ISSN 0015-4628, Fluid Dynamics, 2022, Vol. 57, No. 7, pp. 911–922. © Pleiades Publishing, Ltd., 2022.
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On the Curvature of Boundary Streamlines of an Ideal Gas at Separation and Reattachment Points

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Received February 10, 2022; revised April 15, 2022; accepted April 15, 2022

Abstract—The paper considers the stationary flows of an ideal (nonviscous and nonheat-conducting) gas with streamlines, i.e., boundaries of flowing and stationary media. In the 19th century such boundaries appeared in the problems of the outflow of jets into a flooded space. Until 1903 only jets of incompressible fluid were considered; the main contribution was made by Zhukovsky. In 1903 Chaplygin began studying flat subsonic jets of an ideal gas. In 1949 Ovsyannikov, having solved the problem of the outflow of a "critical" jet, discovered the fascinating properties of a flow with a sonic boundary streamline. Soon, segments of such streamlines, which arose mostly in problems of jet-flow theory, appeared in the construction of bodies subjected to subsonic flows with the largest "critical" Mach numbers M*. For an incident flow with $M_0 < M^*$ M < 1 in the entire flow, there are no shock waves and wave drag. At $M_0 > M^*$ supersonic zones appear, shock waves arise as well as wave drag, increasing with increasing M_0. It turned out that M* is achieved by bodies subject to a flow in which with $M_0 = M^*$ some of the contours are segments of sonic streamlines. It is useful to know their curvature at the separation and reattachment points. Zhukovsky states it to be infinite for a fluid at separation points. The infinity of the curvature of such streamlines in an ideal gas has been established only after 100 years. The following shows how the flow parameters and their derivatives, including the curvature of the streamlines, behave when approaching separation and reattachment points. The curvature of the boundary streamlines at these points is infinite, while the curvature of sonic streamlines when they intersect with a straight sonic transition line is zero.

Keywords: stationary ideal-gas flows, hodograph variables, curvature of subsonic and sonic boundary streamlines at separation and reattachment points

DOI: 10.1134/S0015462822070072

1. INTRODUCTION

The hydro-aero-gas dynamic researchers listed in the annotation were not the only prominent scientists interested in stationary flows of an ideal liquid or gas with isobaric streamlines, i.e., the boundaries of flowing and stationary media. A few more were named by Zhukovsky [1]: "The foundation of solving such problems was laid in 1868 by Helmholtz. ... In the same year, Kirchhoff proposed a general technique for solving such problems. ... A more detailed development was made by Rayleigh in 1876, who published two notes." After noting the contributions of Gerlach (1885), Bobylev (1881), Meshchersky (1886), Kotelnikov (1889), and Voigt (1886), Zhukovsky writes: "Brillouin's work provides a critical assessment of Kirchhoff's method" and further: "In all the works mentioned, the authors adhered to the Kirchhoff method, and an attempt to modify this method was made only by Planck (in 1884), who set out to free the method from the imaginary-variable theory in order to expand it to three-dimensional space. ... But, not to mention the problems of three dimensions, Planck did not solve a single new problem using his method. ... The main drawback of Planck's method is the fact that, as the original Kirchhoff method, it cannot be used to solve a specific problem. The conformal map introduced by Kirchhoff eliminates this difficulty, but introduces into the problem an extra, sometimes very difficult operation. Since the number of known conformal maps of a closed domain into a domain bounded by two parallel lines is small, the number of problems that can be solved by the Kirchhoff method is also small. ... In the proposed (by Zhukovsky, the author) change to the Kirchhoff method, it is possible to proceed to the solution of a specific problem without resorting in advance to conformal mapping ..." As a result, by the "modified Kirchhoff method," in fact, by the original "Zhukovsky method," its creator solved many new problems that were not amenable to the Kirchhoff method. The proposed method and these problems are the subject of two articles by Zhukovsky ([1] published in 1890 and [2], in 1891). Their total volume is one hundred and fifty pages.

The authors listed above limited themselves to an incompressible fluid, more precisely, to the outflow of fluid from plane channels with walls composed of straight line segments. If the origin of the Cartesian coordinates xy is aligned with one of the trailing edges, then when approaching it along such a segment, the inclination angle θ of the flow velocity vector V to the x axis does not change, the velocity V increases, and the pressure p drops so that the derivatives of V and p along the contours on the edge turn into $\pm\infty$. Conversely, after exiting the channel while moving along the boundary streamline, V and p are constant, and the angle θ changes so that the curvature of the boundary becomes infinite as $x \to +0$ (for symmetric jets flowing from left to right, to $+\infty$ on the top boundary and $-\infty$ on the bottom boundary).

From the middle of the twentieth century flows with an isobaric boundary streamline became relevant for the study of viscous boundary-layer separation from curved walls. If the origin of the Cartesian coordinates xy is aligned with the separation point, and the x axis is directed tangentially to the wall as $x \to -0$, then in the incompressible-fluid approximation with vanishing viscosity [3–5], the boundary curvature is the same as above.

Particular interest in the sonic boundary streamline was sparked by an unexpected solution found by Ovsyannikov [6, 7]. According to this solution, the alignment of the "critical" jet of an ideal gas does not occur asymptotically, as for an incompressible liquid, but in a "straight transition line" at a finite distance from the outlet cross section of the channel. When an ideal-gas sonic jet impinges on wedge obstacles [8, 9], there are three straight transition lines. One limits the incoming sonic jet, and the other two (one on each side of the obstacle) limit the outflowing inclined sonic flows.

As we know, the curvature of a curvilinear sonic streamline in plane-parallel and axisymmetric flows is zero at the point of intersection with a straight transition line [10–12]. In general, the analysis of flows with subsonic and sonic boundary streamlines has significantly simplified the approach developed in [13] (see also [14]). The same approach is applied below.

Sonic boundary streamlines became even more interesting after it was established [15] that the segments of such lines form symmetrical profiles, bodies of revolution, head and rear parts of a semi-infinite plate and a circular cylinder, which, under a number of additional restrictions, are subject to an infinite subsonic oncoming flow under zero angle of attack with the highest critical Mach numbers M*. The typical restrictions here are assignment of the profile chord, the length of the body or its head (rear) parts, taken as the linear scale, the half-thickness of the plate, the radius of a circular cylinder, the minimum admissible "longitudinal" profile area or volume of the body of revolution, etc. The simplest examples of such configurations are a plate at zero angle of attack and a straight line segment ("axisymmetric needle") that do not disturb the surrounding uniform sonic flow with $M \equiv M_0 \equiv M^* = 1$. The area of their "longitudinal" sections related to a square of a given length is $S = 0$. If the minimum admissible value $S > 0$ is set in addition to a fixed chord or length, then, according to [15], the contours of critical bodies make up the front and rear ends and the top and symmetrical bottom sonic streamlines that connect them without kinks. At $S \to 0$ the height of the ends tends to zero, M_0 and M* tend to unity and the result is a plate and a needle. In order to get rid of the inevitable separations behind the bodies constructed under the assumption of a nonseparated flow at $S > 0$, a restriction is introduced on the inclination angles of the contours of their rear parts. As a result, the rear end is replaced by a pair of straight segments, and the flat critical configuration becomes a symmetrical airfoil.

In the examples of critical configurations given above and their generalizations [16], each sonic streamline, along with a point of smooth separation, has a point of smooth reattachment. Critical configurations with separation or reattachment points on straight transition lines are possible. Although the structure of planar and axisymmetric critical configurations is fundamentally simple, the techniques for constructing them [17–24] turned out to be rather complex. Therefore, any additional information, for example, on the curvature of boundary streamlines, is useful for constructing critical configurations. The authors of [21–24], without dwelling on the details of a far from simple analysis, state that the curvature of the boundary streamline found by them for an ideal gas at the point of separation from a rectilinear wall is infinite. Below, the approach developed in [13] resulted in formulas that, confirming this conclusion, show how, in different cases, the streamline curvature increases when approaching the separation and reattachment points.

2. PLANE-PARALLEL POTENTIAL FLOWS OF AN IDEAL GAS IN HODOGRAPH VARIABLES. BOUNDARY STREAMLINE NEAR A STRAIGHT TRANSITION LINE

Since plane-parallel isentropic and isoenergetic stationary flows of an ideal gas are potential, they allow the transition to independent variables $V-\theta$. Let us introduce the stream function ψ up to an arbitrary additive constant and a positive factor k by equality

$$d\psi = k\rho V(\cos\theta dy - \sin\theta dx)$$

with the gas density $\rho = \rho(V)$. The stream function satisfies Chaplygin's equation [14]

$$V^2\psi_{VV} + (1+\mathrm{M}^2)V\psi_V + (1-\mathrm{M}^2)\psi_{\theta\theta} = 0, \tag{2.1}$$

in which the Mach number $\mathrm{M} = \mathrm{M}(V)$. So, for a perfect gas with constant heat capacities and their ratio (adiabatic exponent) γ

$$\mathrm{M} = \frac{V\sqrt{1-\varepsilon}}{\sqrt{1-\varepsilon V^2}} \Leftrightarrow 1-\mathrm{M}^2 = \frac{1-V^2}{1-\varepsilon V^2}, \quad \varepsilon = \frac{\gamma-1}{\gamma+1}. \tag{2.2}$$

In the above equations and further, critical values of the velocity and density are taken as their scales. When the stream function $\psi = \psi(V, \theta)$ is obtained, the coordinates x and y are defined by differential equations

$$\begin{aligned} dx &= -\frac{V\psi_V\sin\theta + (1-\mathrm{M}^2)\psi_\theta\cos\theta}{k\rho V^2}dV + \frac{V\psi_V\cos\theta - \psi_\theta\sin\theta}{k\rho V}d\theta, \\ dy &= \frac{V\psi_V\cos\theta - (1-\mathrm{M}^2)\psi_\theta\sin\theta}{k\rho V^2}dV + \frac{V\psi_V\sin\theta + \psi_\theta\cos\theta}{k\rho V}d\theta. \end{aligned} \tag{2.3}$$

These equalities can be integrated over any curve of the V–θ plane, in particular, over the verticals $V=\text{const}$ corresponding to isobaric streamlines or over the horizontals $\theta = \text{const}$ corresponding to straight segments of the wall contours subject to the flow. Integration over the verticals determines the dependence of the x and y coordinates on the inclination angle of the boundary streamline, and after that its curvature. Let us show how this is done, first for the sonic boundary streamline near the point of its intersection with the straight transition line. It makes sense to begin the analysis with such a special situation because in this case the approach developed in [13] and described in [14] is applied without changes.

The upper half of the jet stream including the point e of intersection of the boundary sonic streamline b–e with the straight transition line 0–e is shown in Cartesian coordinates in Fig. 1a, and in the hodograph variables in Fig. 1b. On the curvilinear streamline b–e and on the straight segment 0–e of the transition line, the flow velocity $V = V_e = 1$. Near the point e, where the velocity V and the Mach number are close to unity, Chaplygin's equation (2.1) and equalities (2.3) take the form

$$\psi_{VV} + 2\psi_V + \sigma^2\eta\psi_{\theta\theta} = 0, \quad \eta = 1-V, \quad \sigma^2 = d(1-\mathrm{M}^2)/d\eta\big|_{V=1}, \tag{2.4}$$

$$\begin{aligned} dx &= -\frac{\psi_V\sin\theta + \sigma^2\eta\psi_\theta\cos\theta}{k}dV + \frac{\psi_V\cos\theta - \psi_\theta\sin\theta}{k}d\theta, \\ dy &= \frac{\psi_V\cos\theta - \sigma^2\eta\psi_\theta\sin\theta}{k}dV + \frac{\psi_V\sin\theta + \psi_\theta\cos\theta}{k}d\theta. \end{aligned} \tag{2.5}$$

It is taken into account that $\rho(1) = 1$ due to the choice of the density scale. According to formulas (2.2) for a perfect gas $\sigma^2 = \gamma + 1$.

All streamlines come to the point e of the V–θ plane, due to which ψ changes by a finite amount. By assuming $\psi = 0$ on the plane of symmetry of the channel and the jet (on the x axis) and utilizing the arbitrary choice of the factor k, we make $\psi = 1$ on the channel wall and on the jet boundary. Therefore, near the point e of the plane V–θ, the stream function should be sought in the form

$$\psi = \chi(\omega) + \ldots; \quad \omega = \frac{-\theta}{\eta^n}, \quad 0 \le \chi(\omega) \le 1, \quad 0 \le \omega \le \infty, \quad \chi(0) = 0, \quad \chi(\infty) = 1, \tag{2.6}$$

with the unknown exponent $n > 0$. Denoting the derivatives with respect to ω with a prime, we obtain

$$\psi_V = \frac{n\omega}{\eta}\chi', \quad \psi_{VV} = n\omega\frac{n\omega\chi'' + (n+1)\chi'}{\eta^2}, \quad \psi_\theta = -\frac{\chi'}{\eta^n}, \quad \psi_{\theta\theta} = \frac{\chi''}{\eta^{2n}}. \tag{2.7}$$

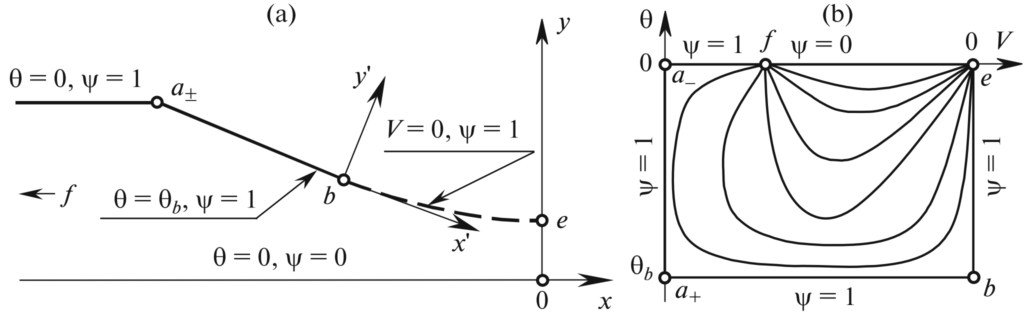

Fig. 1. Jet flow with a sonic boundary streamline *b–e* and a straight transition line 0–*e* ((*a*) in Cartesian coordinates, (b) in hodograph variables).

Substituting the obtained derivatives into equation (2.4) for $\eta \ll 1$ gives the equation

$$n\omega\frac{n\omega\chi'' + (n+1)\chi'}{\eta^2} + \sigma^2\frac{\chi''}{\eta^{2n-1}} = 0,$$

whose solutions for $n \neq 3/2$ do not allow satisfying the conditions (2.6). At $n = 3/2$ we get the equation

$$(9\omega^2 + 4\sigma^2)\chi'' + 15\omega\chi' = 0$$

and its solution

$$\chi'(\omega) = \frac{C}{(9\omega^2 + 4\sigma^2)^{5/6}}, \quad \chi(\omega) = C\int_0^{\omega}\frac{d\omega}{(9\omega^2 + 4\sigma^2)^{5/6}}; \quad \frac{1}{C} = \int_0^{\infty}\frac{d\omega}{(9\omega^2 + 4\sigma^2)^{5/6}}, \tag{2.8}$$

which satisfies all necessary conditions (2.6).

To find the curvature of the boundary streamline near the point *e*, we assume $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ in the coefficients before $d\theta$ of equalities (2.5) and substitute ψ_V and ψ_θ by the expressions from (2.7) with χ' from (2.8). This produces equations

$$dx \approx \frac{-Cd(-\theta)}{3^{5/3}2k(-\theta)^{2/3}}, \quad dy \approx \frac{C(-\theta)^{1/3}}{3^{2/3}2k}d(-\theta),$$

which are satisfied on the vertical *b*—*e* near the point *e*. By integrating them from the point *e*, in which $\theta_e = \chi_e = 0$, we arrive at formulas

$$-\theta = 72\frac{k^3}{C^3}x^3, \quad y - y_e \approx \frac{3^{1/3}C(-\theta)^{4/3}}{8k} \approx 54\frac{k^3}{C^3}x^4, \tag{2.9}$$

that are true for the streamline *b*—*e* near the point *e*. From them, among other things, follows the aforementioned well-known result obtained by other methods [10—12] on the zero curvature of the streamline at the point of intersection with the straight transition line. In the problem of jet impingement on a symmetric wedge-shaped obstacle with a vertex half-angle $0 < \theta_w < \pi$ [8, 9], there are four such points: *e* and *f* in the upper half of the current shown in Fig. 2 and two in its symmetrical lower half.

3. FLOW NEAR THE VANISHING POINT OF A SUBSONIC BOUNDARY STREAMLINE FROM THE STREAMLINED CONTOUR

The simple version of a plane ideal-gas jet considered in this section differs from that shown in Fig. 1 by the fact that for the boundary streamline *b*—*e* the gas velocity $V \equiv V_b$ and the Mach number M_b is less than unity. As a result of this, the flow aligns as $x \to \infty$, and in the solution in the form (2.6) $n = 1$, that is, $\omega = -\theta/\eta$, $\eta = V_b - V$ and $V_b < 1$. In a small neighborhood of the point *b* considered below, we can rewrite Chaplygin's equation (2.1) in the form

$$\psi_{VV} + \alpha^2\psi_V + \beta^2\psi_{\theta\theta} = 0; \quad \alpha^2 = (1 + M_b^2)/V_b, \quad \beta^2 = (1 - M_b^2)/V_b^2. \tag{3.1}$$

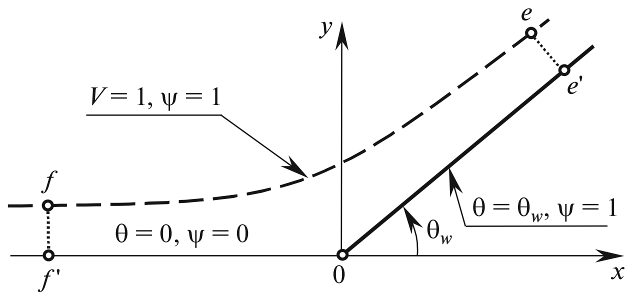

Fig. 2. Impingement of an ideal-gas sonic jet onto a wedge: *e–e*' and *f–f*' are straight transition lines, *e–f* is a curvilinear section of the sonic boundary streamline.

Of the two positive constants that appeared in this equation, only the second is involved in further analysis. In the limit of low subsonic velocities, it increases indefinitely, and at $M_b = 1$ it becomes zero.

At point *b*, in contrast to point *e* studied above, the stream function does not break. Therefore, we will seek a solution in its neighborhood in the form

$$\psi = 1 + \varepsilon\chi(\omega) + \ldots; \quad \omega = \frac{\theta - \theta_b}{\eta^n}, \quad \eta = V_b - V; \quad 0 \le \omega \le \infty, \quad \chi_0 \equiv \chi(0) = 0;$$
$$\chi_\infty \equiv \chi(\infty) = 0, \quad \varepsilon = [(\theta - \theta_b)^2 + \eta^{2n}]^m = (\omega^2 + 1)^m \eta^{2nm}; \quad n > 0, \quad m > 0. \tag{3.2}$$

These relations differ from (2.6), firstly, by the same (zero) conditions for χ at $\omega = 0$ and $\omega = \infty$, and, secondly, by two exponents *n* and *m* that are to be determined. Now instead of (2.7) we get

$$\psi_V = n[(\omega^2 + 1)\omega\chi' - 2m\chi](\omega^2 + 1)^{m-1}\eta^{2mn-1},$$
$$\psi_\theta = [(\omega^2 + 1)\chi' + 2m\omega\chi](\omega^2 + 1)^{m-1}\eta^{2mn-n},$$
$$\psi_{VV} = n\{\omega(\omega^2 + 1)^2[n\omega\chi'' + (n + 1)\chi'] - 4mn(\omega^2 + 1)\omega\chi' \tag{3.3}$$
$$+ 2m[2n(\omega^2 + m) - 1 - \omega^2]\chi\}(\omega^2 + 1)^{m-2}\eta^{2mn-2},$$
$$\psi_{\theta\theta} = \{(\omega^2 + 1)^2\chi'' + 4m(\omega^2 + 1)\omega\chi' + 2m[(2m - 1)\omega^2 + 1]\chi\}(\omega^2 + 1)^{m-2}\eta^{2mn-2n}.$$

Since $\psi_V \ll \psi_{VV}$ at $\eta \ll 1$, then by substituting these derivatives into (3.1) and discarding the second term, we arrive at the equation

$$\omega(\omega^2 + 1)^2[n\omega\chi'' + (n + 1)\chi'] - 4mn(\omega^2 + 1)\omega\chi' + 2m[2n(\omega^2 + m) - 1 - \omega^2]\chi$$
$$+ \beta^2\{(\omega^2 + 1)^2\chi'' + 4m(\omega^2 + 1)\omega\chi' + 2m[(2m - 1)\omega^2 + 1]\chi\}\eta^{2(1-n)}/n = 0.$$

Near the point *b*, the last term of this equation is much less, equal to, or much greater than the others for $n < 1$, $n = 1$ or $n > 1$ respectively, and $\chi(\omega)$ and "eigenvalues" of *m* are determined by the solution of one of the three boundary value problems:

$$n < 1: \quad \chi'' + \frac{(n + 1)(\omega^2 + 1) - 4mn}{n(\omega^2 + 1)\omega}\chi' + 2m\frac{(2n - 1)\omega^2 - 1 + 2nm}{n(\omega^2 + 1)^2\omega^2}\chi = 0,$$

$$n = 1: \quad \chi'' + 2\frac{\omega^2 + 1 + 2m(\beta^2 - 1)}{(\omega^2 + 1)(\omega^2 + \beta^2)}\omega\chi' + 2m\frac{\omega^2 + \beta^2 + (2m - 1)(1 + \beta^2\omega^2)}{(\omega^2 + 1)^2(\omega^2 + \beta^2)}\chi = 0,$$

$$n > 1: \quad \chi'' + \frac{4m\omega}{\omega^2 + 1}\chi' + 2m\frac{(2m - 1)\omega^2 + 1}{(\omega^2 + 1)^2}\chi = 0,$$

with the same boundary conditions $\chi_0 = 0$ and $\chi_\infty = 0$ for each of them.

For $\chi_0 = 0$ and arbitrary *n* and *m* the solutions of the reduced linear homogeneous equations proportional to χ'_0 naturally do not fulfill the condition $\chi_\infty = 0$.

Let us first show how its fulfillment determines m for the case of $n = 1$, for which, taking into account the condition $\chi_0 = 0$, we find:

$$\omega \ll 1: \quad \chi'' + \lambda^2\chi = 0, \quad \chi = A\sin(\lambda\omega) \approx \chi_0'\omega, \quad \lambda^2 = 2m\frac{\beta^2 + 2m - 1}{\beta^2}, \quad \chi_0' = A\lambda;$$

$$\omega \gg 1: \quad \chi'' + 2\frac{\chi'}{\omega} = 0, \quad \chi' = \frac{B}{\omega^2} \to \chi = \chi_\infty - \frac{B}{\omega}.$$

Here the constant χ_0' is arbitrary and B (proportional to χ_0') and χ_∞ are found by numerically solving the complete equation (or the system of two equations for χ and χ') to such large ω, at which the values of $B = \omega^2\chi'$ and $\chi_\infty = \chi + \omega\chi'$ stop changing.

Having fixed $n = 1$, χ_0' and β^2 and carrying out the calculations described above for different $m > 0$, we construct the curve $\chi_\infty = \chi_\infty(m)$ including a point, at which $\chi_\infty(m)$ intersects the m axis. This point gives us the solution. If there are more than one such positive values of m, then the minimum is taken, which, according to representation (3.2), gives the largest value in the expansions in powers of a small "distance" to the point b in the hodograph variables. Calculations carried out for $n = 1$ gave the minimum $m = 1$, which does not depend only on χ_0', as it should be, but also on β^2, which was unexpected. The analysis carried out to clarify this not only confirmed this numerically detected singularity, but also led to a complete analytical solution of the problem. To understand how a solution that does not depend on β^2 is obtained, we rewrite the equation corresponding to $m = n = 1$, leaving β^2 only in the denominator of a single term

$$\chi'' + 2\frac{2\omega\chi' + \chi}{\omega^2 + 1} - \frac{2F}{\omega^2 + \beta^2} = 0, \quad F = \omega\chi' + \frac{\omega^2 - 1}{\omega^2 + 1}\chi. \tag{3.4}$$

The function $F = F(\omega, \chi, \chi')$ is such that $F(0, 0, \chi_0') = 0$. Obtaining the derivative

$$F' = \omega\left[\chi'' + \frac{2\omega\chi'}{\omega^2 + 1} + \frac{4\chi}{(\omega^2 + 1)^2}\right],$$

from here we express χ'' and, substituting it into equation (3.4), we arrive at the equation for F and the solution

$$\frac{F'}{\omega} - \frac{2F}{\omega^2 + \beta^2} = 0 \to F = C(\omega^2 + \beta^2).$$

Since $F(0, 0, \chi_0') = 0$, the constant $C = 0$, $F = 0$, and both the equation

$$F(\omega, \chi, \chi') \equiv (\omega^2 + 1)\omega\chi' + (\omega^2 - 1)\chi = 0$$

and its solution (that does not contain β^2 either) are true

$$n = 1, \quad m = 1, \quad \chi = \frac{\chi_0'\omega}{\omega^2 + 1}, \quad \chi' = \frac{\chi_0'(1 - \omega^2)}{(\omega^2 + 1)^2}. \tag{3.5}$$

According to the performed calculations, $\chi_\infty = 0$ for $n = 1$ for any positive integer m; however, for $m \geq 2$, the solution, for example, the constant B that decreases with increasing m, depends on the value of β^2.

Substituting χ and χ' from (3.5) into formulas (3.2) and (3.3) gives the expressions

$$\psi = 1 + \chi_0'(\theta - \theta_b)(V_b - V) + \ldots, \quad \psi_V = -\chi_0'(\theta - \theta_b), \quad \psi_\theta = \chi_0'(V_b - V). \tag{3.6}$$

In this and subsequent solutions, ψ is a linear function of V and θ, and Chaplygin's equation without ψ_V is satisfied due to the fact that both ψ_{VV} and $\psi_{\theta\theta}$ are equal to zero separately. If we substitute solution (3.5) into formulas (3.3) ψ_{VV} and $\psi_{\theta\theta}$ become zero as well.

To find the curvature of the boundary streamline near the point b, we set $dV = 0$ in equalities (2.3) and, substituting ψ_V and ψ_θ from (3.6), we obtain the equations

$$dx = -\chi_0'\frac{(\theta - \theta_b)\cos\theta_b}{k\rho_b}d\theta, \quad dy = -\chi_0'\frac{(\theta - \theta_b)\sin\theta_b}{k\rho_b}d\theta.$$

After integrating them, we arrive at a parametric representation of the coordinates of the boundary streamline with the angle θ as a parameter

$$x - x_b = -\chi_0' \frac{(\theta - \theta_b)^2}{2k\rho_b} \cos\theta_b, \quad y - y_b = -\chi_0' \frac{(\theta - \theta_b)^2}{2k\rho_b} \sin\theta_b \tag{3.7}$$

As a consequence of these formulas, the dependences of θ and the curvature of the subsonic boundary streamline K are true near the point b from the distance $\tau \geq 0$ to this point

$$\theta = \theta_b + \frac{(2k\rho_b\tau)^{1/2}}{(-\chi_0')^{1/2}}, \quad K = \frac{\partial\theta}{\partial\tau} = \frac{(k\rho_b)^{1/2}}{(-2\chi_0'\tau)^{1/2}}, \quad \tau = \sqrt{(x - x_b)^2 + (y - y_b)^2}. \tag{3.8}$$

It was shown [21] that in an ideal gas, the curvature of the subsonic boundary streamline tends to infinity when approaching the separation point. The same authors, when constructing sonic boundary streamlines [22] separating from circular disks ($\theta_b = \pi/2$), write: "A more detailed analysis establishes that ... near the point b: $y - y_b = O((\pi - \pi/2)^2)$". This is all.

Similarly, by setting $d\theta = 0$ in equalities (2.3), for V and $\partial p/\partial\tau$ when approaching the point b along the wall subject to flow, we obtain the formulas

$$V = V_b - \frac{V_b(2k\rho_b\tau)^{1/2}}{[\chi_0'(\mathrm{M}_b^2 - 1)]^{1/2}}, \quad \frac{\partial p}{\partial\tau} = -\rho V\frac{\partial V}{\partial\tau} = \frac{\rho_b V_b^2 (k\rho_b)^{1/2}}{[2\chi_0'(\mathrm{M}_b^2 - 1)\tau]^{1/2}}. \tag{3.9}$$

Let us find V and θ as functions $\nu = -y'$ in coordinates x', y' (Fig. 1a), that is, their change along the normal to b–e at point b. For the tangential $\mathbf{t}$ ($\cos\theta$, $\sin\theta$) and normal $\mathbf{n}$ ($\sin\theta$, $-\cos\theta$) unit vectors directed along x' and $-y'$ at this point: $\nu = (x - x_b)\sin\theta_b - (y - y_b)\cos\theta_b$ and $dy = -\cot\theta_b dx$ on $\mathbf{n}$. Substituting dx and dy from (2.3) by ψ_V and ψ_θ from (3.6) into the last equality gives a differential equation, by integrating which we arrive at the solution and its consequences for small $\theta - \theta_b$ and $V_b - V$:

$$\theta - \theta_b = \frac{(1 - \mathrm{M}_b^2)^{1/2}}{V_b}(V_b - V), \quad V_b - V = \frac{V_b(k\rho_b\nu)^{1/2}}{(-\chi_0')^{1/2}(1 - \mathrm{M}_b^2)^{1/4}},$$
$$\theta - \theta_b = \frac{(1 - \mathrm{M}_b^2)^{1/4}(k\rho_b\nu)^{1/2}}{(-\chi_0')^{1/2}}, \quad \nu = (x - x_b)\sin\theta_b - (y - y_b)\cos\theta_b \ll 1. \tag{3.10}$$

The second and third formulas are the result of integrating the equations (2.3) with ψ_V and ψ_θ from (3.6) and with c $\theta - \theta_b$ from the first formula.

Let us preface the numerical solution of boundary-value problems corresponding to the values of the exponent $n < 1$ by considering small and large ω, for which

$$n < 1: \quad \chi'' + \frac{(n+1)(\omega^2 + 1) - 4mn}{n(\omega^2 + 1)\omega}\chi' + 2m\frac{(2n - 1)\omega^2 - 1 + 2nm}{n(\omega^2 + 1)^2\omega^2}\chi = 0,$$

$$\omega \ll 1 \to \chi = \frac{A}{l}\omega^l, \quad \chi' = A\omega^{l-1}, \quad l = \frac{2m(1 - 2nm)}{n + 1 - 4mn} > 0,$$

$$\omega \gg 1 \to \chi' = \frac{-B}{\omega^{1+1/n}} \to B = -\omega^{1+1/n}\chi', \quad \chi = \chi_\infty + \frac{nB}{\omega^{1/n}} \to \chi_\infty = \chi + n\omega\chi'.$$

The numerical solution of this equation for different $0 < n < 1$ invariably led to $\chi_\infty \approx 0$ for $m \approx (n + 1)/2n$. Substituting $m = (n + 1)/2n$ in the formula for l gives $l = 1$, $A = \chi_0'$ and the equation

$$\chi'' + \frac{(1 + n)(\omega^2 - 1)}{n(\omega^2 + 1)\omega}\chi' + (1 + n)\frac{(2n - 1)\omega^2 + n}{n^2(\omega^2 + 1)^2\omega^2}\chi = 0.$$

By writing it in the form $f\chi'' + g\chi' + h\chi = 0$ with rational coefficients:

$$f = n^2(\omega^2 + 1)^2\omega^2, \quad g = n(1 + n)(\omega^4 - 1)\omega, \quad h = (1 + n)[(2n - 1)\omega^2 + n],$$

reducing it to the "normal form" and solving according to [25], we obtain

$$\chi = u(\omega)\exp\left(\frac{1}{2}\int\frac{g}{f}d\omega\right) = u(\omega)\left(\frac{\omega}{\omega^2+1}\right)^{(1+n)/(2n)}, \quad u'' = -Iu,$$

$$I = \frac{h}{f} - \frac{1}{4}\left(\frac{g}{f}\right)^2 - \frac{1}{2}\left(\frac{g}{f}\right)' = \frac{c}{\omega^2}, \quad c = \frac{1-n^2}{4n^2}, \quad s = \frac{\sqrt{4c+1}}{2} = \frac{1}{2n},$$

$$u = C_1\omega^{1/2+s} + C_2\omega^{1/2-s} = C_1\omega^{(1+n)/(2n)} + C_2\omega^{(n-1)/(2n)}.$$

Then, taking into account the conditions $\chi_0 = \chi_\infty = 0$, we obtain

$$n < 1, \quad m = \frac{1+n}{2n}, \quad \chi = \frac{\chi_0'\omega}{(\omega^2+1)^m}, \quad \chi' = \frac{\chi_0'(n-\omega^2)}{n(\omega^2+1)^{m+1}}. \tag{3.11}$$

By substituting these m, χ, and χ' into (3.2) and (3.3) for ψ, ψ_V and ψ_θ, and ψ_V and ψ_θ, into equations (2.3) we obtain the same equations (3.6)–(3.9), that were obtained for $n = 1$.

The equality $l = 1$ leads to a quadratic equation for m. Its second root $m = 1/2$ corresponds to the boundary-value problem

$$n < 1, \quad m = \frac{1}{2}: \quad \chi'' + \frac{(n+1)\omega^2 + 1 - n}{n(\omega^2+1)\omega}\chi' + \frac{(2n-1)\omega^2 + n - 1}{n(\omega^2+1)^2\omega^2}\chi = 0.$$
$$\chi_0 = 0, \quad \chi_\infty = 0.$$

Dealing with it in the same way as with the problem for the first root, we first find

$$\chi = \frac{u(\omega)}{\omega^{(1-n)/(2n)}(\omega^2+1)^{1/2}}, \quad u'' = \frac{c}{\omega^2}u, \quad c = \frac{1-n^2}{4n^2}, \quad s = \frac{\sqrt{4c+1}}{2} = \frac{1}{2n},$$
$$u = C_1\omega^{1/2+s} + C_2\omega^{1/2-s} = C_1\omega^{(1+n)/(2n)} + C_2\omega^{(n-1)/(2n)},$$

but then, under the same boundary conditions $\chi_0 = \chi_\infty = 0$ there is only the trivial solution $\chi \equiv 0$.

The boundary-value problem

$$n > 1: \quad \chi'' + \frac{4m\omega}{\omega^2+1}\chi' + 2m\frac{(2m-1)\omega^2+1}{(\omega^2+1)^2}\chi = 0; \quad \chi_0 = 0, \quad \chi_\infty = 0$$

is solved by reducing the equation to the "normal" form [25] even more simply than in the cases already considered. By obtaining

$$\chi = \frac{u(\omega)}{(\omega^2+1)^m}, \quad I = 0, \quad u'' = -Iu = 0, \quad u(\omega) = C_1 + C_2\omega,$$

due to boundary conditions $\chi_0 = \chi_\infty = 0$, we get

$$n > 1, \quad m > \frac{1}{2}, \quad \chi = \frac{\chi_0'\omega}{(\omega^2+1)^m}, \quad \chi' = \chi_0'\frac{1+(1-2m)\omega^2}{(\omega^2+1)^{m+1}}.$$

Formulas (3.6)–(3.10) turn out to be a consequence of this solution not for any $m > 1/2$, but only for $m = (n + 1)/2n$. Therefore, the solution for $n > 1$ is

$$n > 1, \quad m = \frac{1+n}{2n} > \frac{1}{2}, \quad \chi = \frac{\chi_0'\omega}{(\omega^2+1)^m}, \quad \chi' = \frac{\chi_0'(n-\omega^2)}{n(\omega^2+1)^{m+1}}. \tag{3.12}$$

4. FLOWS NEAR THE POINTS OF SEPARATION AND REATTACHMENT OF SONIC BOUNDARY STREAMLINES

Let us begin this topic with the flow near the separation point of the sonic streamline in Fig. 1. Other examples of such flows and flows with a sonic streamline reattaching to the contour are shown in Fig. 3, from which one can get an idea of the planar and axisymmetric configurations subject to a subsonic flow at zero angle of attack with the largest M*. Here, these are symmetrical profiles and bodies of revolution

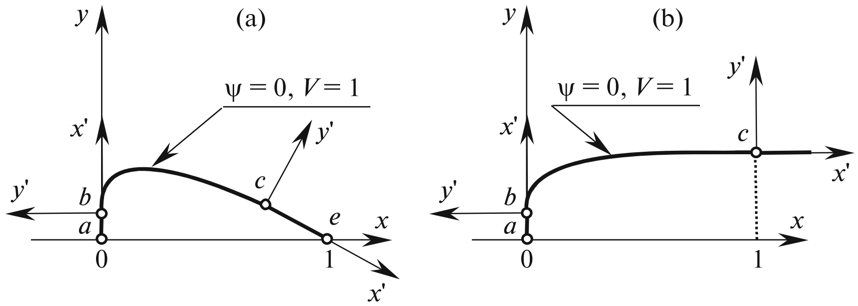

Fig. 3. Symmetrical profile or a body of revolution (a) and the forebody of a plate or a circular cylinder (b), subject to a flow with the highest critical Mach numbers (b–c is a segment of a sonic streamline).

(Fig. 3a) and forebodies of a semi-infinite plate or a circular cylinder (Fig. 3b). As already noted, in their construction, in addition to the usually given length (for the profile–chord length), the "longitudinal" area is fixed for the bodies of Fig. 3a (in the xy plane of Cartesian or cylindrical coordinates), and the half-height of the plate or the radius of the cylinder, for the forebodies. All contours in Fig. 3 have a segment b–c of a sonic streamline with smooth separation and reattachment at points b and c of the wall subject to the flow. Outside of segment b–c almost always $\mathrm{M} < 1$. Exceptions are internal flows with point b or c lying on the normal x axis of the straight transition line [16]. When approaching such a point, the curvature of the sonic streamline $K \to 0$ due to the solution (2.9).

The solution of the transonic Chaplygin's equation (2.4) near the point b in Fig. 1 is sought in the form (3.2) with $V_b = 1$, and for the flows in Fig. 3, the formulas for ψ and ω are changed to $\psi = \varepsilon\chi(\omega) + \ldots$, $\omega = (\theta_b - \theta)/\eta^n$. It does not, however, affect the equation

$$\omega(\omega^2+1)^2[n\omega\chi'' + (n+1)\chi'] - 4mn(\omega^2+1)\omega\chi' + 2m[2n(\omega^2+m) - 1 - \omega^2]\chi$$
$$+ \sigma^2\{(\omega^2+1)^2\chi'' + 4m(\omega^2+1)\omega\chi' + 2m[(2m-1)\omega^2+1]\chi\}\eta^{3-2n}/n = 0,$$

that defines χ at $\varepsilon \ll 1$. At $\eta \ll 1$ its last term is much less than, equal to, or much greater than the rest for $n < 3/2$, $n = 3/2$ or $n > 3/2$, and $\chi(\omega)$ and "eigenvalues" of m are determined by the solution of one of three boundary-value problems:

$$n < \frac{3}{2}: \quad \chi'' + \frac{(n+1)(\omega^2+1) - 4mn}{n(\omega^2+1)\omega}\chi' + 2m\frac{(2n-1)\omega^2 - 1 + 2nm}{n(\omega^2+1)^2\omega^2}\chi = 0,$$

$$n = \frac{3}{2}: \quad \chi'' + \frac{15(\omega^2+1) + 4m(4\sigma^2 - 9)}{(\omega^2+1)(9\omega^2+4\sigma^2)}\omega\chi' + 4m\frac{[6 + (4m-2)\sigma^2]\omega^2 + 9m - 3 + 2\sigma^2}{(\omega^2+1)^2(9\omega^2+4\sigma^2)}\chi = 0,$$

$$n > \frac{3}{2}: \quad \chi'' + \frac{4m\omega}{\omega^2+1}\chi' + 2m\frac{(2m-1)\omega^2+1}{(\omega^2+1)^2}\chi = 0,$$

with the same boundary conditions $\chi_0 = 0$ and $\chi_\infty = 0$.

Numerical solution of the problem corresponding to $n = 3/2$ led to a result similar to that obtained for $n = 1$ in Section 3. The minimum exponent $m \approx 5/6$ was now obtained for any σ^2. Experience with equation (3.4) helped to understand how this is possible.

Let us rewrite the equation corresponding to $n = 3/2$ and $m = 5/6$, leaving σ^2 only in the denominator of one of the terms

$$\chi'' + \frac{10\omega\chi'}{3(\omega^2+1)} + \frac{5(2\omega^2+3)\chi}{9(\omega^2+1)^2} - \frac{F}{9\omega^2+4\sigma^2} = 0, \quad F = 15\omega\chi' + \frac{5(2\omega^2-3)}{\omega^2+1}\chi.$$

As in (3.4), function $F(\omega, \chi, \chi')$ is such that $F(0,0,\chi'_0) = 0$. By finding the derivative

$$F' = 15\omega\left[\chi'' + \frac{5\omega\chi'}{3(\omega^2+1)} + \frac{10\chi}{3(\omega^2+1)^2}\right],$$

we express χ'' from this equality and by substituting it into the original equation, we arrive at the equation for F and the solution

$$\frac{F'}{15\omega}+\frac{F}{9(\omega^2+1)}-\frac{F}{9\omega^2+4\sigma^2}=0\to F=C\left(\frac{9\omega^2+4\sigma^2}{\omega^2+1}\right)^{5/6}.$$

Since $F(0,0,\chi_0')=0$, the constant $C=0$, $F=0$, and the following equation is true:

$$\chi'=\frac{(3-2\omega^2)\chi}{3(\omega^2+1)\omega}.$$

Solving it, we get

$$n=\frac{3}{2},\quad m=\frac{5}{6}:\quad \chi=\frac{\chi_0'\omega}{(\omega^2+1)^{5/6}},\quad \chi'=\frac{\chi_0'(3-2\omega^2)}{3(\omega^2+1)^{11/6}}. \tag{4.1}$$

According to the performed calculations, the eigenvalues of the boundary-value problem corresponding to $n=3/2$, along with $m=5/6$, are $m=2l+5/6$, $l=1,2,...$. For the neighborhood of point b of the flow shown in Fig. 1, substituting χ and χ' from (4.1) into formulas for ψ, ψ_V, and ψ_θ results in expressions

$$\psi=1+\chi_0'(\theta-\theta_b)(1-V)+\ldots,\quad \psi_V=-\chi_0'(\theta-\theta_b),\quad \psi_\theta=\chi_0'(1-V),\quad \chi_0'<0. \tag{4.2}$$

Let us substitute these ψ_V and ψ_θ into equations (2.5), written for b–e with $V=1$ and $dV=0$. Proceeding further, as in Section 3, we arrive at formulas (3.7) and (3.8) with $\rho_b=\rho_*=1$. For V, $\partial V/\partial\tau$, and $\partial p/\partial\tau$ when approaching point b along the wall, we get the formulas

$$V=1+\frac{(3k\tau)^{1/3}}{(\sigma^2\chi_0')^{1/3}},\quad \frac{\partial p}{\partial\tau}=-\frac{\partial V}{\partial\tau}=-\frac{(k/\chi_0')^{1/3}}{(3\sigma\tau)^{2/3}}, \tag{4.3}$$

whose essential difference from (3.9) is different powers of τ.

For the flows shown in Fig. 3,

$$\psi=\chi_0'(\theta_b-\theta)(1-V)+\ldots,\quad \psi_V=-\chi_0'(\theta_b-\theta),\quad \psi_\theta=-\chi_0'(1-V);\quad \chi_0'>0, \tag{4.4}$$

$\nu=y'$, and instead of (3.7), (3.8), and (4.3) we get similar versions:

$$\begin{gathered} x'>0:\quad x-x_b=\chi_0'\frac{(\theta_b-\theta)^2}{2k}\cos\theta_b,\quad y-y_b=\chi_0'\frac{(\theta_b-\theta)^2}{2k}\sin\theta_b,\\ \theta=\theta_b-\frac{(2k\tau)^{1/2}}{\chi_0'^{1/2}},\quad K=\frac{\partial\theta}{\partial\tau}=\frac{-k^{1/2}}{(2\chi_0'\tau)^{1/2}},\\ x'<0:\quad V=1-\frac{(3k\tau)^{1/3}}{(\chi_0'\sigma^2)^{1/3}},\quad \frac{\partial p}{\partial\tau}=-\frac{\partial V}{\partial\tau}=\frac{(k/\chi_0')^{1/3}}{(3\sigma\tau)^{2/3}}. \end{gathered} \tag{4.5}$$

The difference from (3.10) formulas for the change in V and θ along the normal to the streamline:

$$\begin{gathered} (\theta-\theta_{b,c})^2=\frac{2}{3}\sigma^2(1-V)^3,\quad 1-V=\frac{(3/2)^{1/5}\nu^{2/5}}{(\sigma\chi_0'/k)^{2/5}}\to\theta=\theta_{b,c}-\frac{(2\sigma^2/3)^{1/5}\nu^{3/5}}{(\chi_0'/k)^{3/5}},\\ \nu=(y-y_{b,c})\cos\theta_{b,c}-(x-x_{b,c})\sin\theta_{b,c}\ll 1. \end{gathered} \tag{4.6}$$

is expected. The difference because of the different direction of the normal corresponding to Fig. 3 formulas for ν compared to the formula from (3.10) in the case of Fig. 1 is not significant. The equations corresponding to $n<3/2$ and $n>3/2$ coincide with the equations of Section 3, which in that case corresponded to $n<1$ and $n>1$. Solving them provides the results:

$$n<\frac{3}{2},\quad m=\frac{n+1}{2n}:\quad \chi=\frac{\chi_0'\omega}{(\omega^2+1)^m},\quad \chi'=\frac{\chi_0'(n-\omega^2)}{n(\omega^2+1)^{m+1}},$$

$$n > \frac{3}{2}, \quad m = \frac{n+1}{2n} > \frac{1}{2}: \quad \chi = \frac{\chi_0'\omega}{(\omega^2+1)^m}, \quad \chi' = \frac{\chi_0'(n-\omega^2)}{n(\omega^2+1)^{m+1}}$$

that differ from (3.11) and (3.12) only by the values of $n = 3/2$. Their consequences for the separation and reattachment points of flow streamlines coincide or almost coincide with those given in (4.2)–(4.6).

For axisymmetric flows, the right-hand side appears in Chaplygin's equation. Its terms, proportional to the first–third powers of ψ_V and ψ_θ, in all studied solutions have a higher order of smallness over η, than ψ_{VV} and $\psi_{\theta\theta}$. The additional terms in the equations determining the x and y coordinates are also small. As a consequence, all the results obtained can be transferred to axisymmetric flows with a single substitution: k to ky_b.

CONCLUSIONS

The performed study, which initially had some relation to the construction of bodies subject to flow with the highest critical Mach numbers, unexpectedly led the authors to the practically forgotten works of Zhukovsky on jets of an ideal fluid. It turned out that in the 19 century the problems that attracted his attention were of interest to many famous physicists, including Helmholtz, Kirchhoff, Rayleigh, Brillouin, Meshchersky, and Planck. However, none of them managed to advance as far as Zhukovsky in only two papers of 1890–1891. The desire to remind our contemporaries of this and the understanding of whose work is being continued here helped to complete this work.

ACKNOWLEDGMENTS

We are grateful to A.M. Gaifullin for useful information.

FUNDING

The work was supported by the Russian Foundation for Basic Research (project 20-01-00100).

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Translated by L. Trubitsyna