decrease to $\frac{\tau_{nc}}{\tau_{0kc}} = 1.5$.

It should be noted that in the case $De = 0.1$, at the low oscillation frequencies, a sharp decrease in the shear stress is observed, except for the case $K_H = 0$. This suggests that, at large values of the relaxation time, reverse fluid flows can develop at the low oscillation frequencies. Then, with increase in the oscillation frequency, all curves showing variations in the ratio of the non-stationary shear stress on the channel wall asymptotically approach, with oscillations, the value $\frac{\tau_{nc}}{\tau_{0kc}} = 1$ depending on the fluid acceleration (Fig. 4).

Thus, the considered features in variations in the shear stress on the wall at a given harmonic fluctuation of the fluid flow rate are caused by the violation of the parabolic law of the local velocity distribution over the channel cross-section. The calculations show that in the near-wall layer the velocities vary in phase with the variation in the shear stress on the wall along the channel, while in the central part of flow they remain in phase with the phase of the tangential shear stress on the wall. Therefore, the viscoelastic properties of the fluid, as well as its acceleration, are limiting factors for using the quasi-stationary approach. In addition, the found formulas for determination of the transfer function for the flow of a viscoelastic fluid in a non-stationary flow make it possible to find the dissipation of mechanical energy in a non-stationary flow of the medium, which are important for control of the hydraulic and pneumatic systems.

SUMMARY

The problems of oscillatory flow of a viscoelastic incompressible fluid in a plane channel are solved for a given harmonic oscillation of the fluid flow rate. The transfer function (amplitude–phase frequency response) is found. Using this function, the influence of the acceleration oscillation frequency and the relaxation properties of fluid on the ratio of the tangential shear stress on the channel wall to the average cross-sectional velocity is determined. The calculations show that the non-stationary shear stress on the channel wall in the flow of a viscoelastic fluid increases nonmonotonically with the acceleration of fluid particle at the low oscillation frequencies. Initially, it reaches a maximum, then decreases with increase in the dimensionless oscillation frequency, and asymptotically approaches, with oscillations, the values characteristic of flow without acceleration. It is shown that the viscoelastic properties of the fluid, as well as its acceleration, are the limiting factors for using the quasi-stationary approach. The found formulas for determining the transfer function during the flow of a viscoelastic fluid in a non-stationary flow make it possible to determine the dissipation of mechanical energy in a non-stationary flow of the medium, which are of great importance in the calculation of control of the hydraulic and pneumatic systems.

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