

# Weakly Nonlinear Gravity Three-Dimensional Unbounded Interfacial Waves: Perturbation Method and Variational Formulation

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**Abstract**—Weakly non-linear behaviour of interfacial short-crested waves with current is presented in this paper. Two approaches are used to determine analytical solutions. First, a perturbation method was applied to determine the fifth-order solutions. The advantage of this method is that it allows for the determination of the harmonic resonance condition which is one of the major short-crested waves characteristics. The second method is Whitham's Lagrangian approach. From this method, we obtained a quadratic dispersion equation. In the linear case, we have shown that there is a critical current beyond which steady wave solutions cannot exist. This critical current is associated with the emergence of instability. For the non-linear case, the critical current increases with the wave amplitude as in the two-dimensional case.

**Keywords:** harmonic resonance, short crested interfacial wave, perturbation method, variational formulation, current

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For several years great efforts have been devoted to the study of interfacial waves due to their implication in many practical viewpoints. One example is related to the modelling of two-phase flow problems like gas-liquid flow in pipelines. The problems dealing with interfacial waves with current have been focused for long time on two-dimensional wave fields. Saffman and Yuen [1] considered the problem of progressive interfacial waves of permanent form propagating at the interface of two infinite fluid layers. They have obtained valid solutions for small to moderate wave amplitudes. It was shown that there is a critical current  $U_c$  beyond which steady solutions no longer exist. Furthermore, the critical current velocity increases with the wave-amplitude. Bontozoglou and Hanratty [2] extended Saffman's analysis to include the effect of finite fluid depth. In particular, it was shown that the critical current is a decreasing function of the wave amplitude for some depths.

In the case of three-dimensional forms, most of the published work concerns surface waves. Due to the complexity of the problem, short-crested waves that represent the simplest form are often considered. Hsu et al. [3] have obtained third-order solutions which remain valid for weakly nonlinear waves. Roberts [4] studied the case of infinite depth by using a perturbation method. Numerical solutions up to the 27th order were calculated and the phenomenon of harmonic resonance was highlighted. Marchant and Roberts [5] extended the method [4] to calculate the 35th-order solutions for waves propagating on the finite depth. Tsai and Jeng [6] presented the Fourier method to calculate the highest short-crested waves. Recently, studies investigating on short crested wave breaking using numerical models have been carried out. Wei et al. [7] studied short crested waves in the surf zone by using Smoothed Particle Hydrodynamics (SPH) approaches. Kirby and Derakhti [8] were performed calculations using the FUNWAVE-TVD which is the Total Variation Diminishing (TVD) version of the fully nonlinear Boussinesq wave model. In this way,

these authors have revealed the formation of rip currents and the generation of vertical vorticity. Debiante and Kharif [9] studied short-crested gravity-capillarity waves in infinite depth using the Lagrangian method. This technique has allowed for the calculation of the resonance waves. Concus [10] ran into a problem that the set of fluid depths at which a resonance condition invalidates the standing wave perturbation expansion is a dense set, calling into question the whole approach. This issue was finally resolved by Plotnikov and Toland [11] by using Nash–Moser theory to overcome the small divisor problem. For three dimensional travelling waves, similar to the short-crested waves, Iooss and Plotnikov [12] resolved the small-divisor issue using Nash–Moser theory for non-resonant angles. They also work out a formal asymptotic expansion. Jian et al. [13], for their part, calculated analytical solutions of gravity-capillarity waves at finite depths in the presence of a uniform current. They focused on the influence of the current on the characteristics of the wave (profile, frequency, pressure, etc.). In particular, their results show that the wave becomes steeper as the current increases.

Very few studies have focused on three-dimensional interfacial waves and those that exist do only for solitary waves [14–17]. Otherwise, Allalou et al. [18] studied short-crested interfacial waves, that are the simplest three-dimensional and progressive shapes, in details. They obtained third-order analytical solutions and 27th order numerical solutions. This study determines the resonance condition in the case of three-dimensional interfacial waves and represents the generalization of the work of Roberts et al. [4, 5]. More recent example comprising the study of interfacial waves is the theoretical investigation by Li et al. [19]. They have conducted a detailed investigation of steady-state resonant interfacial waves in a two layer fluids within a frictionless duct.

The objective of the present work is to extent the study of Allalou et al. [18] for case where the current is present. It can also be seen as the generalization of Saffman’s [1] work to the three-dimensional case. In Section 1, the physical and the mathematical description of short crested wave problem in the presence of current is exposed. In Section 2, the perturbation method is used to determine analytical solutions up to the fifth-order with emphasis on the phenomenon of harmonic resonance. Then, the Lagrangian approach, applied to the same problem, is presented in Section 3. Finally, in Section 4, the obtained results are discussed.

## 1. FORMULATION OF THE PROBLEM

We will consider a two-layer unbounded fluid system with parallel uniform current  $U$  as shown in Fig. 1a. The top layer has a density of  $\rho_1$  and the lower has density  $\rho_2$ . The two layers are supposed to be horizontal with infinite depths. The defined cartesian coordinates system is composed of the horizontal plane ( $xOy$ ) and the  $z$ -axis which is pointed vertically upward. Its origin is placed on the unperturbed interface between the two layers as shown in Fig. 1a. Herein, short crested-wave field are generated from the non-linear interaction of two wave trains propagating toward each other with the same characteristics.  $L$  is the incident wave’s wavelength,  $\theta$  is the angle between its direction of incidence and the normal to the wall as shown in Fig. 1b. The two fluids are assumed inviscid, incompressible, homogeneous and the motion is irrotational. Both fluids are considered to be stably stratified by gravity, i.e.,  $\rho_1 < \rho_2$ .

With the above assumptions, the velocity field of the upper layer is then the sum of two components:

$$\vec{V}_1 = U\vec{i} + \vec{\nabla}\phi_1. \quad (1.1)$$

The first component is due to the presence of a uniform current and the second one is an irrotational component and thus derives from a potential. The velocity vector of the lower layer is given by:

$$\vec{V}_2 = \vec{\nabla}\phi_2. \quad (1.2)$$

The fluid motion can be described by the velocity potentials  $\phi_1(x, y, z, t)$  and  $\phi_2(x, y, z, t)$  which must satisfy the Laplace equation in the two fluid layers domain, as written as follow:

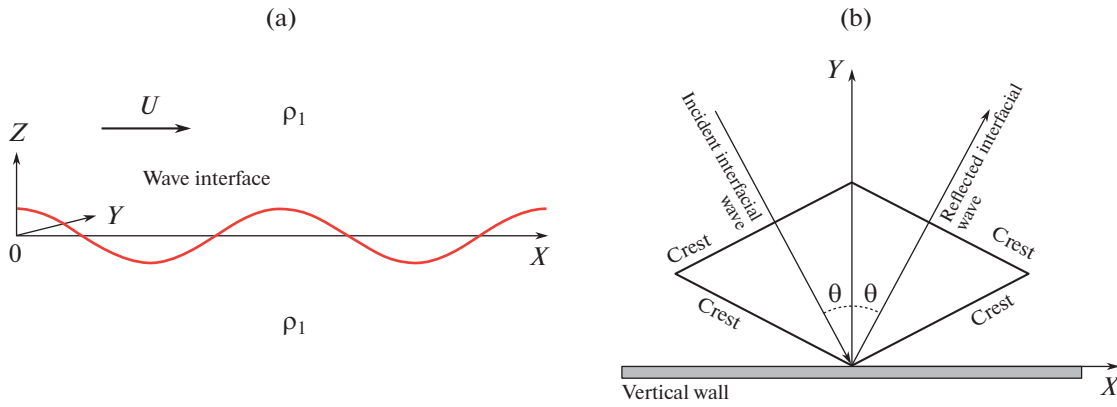
$$\Delta\phi_1 = 0 \quad \text{for} \quad \eta(x, y, t) < z < +\infty, \quad (1.3)$$

$$\Delta\phi_2 = 0 \quad \text{for} \quad -\infty < z < \eta(x, y, t). \quad (1.4)$$

The kinematic boundary conditions, at the interface, for the two fluids are:

$$\eta_t + \eta_x\phi_{1x} + U\eta_x + \eta_y\phi_{1y} - \phi_{1z} = 0 \quad \text{as} \quad z = \eta(x, y, t), \quad (1.5)$$

$$\eta_t + \eta_x\phi_{2x} + \eta_y\phi_{2y} - \phi_{2z} = 0 \quad \text{as} \quad z = \eta(x, y, t). \quad (1.6)$$



**Fig. 1.** Schematic representation of an interfacial wave in the presence of a current parallel to the  $Ox$  axis (a). Reflection of a plane wave from a vertical wall (b).

The dynamic boundary condition at the interface is:

$$\left( \rho_2(\phi_{2t} + g\eta + \frac{1}{2}(\phi_{2x}^2 + \phi_{2y}^2 + \phi_{2z}^2)) \right) - \left( \rho_1(\phi_{1t} + g\eta + \frac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2 + \phi_{1z}^2)) \right) + C = 0$$

as  $z = \eta(x, y, t)$ . (1.7)

It can be derived from Bernoulli's equation and describes the balance of pressure forces on either side of the interface. Here,  $C$  is the Bernoulli constant.

The conditions that the normal velocities to be zero far below and above the interface give:

$$\phi_{1z} = 0 \quad \text{for} \quad z \rightarrow +\infty, \quad (1.8)$$

$$\phi_{2z} = 0 \quad \text{for} \quad z \rightarrow -\infty. \quad (1.9)$$

Dimensionless equations are obtained by defining a reference time  $1/\sqrt{gk}$  and a reference length  $1/k$ ,  $k$  is the wave number  $2\pi/L$ ,  $L$  being the wavelength of the incident wave and  $g$  is the gravity acceleration. The following dimensionless quantities can then be introduced:

$$\tilde{x} = kx, \quad \tilde{y} = ky, \quad \tilde{z} = kz, \quad \tilde{\eta} = k\eta, \quad \tilde{t} = t\sqrt{gk}, \quad \tilde{U} = U\sqrt{k/g}.$$

The tildes that denote dimensionless quantities will now be omitted for the sake of simplicity. Due to the periodicity of the solutions in both  $(ox)$  and  $(oy)$  directions with a period of  $2\pi$  and the fact that the waves are propagating without change of shape we can define:

$$X = px - \omega t, \quad Y = qy, \quad Z = z,$$

where  $p$  and  $q$  are the two wavenumbers following the  $(ox)$  and  $(oy)$  directions, respectively, and are defined by:

$$p = \sin \theta, \quad q = \cos \theta.$$

The governing equations may now be written in terms of these dimensionless quantities:

$$p^2\phi_{iXX} + q^2\phi_{iYY} + \phi_{iZZ} = 0, \quad i = (1, 2), \quad (1.10)$$

$$(pU - \omega)\eta_X + p^2\eta_X\phi_{1X} + q^2\eta_Y\phi_{1Y} - \phi_{1Z} = 0 \quad \text{as} \quad Z = \eta, \quad (1.11)$$

$$-\omega\eta_X + p^2\eta_X\phi_{2X} + q^2\eta_Y\phi_{2Y} - \phi_{2Z} = 0 \quad \text{as} \quad Z = \eta, \quad (1.12)$$

$$\phi_{1Z} = 0 \quad \text{as} \quad Z \rightarrow +\infty, \quad (1.13)$$

$$\phi_{2Z} = 0 \quad \text{as} \quad Z \rightarrow -\infty, \quad (1.14)$$

$$\mu \left\{ (pU - \omega)\phi_{1X} + \eta + \frac{1}{2}(p^2\phi_{1X}^2 + q^2\phi_{1Y}^2 + \phi_{1Z}^2) \right\} + \omega\phi_{2X} - \eta$$

$$-\frac{1}{2}(p^2\phi_{2X}^2 + q^2\phi_{2Y}^2 + \phi_{2Z}^2) + C = 0 \quad \text{as} \quad Z = \eta, \quad (1.15)$$

where  $\mu = \rho_1/\rho_2$  is the density ratio.

## 2. PERTURBATION METHOD

### 2.1. Five-Order Solution

In this section, the problem is solved by the perturbation method. The parameters of the problem are developed in power series based on a small perturbation parameter  $h$  as follow:

$$\left\{ \begin{array}{l} \phi_i = \sum_{r=1}^{\infty} h^r \phi_i^{(r)}(X, Y, Z) \quad i = (1, 2), \\ \eta = \sum_{r=1}^{\infty} h^r \eta^{(r)}(X, Y), \\ \omega = \sum_{r=1}^{\infty} h^r \omega_r, \\ C = \sum_{r=1}^{\infty} h^r C^{(r)}, \end{array} \right. \quad (2.1)$$

where  $h$  is the wave steepness defined by  $h = 1/2(\eta(0, 0) - \eta(\pi, 0))$ . The boundary conditions are applied at the interface  $Z = \eta(X, Y)$ , which is a priori unknown. In order to overcome this difficulty, the interface boundary conditions are applied at  $Z=0$ . This is accomplished by expanding the velocity potentials  $\phi_1$  and  $\phi_2$  in Taylor series about the  $Z=0$ :

$$\phi_i(X, Y, \eta) = \phi_i(X, Y, 0) + \phi_{iZ}|_{Z=0} \eta + \frac{1}{2} \phi_{iZZ}|_{Z=0} \eta^2 + \dots \quad (2.2)$$

By inserting Eqs. (2.1) and (2.2) into the basic equations (1.10)–(1.14), the first order equations of approximations become:

$$\left\{ \begin{array}{l} p^2 \phi_{iXX}^{(1)} + q^2 \phi_{iYY}^{(1)} + \phi_{iZZ}^{(1)} = 0, \quad i = (1, 2), \\ \omega_0 \eta_X^{(1)} + \phi_{2Z}^{(1)} = 0, \\ (\omega_0 - pU) \eta_X^{(1)} + \phi_{1Z}^{(1)} = 0, \\ \omega_0 (\phi_{2X}^{(1)} - \mu \phi_{1X}^{(1)}) + (\mu - 1) \eta^{(1)} + \mu p U \phi_{1X}^{(1)} = 0, \\ \phi_{1Z}^{(1)} = 0, Z \rightarrow +\infty, \\ \phi_{2Z}^{(1)} = 0, Z \rightarrow -\infty. \end{array} \right. \quad (2.3)$$

After solving the system of equations (2.3), the following linear solutions are obtained:

$$\left\{ \begin{array}{l} \eta^{(1)} = \cos X \cos Y, \\ \phi_1^{(1)} = (-\omega_0 + pU) \sin X \cos Y e^{-Z}, \\ \phi_2^{(1)} = \omega_0 \sin X \cos Y e^Z, \\ \omega_0^2 + \mu(\omega_0 - pU)^2 = 1 - \mu, \\ C^{(1)} = 0. \end{array} \right. \quad (2.4)$$

This solution is reduced to that obtained by Allalou et al. [18] when  $U = 0$ . The examination of the linear dispersion relation shows that there are two modes  $\omega_0^+$  and  $\omega_0^-$  with  $\omega_0^+ > \omega_0^-$ . These two modes are given by:

$$\omega_0^+ = \frac{\mu p U + \sqrt{1 - \mu^2 - \mu p^2 U^2}}{\mu + 1}, \quad (2.5)$$

**Table 1.** Second-order solution obtained from the perturbation method

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$$a_{2,2}^{(2)} = -\frac{1 - \omega_0^2 - 2\mu p U \omega_0 + \mu \omega^2 + \mu p^2 U^2}{4 \mu - 1 + 2\omega_0^2 + 2\mu(-\omega_0 + pU)^2},$$

$$a_{02}^{(2)} = -\frac{1}{4} \frac{(p-1)(p+1)(\mu(pU - \omega_0)^2 - \omega_0^2)}{\mu - 1},$$

$$a_{20}^{(2)} = -\frac{1}{4} \frac{p(p^2 - 2 - 2p)(\mu(pU - \omega_0)^2 - \omega_0^2)}{p(\mu - 1) + 2\omega_0^2 + 2\mu(-\omega_0 + pU)^2},$$

$$b_{22}^{(2)} = \frac{1}{4} \frac{(-\omega_0 + pU)(\mu(-\omega_0 + pU)^2) - 1 + \mu + 3\omega_0^2}{\mu - 1 + 2\omega_0^2 + 2\mu(-\omega_0 + pU)^2},$$

$$b_{20}^{(2)} = -\frac{1}{4} \frac{(-\omega_0 + pU)(\mu(p^2 - 2)(pU - \omega_0)^2) - p(p\omega_0)^2 - p + \mu p + 4\omega_0^2 + 2\omega_0^2}{p(\mu - 1) + 2\omega_0^2 + 2\mu(-\omega_0 + pU)^2},$$

$$c_{22}^{(2)} = -\frac{1}{4} \frac{\omega_0(\omega_0)^2 + 3\mu(-\omega_0 + pU)^2 + \mu - 1}{\mu - 1 + 2\omega_0^2 + 2\mu(-\omega_0 + pU)^2},$$

$$c_{20}^{(2)} = -\frac{1}{4} \frac{\mu(p^2 + 4p - 2)(-\omega_0 + pU)^2 + 2\omega_0^2 + p^2(-\omega_0^2 - 1 + \mu)}{p(\mu - 1) + 2\omega_0^2 + 2\mu(-\omega_0 + pU)^2},$$

$$\omega_1 = 0$$


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$$\omega_0^- = \frac{\mu p U - \sqrt{1 - \mu^2 - \mu p^2 U^2}}{\mu + 1}. \quad (2.6)$$

The second order approximations are:

$$p^2 \phi_{iXX}^{(2)} + q^2 \phi_{iYY}^{(2)} + \phi_{iZZ}^{(2)} = 0, \quad i = (1, 2), \quad (2.7)$$

$$(\omega_0 - Up) \eta_X^{(2)} + \phi_{1Z}^{(2)} = q^2 \phi_{1Y}^{(1)} \eta_Y^{(1)} + p^2 \phi_X^{(1)} \eta_X^{(1)} - \phi_{1ZZ}^{(1)} \eta^{(1)} - \omega_1 \eta_X^{(1)} \quad \text{as } Z = 0, \quad (2.8)$$

$$\omega_0 \eta_X^{(2)} + \phi_{2Z}^{(2)} = q^2 \phi_{2Y}^{(1)} \eta_Y^{(1)} + p^2 \phi_{2X}^{(1)} \eta_X^{(1)} - \phi_{2ZZ}^{(1)} \eta^{(1)} - \omega_1 \eta_X^{(1)} \quad \text{as } Z = 0, \quad (2.9)$$

$$\begin{aligned} \omega_0(\mu \phi_{1X}^{(2)} - \phi_{2X}^{(2)}) + \eta^{(2)}(1 - \mu) - \mu p U \phi_{1X}^{(2)} &= \frac{1}{2} \mu [p^2 (\phi_{1X}^{(1)})^2 + q^2 (\phi_{1Y}^{(1)})^2 + (\phi_{1Z}^{(1)})^2] \\ &- \frac{1}{2} [p^2 (\phi_{2X}^{(1)})^2 + q^2 (\phi_{2Y}^{(1)})^2 + (\phi_{2Z}^{(1)})^2] + \omega_0 \eta^{(1)} (\phi_{2XZ}^{(1)} - \mu \phi_{1XZ}^{(1)} + \mu p U \phi_{1XZ}^{(1)}) \\ &+ \omega_1 (\phi_{2X}^{(1)} - \mu \phi_{1X}^{(1)}), \quad \text{as } Z = 0, \end{aligned} \quad (2.10)$$

$$\phi_{1Z}^{(2)} = 0, \quad Z \rightarrow +\infty, \quad (2.11)$$

$$\phi_{2Z}^{(2)} = 0, \quad Z \rightarrow -\infty. \quad (2.12)$$

We will seek solutions in the following form:

$$\begin{cases} \eta^{(2)} = a_{20}^{(2)} \cos(2X) + a_{02}^{(2)} \cos(2Y) + a_{22}^{(2)} \cos(2X) \cos(2Y), \\ \phi_1^{(2)} = b_{02}^{(2)} \cos(2Y) \exp(-2\alpha_{02}Z) + b_{22}^{(2)} \sin(2X) \cos(2Y) \exp(-2\alpha_{22}Z), \\ \phi_2^{(2)} = c_{02}^{(2)} \cos(2Y) \exp(2\alpha_{02}Z) + c_{22}^{(2)} \sin(2X) \cos(2Y) \exp(2\alpha_{22}Z), \end{cases} \quad (2.13)$$

where  $\alpha_{mn}^2 = (mp)^2 + (nq)^2$  and  $m$  and  $n$  are integers of the same parity.

Obviously,  $\phi_1^{(2)}$  and  $\phi_2^{(2)}$  must satisfy the Laplace equation and the boundary conditions. After substituting the first-order solutions in Eqs. (2.8), (2.9), and (2.10) and some algebraic calculation, the second-order coefficients are obtained. These coefficients are given in Table 1. It is possible to verify that in the particular case  $U = 0$ , the solutions of order two coincides with that of Allalou et al. [18].

Similarly, the third-order system of equations is as follows:

$$p^2\phi_{iXX}^{(3)} + q^2\phi_{iYY}^{(3)} + \phi_{iZZ}^{(3)} = 0, \quad i = (1, 2), \tag{2.14}$$

$$\begin{aligned} (\omega_0 - Up)\eta_X^{(3)} + \phi_{1Z}^{(3)} &= \eta^{(1)}(p^2\phi_{1XZ}\eta_X^{(1)} + q^2\phi_{1YZ}\eta_Y^{(1)} - \phi_{1ZZ}^{(2)}) + p^2\phi_{1X}^{(2)}\eta_X^{(1)} + q^2\phi_{1Y}^{(2)}\eta_Y^{(1)} \\ &- \frac{1}{2}\phi_{1ZZZ}^{(1)}(\eta^{(1)})^2 + p^2\phi_{1X}^{(1)}\eta_X^{(2)} + q^2\phi_{1Y}^{(1)}\eta_Y^{(2)} - \phi_{1ZZ}^{(1)}\eta^{(2)} - \omega_2\eta_X^{(1)} \quad \text{as } Z = 0, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \omega_0\eta_X^{(3)} + \phi_{2Z}^{(3)} &= \eta^{(1)}(p^2\phi_{2XZ}\eta_X^{(1)} + q^2\phi_{2YZ}\eta_Y^{(1)} - \phi_{2ZZ}^{(2)}) + p^2\phi_{2X}^{(2)}\eta_X^{(1)} + q^2\phi_{2Y}^{(2)}\eta_Y^{(1)} - \frac{1}{2}\phi_{2ZZZ}^{(1)}(\eta^{(1)})^2 \\ &+ p^2\phi_{2X}^{(1)}\eta_X^{(2)} + q^2\phi_{2Y}^{(1)}\eta_Y^{(2)} - \phi_{2ZZ}^{(1)}\eta^{(2)} - \omega_2\eta_X^{(1)} \quad \text{as } Z = 0, \end{aligned} \tag{2.16}$$

$$\begin{aligned} \omega_0(\mu\phi_{1X}^{(3)} - \phi_{2X}^{(3)}) + \eta^{(3)}(1 - \mu) - \mu pU\phi_{1X}^{(3)} &= -(p^2\phi_{2X}^{(1)}\phi_{2X}^{(2)} + q^2\phi_{2Y}^{(1)}\phi_{2Y}^{(2)} + \phi_{2ZZ}^{(1)}\phi_{2Z}^{(2)}) \\ + \eta^{(1)}\mu(p^2\phi_{1X}^{(1)}\phi_{1XZ}^{(1)} + q^2\phi_{1Y}^{(1)}\phi_{1YZ}^{(1)} + \phi_{1ZZ}^{(1)}\phi_{1Z}^{(1)}) - \eta^{(1)}(p^2\phi_{2X}^{(1)}\phi_{2XZ}^{(1)} + q^2\phi_{2Y}^{(1)}\phi_{2YZ}^{(1)} + \phi_{2ZZ}^{(1)}\phi_{2Z}^{(1)}) \\ + \eta^{(1)}\omega_0(\phi_{2XZ}^{(2)} - \mu\phi_{1XZ}^{(2)}) + \mu(p^2\phi_{1X}^{(1)}\phi_{1X}^{(2)} + q^2\phi_{1Y}^{(1)}\phi_{1Y}^{(2)} + \phi_{1Z}^{(1)}\phi_{1Z}^{(2)}) + \omega_2(\phi_{2X}^{(1)} - \mu\phi_{1X}^{(1)}) \\ + \frac{1}{2}\omega_0(\eta^{(1)})^2(\phi_{2XZZ}^{(1)} - \mu\phi_{1XZZ}^{(1)}) + \omega_0\eta^{(2)}(\phi_{2XZ}^{(1)} - \mu\phi_{1XZ}^{(1)}) \\ + \mu pU\left(\phi_{1XZ}^{(2)}\eta^{(1)} + \phi_{1XZ}^{(1)}\eta^{(2)} + \frac{1}{2}\phi_{1XZZ}^{(2)}(\eta^{(1)})^2\right) \quad \text{as } Z = 0, \end{aligned} \tag{2.17}$$

$$\phi_{1Z}^{(3)} = 0 \quad Z \rightarrow +\infty, \tag{2.18}$$

$$\phi_{2Z}^{(3)} = 0 \quad Z \rightarrow -\infty. \tag{2.19}$$

The coefficients for this order of approximation are given in the Appendix.

The fourth and fifth orders proceed in a similar manner. However, because of the length of their expressions, it was not possible to give them in this article.

### 2.2. Harmonic Resonance

Roberts [4] has shown that for some values of the angle  $\theta$ , the surface wave problem has no unique solution. This is due to the phenomenon of harmonic resonance between the fundamental and higher order modes. Allalou et al. [18] have obtained the resonance relation in the case of short-crested interfacial waves. We will determine the resonance relation in the case of short-crested interfacial waves in the presence of a uniform current. We first note that kinematic and dynamic equations of the order  $r$  can be written in the following form:

$$\begin{cases} (\omega_0 - Up)\eta_X^{(r)} + \phi_{1Z}^{(r)} = I_1^{(r)} & \text{as } Z = 0, \\ \omega_0\eta_X^{(r)} + \phi_{2Z}^{(r)} = I_2^{(r)} & \text{as } Z = 0, \\ \omega_0(\mu\phi_{1X}^{(r)} - \phi_{2X}^{(r)}) + \eta^{(r)}(1 - \mu) - \mu pU\phi_{1X}^{(r)} = II^{(r)} & \text{as } Z = 0, \end{cases} \tag{2.20}$$

where the terms  $I_1^{(r)}$ ,  $I_2^{(r)}$ , and  $II^{(r)}$  depend on the solutions of the  $(r - 1)$ th order. We look for solutions of the following form:

$$\begin{cases} \eta^{(r)} = \sum_{mn}^r a_{mn}^{(r)} \cos(mX)\cos(nY), \\ \phi_1^{(r)} = \sum_{mn}^r b_{mn}^{(r)} \sin(mX)\cos(nY)\exp(-\alpha_{mn}Z), \\ \phi_2^{(r)} = \sum_{mn}^r c_{mn}^{(r)} \sin(mX)\cos(nY)\exp(\alpha_{mn}Z). \end{cases} \tag{2.21}$$

Furthermore, it is possible to show that the terms  $I_1^{(r)}$ ,  $I_2^{(r)}$ , and  $II^{(r)}$  can be written as follows:

$$\begin{cases} I_1^{(r)} = \sum_{mn}^r F_{1mn}^{(r)} \sin(mX) \cos(nY), \\ I_2^{(r)} = \sum_{mn}^r F_{2mn}^{(r)} \sin(mX) \cos(nY), \\ II^{(r)} = \sum_{mn}^r G_{mn}^{(r)} \cos(mX) \cos(nY). \end{cases} \quad (2.22)$$

After substituting the expressions (2.21) and (2.22) in the system of equations (2.20), we obtain

$$\begin{cases} -m(\omega_0 - pU) a_{mn}^{(r)} - \alpha_{mn} b_{mn}^{(r)} = F_{1mn}^{(r)}, \\ -m\omega_0 a_{mn}^{(r)} + \alpha_{mn} c_{mn}^{(r)} = F_{2mn}^{(r)}, \\ \omega_0 m(\mu b_{mn}^{(r)} - c_{mn}^{(r)}) + (1 - \mu) a_{mn}^{(r)} - \mu p U m b_{mn}^{(r)} = G_{mn}^{(r)}, \end{cases} \quad (2.23)$$

which has the solution

$$\begin{cases} a_{mn}^{(r)} = -\frac{m\mu(\omega_0 - pU) F_{1mn}^{(r)} + m\omega_0 F_{2mn}^{(r)} + \alpha_{mn} G_{mn}^{(r)}}{\alpha_{mn}(\mu - 1) + m^2(\omega_0^2 + \mu(\omega_0 - pU)^2)}, \\ b_{mn}^{(r)} = -\frac{(m^2\omega_0^2 + \alpha_{mn}(\mu - 1)) F_{1mn}^{(r)} + m^2\omega_0(pU - \omega_0) F_{2mn}^{(r)} + m\alpha_{mn}(-\omega_0 + pU) G_{mn}^{(r)}}{\alpha_{mn}(\alpha_{mn}(\mu - 1) + m^2(\omega_0^2 + \mu(\omega_0 - pU)^2))}, \\ c_{mn}^{(r)} = -\frac{-m^2\omega_0\mu(-\omega_0 + pU) F_{1mn}^{(r)} - (\alpha_{mn}(-1 + \mu) + \mu m^2(-\omega_0 + pU)^2) F_{2mn}^{(r)} + m\omega_0\alpha_{mn} G_{mn}^{(r)}}{\alpha_{mn}(\alpha_{mn}(\mu - 1) + m^2(\omega_0^2 + \mu(\omega_0 - pU)^2))}. \end{cases} \quad (2.24)$$

The coefficients in Eq. (2.24) involve divisions by zero at certain parameters values. These zero divisors correspond to the occurrence of *harmonic resonances* and it occurs when:

$$\alpha_{mn}(\mu - 1) + m^2(\omega_0^2 + \mu(\omega_0 - pU)^2) = 0. \quad (2.25)$$

Using the linear dispersion relation, expression (2.25) is simplified to:

$$m^2 = \alpha_{mn}. \quad (2.26)$$

From this expression, the critical angle of resonance  $\theta_c$  is then deduced:

$$\cos^2 \theta_c = \frac{m^4 - m^2}{n^2 - m^2}, \quad (2.27)$$

which is identical to that found in [4]. The angle at which harmonic resonance occurs is independent of the current  $U$  and density ratio  $\mu$ .

### 3. VARIATIONAL PRINCIPLE

For gravity short-crested interfacial waves with a current  $U$  in two unbounded fluids, the average Lagrangian is given by

$$\begin{aligned} L = & \overline{\left[ \int_{-\infty}^{\eta} \frac{1}{2} (p^2 \phi_{2X}^2 + q^2 \phi_{2Y}^2 + \phi_{2Z}^2) - \omega \phi_{2X} + \eta \right] dZ} \\ & - \mu \overline{\left[ \int_{\eta}^{+\infty} \frac{1}{2} ((p\phi_{1X} + U)^2 + q^2 \phi_{1Y}^2 + \phi_{1Z}^2) - \omega \phi_{1X} + \eta \right] dZ} - L_0, \end{aligned} \quad (3.1)$$

where the overbar denotes a space average, and  $L_0$  being the Lagrangian density of the undisturbed flow (see [1]). It is introduced in order to ensure convergence value for  $L$  and is given by

$$L_0 = -\int_{-\infty}^0 \eta dZ - \int_0^{\infty} \left( \mu \eta dZ + \frac{1}{2} \mu U^2 \right) dZ. \tag{3.2}$$

Note that the expression (3.1) is the generalization of the study [1] to the three-dimensional case.

Following Whitham [20], interfacial wave  $\eta$  and velocity potentials,  $\phi_1$  and  $\phi_2$ , are written in the following truncated form:

$$\begin{cases} \eta(X, Y) = A_{11} \cos X \cos Y + A_{22} \cos 2X \cos 2Y + A_{20} \cos 2X + A_{02} \cos 2Y, \\ \phi_1(X, Y, Z) = B_{11} \sin X \cos Y e^{-\alpha_{11}Z} + B_{22} \sin 2X \cos 2Y e^{-\alpha_{22}Z} + B_{20} \sin 2X e^{-\alpha_{20}Z}, \\ \phi_2(X, Y, Z) = C_{11} \sin X \cos Y e^{\alpha_{11}Z} + C_{22} \sin 2X \cos 2Y e^{\alpha_{22}Z} + C_{20} \sin 2X e^{\alpha_{20}Z}. \end{cases} \tag{3.3}$$

Note that  $A_{11}$ ,  $B_{11}$ , and  $C_{11}$  are the quantities of the first order in wave amplitude while the other coefficients are of the second order. We substitute these expressions in (3.1), calculate  $L$ , resolve for  $L$ , and eliminate  $B_{11}$ ,  $B_{20}$ ,  $B_{22}$ ,  $C_{11}$ ,  $C_{20}$ , and  $C_{22}$ , using the equations:

$$\frac{\partial L}{\partial B_{11}} = \frac{\partial L}{\partial B_{20}} = \frac{\partial L}{\partial B_{22}} = \frac{\partial L}{\partial C_{11}} = \frac{\partial L}{\partial C_{20}} = \frac{\partial L}{\partial C_{22}} = 0, \tag{3.4}$$

and after some algebraic operations, we get:

$$\begin{aligned} L = & \frac{1}{4p} (-p + p\mu + 2\omega^2 + 2\mu(\omega - pU)^2) A_{20}^2 + \frac{1}{8} ((2(\omega - pU)^2 + 1)\mu + 2\omega^2 + \mu - 1) A_{22}^2 \\ & + \frac{1}{8} (((\omega - pU)^2)\mu + \omega^2 + \mu - 1) A_{11}^2 + \frac{1}{8} (p - 1)(p + 1)(\mu(\omega - pU)^2 - \omega^2) A_{11}^2 A_{02} \\ & + \frac{1}{64} (-1 + 2p^3 - 4p^2)(\mu(\omega - pU)^2 + \omega^2) A_{11}^4 + \frac{1}{16} (\mu(\omega - pU)^2 - \omega^2) A_{11}^2 A_{22} \\ & + \frac{1}{8} (-2 + p^2 + 2p)(\mu(\omega - pU)^2 - \omega^2) A_{11}^2 A_{20} + \frac{1}{4} (-1 + \mu) A_{02}^2 + \mathcal{O}(A_{11}^6). \end{aligned} \tag{3.5}$$

The lowest order gives the linear dispersion relation for the variation of  $L$  with respect to  $A_{11}$

$$\omega = \omega_0 + \mathcal{O}(A_{11}^2), \quad \text{where} \quad \mu(\omega_0 - pU)^2 + \omega_0^2 + \mu - 1 = 0, \tag{3.6}$$

which is identical to that found by the perturbation method.

By introducing the last relation in the Lagrangian expression, the values of  $A_{22}$ ,  $A_{20}$ , and  $A_{02}$  can be found from the relations

$$\frac{\partial L}{\partial A_{22}} = \frac{\partial L}{\partial A_{20}} = \frac{\partial L}{\partial A_{02}} = 0, \tag{3.7}$$

which gives:

$$A_{22} = -\frac{1}{4} \frac{A_{11}^2 (\mu + 2\omega_0^2 - 1)}{-1 + \mu}, \tag{3.8}$$

$$A_{20} = \frac{1}{4} \frac{A_{11}^2 p (-2 + p^2 + 2p)(\mu + 2\omega_0^2 - 1)}{(p - 2)(-1 + \mu)}, \tag{3.9}$$

$$A_{02} = \frac{1}{4} \frac{A_{11}^2 (p - 1)(p + 1)(\mu + 2\omega_0^2 - 1)}{-1 + \mu}. \tag{3.10}$$

It is easy to verify that these coefficients are similar to those found by the perturbation method.

After the substitution, the expressions of  $A_{22}$ ,  $A_{02}$ , and  $A_{20}$ , the Lagrangian can finally be simplified to:

$$L = \frac{1}{64} \frac{A_{11}^4 p (p^2 - 2 + 2p)^2 (\mu(\omega_0 - pU)^2 - \omega_0^2)^2}{-p + p\mu + 2\omega_0^2 + 2\mu(\omega_0 - pU)^2} + \frac{1}{128} \frac{A_{11}^4 (\mu(\omega_0 - pU)^2 - \omega_0^2)^2}{2\omega_0^2 + 2\mu(\omega_0 - pU)^2 + \mu - 1}$$



$$\begin{aligned}
& + \frac{1}{64} \frac{A_{11}^4 (p-1)^2 (p+1)^2 (\mu(\omega_0 - pU)^2 - \omega_0^2)^2}{-1 + \mu} - \frac{1}{8} A_{11}^2 (-1 + \mu + \omega^2 + \mu(\omega - pU)^2) \\
& - \frac{1}{64} A_{11}^4 (-1 - 4p^2 + 2p^3)(\omega_0^2 + \mu(\omega_0 - pU)^2) + \mathcal{O}(A_{11}^6). \tag{3.11}
\end{aligned}$$

The non-linear dispersion relation is then obtained from  $\partial L / \partial A_{11} = 0$ , giving:

$$\omega^2 + \mu(\omega - pU)^2 = (1 - \mu) \left[ 1 + A_{11}^2 \left( \frac{\lambda^2 4p^5 + 4p^4 - 4p^3 - 8p^2 + 9p - 2}{8(2-p)(1-\mu)^2} - \frac{1}{4}(2p^3 - 4p^2 - 1) \right) \right], \tag{3.12}$$

where  $\lambda = \omega_0^2 - \mu(\omega_0 - pU)^2$ .

## 4. RESULTS AND DISCUSSION

### 4.1. Dynamical Limit

The *dynamical limit* is associated with the existence of a critical current beyond which the problem does not admit steady wave solutions. Saffman and Yuen [1] have determined the value of the critical current in the case of two-dimensional interfacial waves at infinite depths. They calculated it in both its linear and weakly non-linear form, using the dispersion relation. In general, the critical current  $U_c$  is a function of the density ratio  $\mu$ , the angle  $\theta$ , and  $A_{11}$  which is the wave amplitude. We will extend this study to the case of three-dimensional interfacial waves. As  $A_{11} \rightarrow 0$ , equation (3.12) becomes linear. Its examination for fixed values of the density ratio  $\mu$ , the current  $U$  and the angle  $\theta$  shows that there are two solutions to the quadratic equation in  $\omega$ . These two distinct solutions are  $\omega_0^+$  and  $\omega_0^-$  with  $\omega_0^+ > \omega_0^-$ .

The linear critical current is defined when  $\omega_0^+ = \omega_0^-$ , it is given by the following expression:

$$U_{lc} = \frac{\sqrt{\mu(1-\mu^2)}}{\mu p}, \tag{4.1}$$

where the second subscript  $l$  stands for the linear case. In this case, we have:

$$\omega_0^+ = \omega_0^- = \frac{\mu p U_{lc}}{\mu + 1}. \tag{4.2}$$

In the non-linear case ( $A_{11} \neq 0$ ), there are two families of solutions denoted  $\omega_+(A_{11}, \mu, U, \theta)$  and  $\omega_-(A_{11}, \mu, U, \theta)$  that satisfy the non-linear dispersion equation Eq. (3.12). For fixed values of  $A_{11}$ ,  $\mu$ , and  $\theta$ , there is again a critical current  $U_c$  determined by the equality  $\omega_+(A_{11}, \mu, U_c, \theta) = \omega_-(A_{11}, \mu, U_c, \theta)$ , beyond which the problem does not admit steady solutions. The value of  $U_c$ , corrected to the second order, is determined by cancelling the discriminator in Eq. (3.12)

$$U_{c2} = U_{lc} \left[ 1 + A_{11}^2 \left( \frac{(1-\mu)^2 (9p - 8p^2 + 4p^4 + 4p^5 - 2 - 4p^3)}{8(1+\mu)^2 (2-p)} + \frac{1}{4} - \frac{1}{2} p^3 + p^2 \right) \right]^{1/2}, \tag{4.3}$$

with:

$$\omega_{+,2} = \omega_{-,2} = \frac{\mu p U_{c2}}{1 + \mu}, \tag{4.4}$$

where the subscript 2 denotes the second-order approximation.

For given values of  $\mu$  and  $\theta$ , relation (4.3) shows that the critical current  $U_{c2}$  increases with the wave amplitude  $A_{11}$ . This means that the wave amplitude has a stabilizing effect on the propagation of interfacial waves. This relation is shown in Fig. 2 for four different values of  $\theta$ .

#### 4.2. Frequency Variation

One of the important characteristic parameters of short-crested waves is the frequency. The perturbation method was used to calculate solutions up to the fifth-order where the frequency is given by the following series:

$$\omega = \omega_0 + h^2\omega_2 + h^4\omega_4 + \mathcal{O}(h^6). \quad (4.5)$$

For low values of the wave steepness  $h$  it is possible to limit the development to third-order. This procedure allows us to simplify the study of the variation of  $\omega$  as a function of  $h$  simply examining the sign of  $\omega$ . In fact, the expression for  $\omega$  in the third order is given by:

$$\omega = \omega_0 + h^2\omega_2 + \mathcal{O}(h^4). \quad (4.6)$$

This expression shows that the monotonicity of  $\omega(h)$  is given by the sign of  $\omega_2$ . The sign of  $\omega_2$  in the  $(\mu, \theta)$  plane for a current  $U = 1$  is shown in Fig. 3. The dynamical condition above which steady solutions do not exist is also represented. There are two regions for which  $\omega_2$  can be negative or positive. The two regions are separated by a line that corresponds to  $\omega_2 = 0$ . Along this line  $\omega = \omega_0$  and therefore  $\omega$  is independent of the wave steepness  $h$ . On the other hand, for  $\omega_2 < 0$ , the frequency  $\omega$  is a decreasing function of the wave steepness  $h$  whereas for  $\omega_2 > 0$ ,  $\omega$  increases with  $h$ . As an example, Fig. 4 illustrates the variation of  $\omega$  as a function of  $h$  for  $\mu = 0.1$  and  $U = 1$ . Change in monotonicity occurs between  $20^\circ < \theta < 30^\circ$ . This angle of frequency reversal can be determined from Fig. 3 and its value for  $\mu = 0.1$  is equal to  $\theta_c = 22.1^\circ$ .

#### 4.3. Kelvin–Helmholtz Instability

The existence of the critical value of  $U$  is associated with occurrence of instability as will be shown in the following. The used formulation follows that of Saffman and Yuen [1] to the three dimensional case. They have demonstrated that for a very small wave amplitude and sufficiently large values of the current, Kelvin-Helmholtz instability occurs. They also interpreted this result as the non-existence of steady linear waves. Note that the roots of Eq. (3.6) can be either real or complex conjugates. In the first case, the flow is stable. In the second case, the flow is unstable and the imaginary parts of the solutions of this equation in  $\omega$  correspond to the growth rate that increases as  $e^{\omega_i t}$ , where  $\omega_i = \text{Im}(\omega)$ . As an example, in Fig. 5 we have shown the variation of the roots of the linear dispersion relation with respect to the current  $U$  for a density ratio  $\mu = 0.1$  and an angle  $\theta = 40^\circ$ . The coalescence of the two real roots gives rise to the emergence of an instability that corresponds to complex conjugate roots. These complex roots occur when  $U \geq U_{lc}$ .

There is a region of values  $(p, U)$  for which  $\omega_0$  is imaginary. This region is determined by sign of the discriminator of the quadratic equation Eq. (3.6) which is given by the expression:

$$\Delta = 4(1 - \mu^2) - 4\mu p^2 U^2. \quad (4.7)$$

Figure 6 shows the stable and unstable areas that correspond to signs of the discriminator. If  $\Delta > 0$ , the mode is stable, otherwise if  $\Delta < 0$  the mode is considered to be unstable. Solid curve separating the two zones gives the values of  $U_{lc}$ . Here, there are two limiting cases of interest. The first one is when  $\theta = 0^\circ$  and it corresponds to standing interfacial waves. In this case, the linear critical current  $U_{lc} \rightarrow \infty$ . This physically means that standing interfacial waves are always stable in the presence of a current. The second limiting case corresponds to  $\theta = 90^\circ$  and we obtain the two-dimensional progressive waves. In this case, there is a limit current value where the state becomes unstable and therefore progressive waves of permanent form cannot be observed. This region is determined when:

$$U \geq U_{lc} \quad \text{with} \quad U_{lc} = \frac{\sqrt{\mu(1 - \mu^2)}}{\mu p}. \quad (4.8)$$

#### 4.4. Influence of Current on Interfacial Profile

Representative three dimensional profiles and their cross-sectional views in the planes  $Y = 0$  and  $X = 0$  for interfacial short crested wave in the presence of a current are illustrated in Fig. 7. These profiles

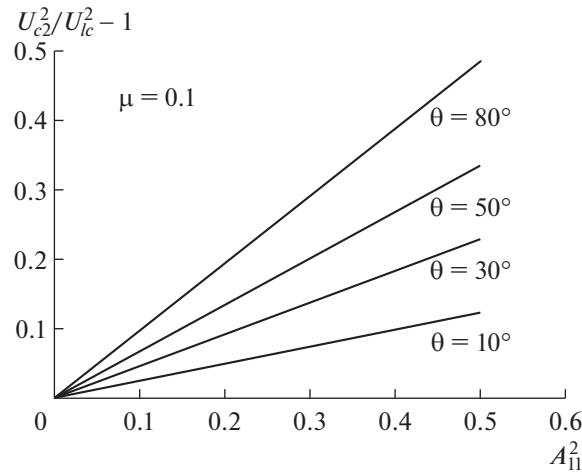


Fig. 2. Plot of  $U_c^2 / U_c^2 - 1$  versus  $A_1^2$  for a fixed value of  $\mu$  and various values of  $\theta$ .

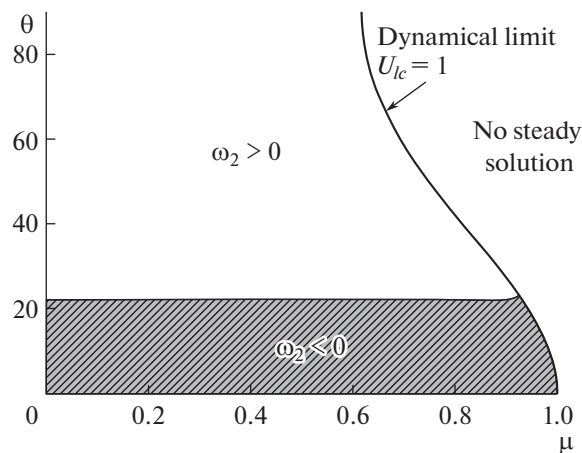


Fig. 3. Sign of  $\omega_2$  in the  $(\mu, \theta)$  plane for  $U_c = 1$ . The dynamical limit is also represented.

correspond to fully three dimensional case ( $\theta = 45^\circ$ ) with  $\mu = 0.2$  and  $h = 0.30$  and for various values of the current  $U$ . As shown in Fig. 7, the shape of the wave is strongly affected by  $U$ . In fact, for low values of current  $U = 1$  the profile is typical of a short-crested wave with a sharp crest and a rounded trough. As the current  $U$  increases, the crest becomes more flattened as it seen at  $U = 3.09$ . As the critical current reaches  $U \approx U_c$ , a secondary peak appears in both directions along the axes ( $Ox$ ) and ( $Oy$ ) being located at  $(X, Y) = (\pi, \pi)$ . The appearance of these secondary peaks is attributable the fact that waves of regular shape cannot exist in the vicinity of the critical current.

## SUMMARY

In this paper, we have presented analytical study of short-crested interfacial waves with current. Two different methods of solution have been proposed and the advantage of each one was highlighted. As a first step, we applied the perturbation method to determine solutions up to the fifth-order. The merits of this method are determination of the harmonic resonance condition which is one of the most important characteristics of short-crested waves. It is important to note that the perturbation method is no longer valid in the vicinity of singularities due to harmonic resonances. In addition, we determined the frequency of reversal where  $\omega_2$  changes its sign. In the second method, we applied the Lagrangian formulation to derive second-order solutions. This technique allows the determination of the critical current associated

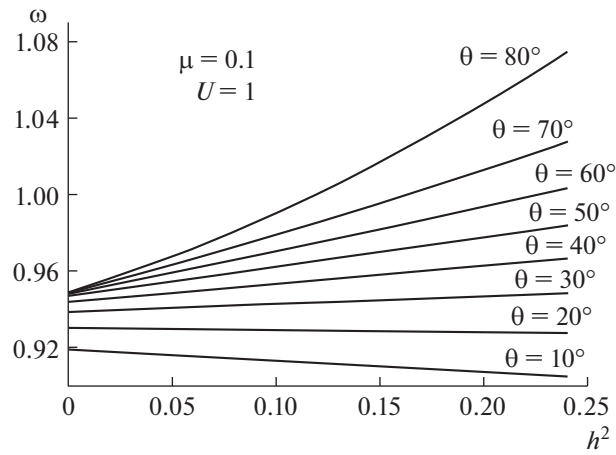


Fig. 4. Frequency  $\omega$  as a function of wave steepness squared  $h^2$  for  $U = 1$ ,  $\mu = 0.1$  and for various values of the angle  $\theta$ .

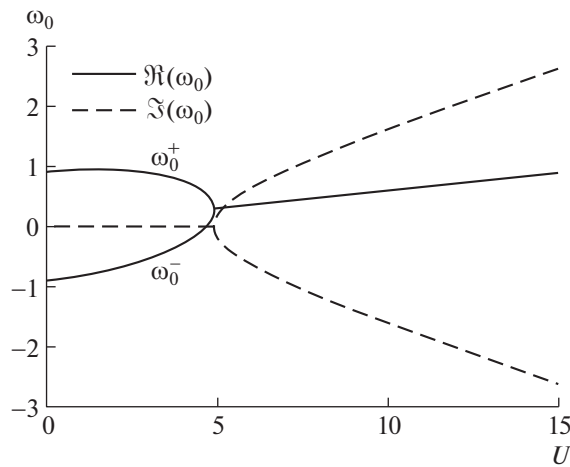


Fig. 5. Two roots of the linear dispersion equation for  $\mu = 0.1$  and  $\theta = 40^\circ$ . Solid curve gives the real parts and dashed curve gives the imaginary parts.

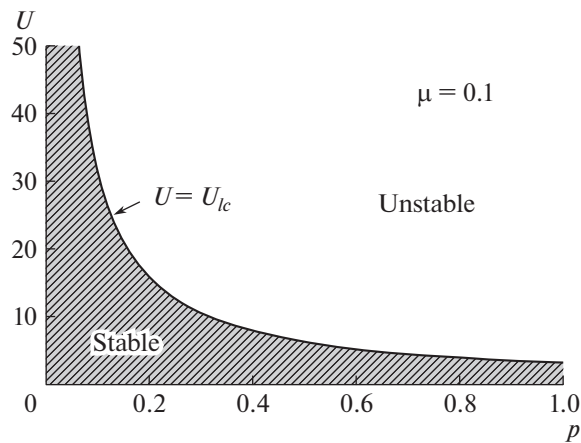


Fig. 6. Stable and unstable regions in the  $(U, p)$  plane. The boundary curve between stable and unstable regions corresponds to the linear critical current values.

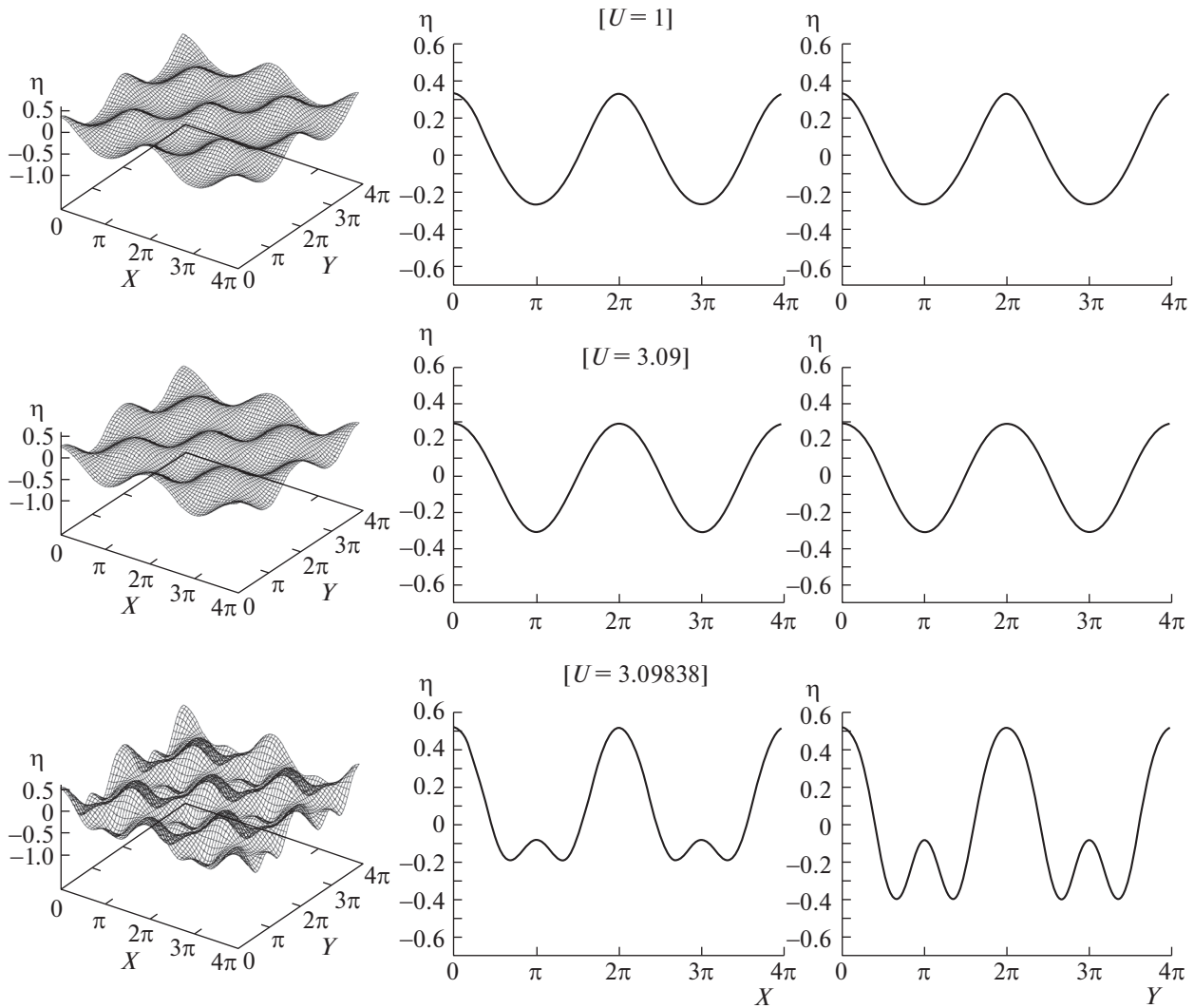


Fig. 7. 3D wave profile and their cross-sectional view  $Y = 0$  and  $X = 0$  on two wavelengths for a selected value of the current  $U$  and the fixed values of  $h = 0.30$ ,  $\mu = 0.2$ , and  $\theta = 45^\circ$ .

with the dynamical condition. In the linear case it was shown that the critical current is related to the emergence of instability. Moreover, the critical current depends on the three-dimensionality of the problem through the parameter  $\theta$ . In the non-linear case the critical current increases with the curvature which can be interpreted as a stabilizing effect.

## APPENDIX

Expressions for the third-order coefficients are given below:

$$\begin{aligned}
 a_{3,1}^{(3)} = & \frac{1}{D_{3,1}} [(-48\mu\omega_0^2 - 32\mu p U \omega_0 \alpha_{3,1} + 16\mu\omega_0^2 \alpha_{3,1} + 16\mu p^2 U^2 \alpha_{3,1} - 16\omega_0^2 \alpha_{3,1} - 48 p^2 U^2 \mu \\
 & - 96 p^4 U^2 \mu + 96 p U \mu \omega_0 + 96 p^2 \omega_0^2 + 192 p^3 U \mu \omega_0 - 96 \mu p^2 \omega_0^2 + 48 \omega_0^2) a_{2,0}^{(2)} + (-8 \omega_0^2 \alpha_{3,1} \\
 & - 96 \mu p^2 \omega_0^2 + 8 \mu p^2 U^2 \alpha_{3,1} + 24 \mu \omega_0^2 + 8 \mu \omega_0^2 \alpha_{3,1} - 16 \mu p U \omega_0 \alpha_{3,1} - 96 p^4 U^2 \mu + 24 p^2 U^2 \mu \\
 & + 96 p^2 \omega_0^2 + 192 p^3 U \mu \omega_0 - 48 p U \mu \omega_0 - 24 \omega_0^2) a_{2,2}^{(2)} + (-96 \mu p \omega_0 \alpha_{3,1} + 96 \mu p^2 U \alpha_{3,1} \\
 & - 32 \mu p^3 U \alpha_{3,1} + 32 \mu p^2 \omega_0 \alpha_{3,1} - 288 p^3 U \mu + 288 \mu p^2 \omega_0) b_{2,0}^{(2)} + (64 \mu p U \alpha_{3,1} - 64 \mu \omega_0 \alpha_{3,1}
 \end{aligned}$$

$$\begin{aligned}
& + 48\mu\omega_0 + 32\mu p^2\omega_0\alpha_{3,1} + 96\mu p^2\omega_0 - 96p^3U\mu - 48pU\mu - 32\mu p^3U\alpha_{3,1})b_{2,2}^{(2)} + (-96\omega_0 p\alpha_{3,1} \\
& + 32p^2\omega_0\alpha_{3,1} + 288p^2\omega_0)c_{2,0}^{(2)} + (96p^2\omega_0 + 32p^2\omega_0\alpha_{3,1} + 48\omega_0 - 64\omega_0\alpha_{3,1})c_{2,2}^{(2)} + 3\mu\omega_0^2 \\
& + 24p^2\omega_0^2 + 3\omega_0^2 - 11\omega_0^2\alpha_{3,1} - 11\mu\omega_0^2\alpha_{3,1} - 11\mu p^2U^2\alpha_{3,1} + 3p^2U^2\mu + 8p^2\omega_0^2\alpha_{3,1} \\
& + 8\mu p^4U^2\alpha_{3,1} - 6pU\mu\omega_0 + 8\mu p^2\omega_0^2\alpha_{3,1} - 48p^3U\mu\omega_0 + 22\mu pU\omega_0\alpha_{3,1} + 24p^4U^2\mu \\
& - 16\mu p^3U\omega_0\alpha_{3,1} + 24\mu p^2\omega_0^2], \\
a_{1,3}^{(3)} &= \frac{1}{D_{1,3}}[(16\mu\omega_0^2\alpha_{1,3} - 64p^3U\mu\omega_0 - 32p^2\omega_0^2 - 48\mu\omega_0^2 + 32p^4U^2 - 32pU\mu\omega_0\alpha_{1,3} \\
& + 16\mu p^2U^2\alpha_{1,3} + 96pU\mu\omega_0 - 48p^2U^2\mu - 16\omega_0^2\alpha_{1,3} + 48\omega_0^2 + 32\mu p^2\omega_0^2)a_{0,2}^{(2)} + (24p^2U^2\mu \\
& - 32\mu p^2\omega_0^2 - 32p^4U^2\mu + 24\mu\omega_0^2 + 8\mu\omega_0^2\alpha_{1,3} - 24\omega_0^2 + 8\mu p^2U^2\alpha_{1,3} + 32p^2\omega_0^2 \\
& - 16pU\mu\omega_0\alpha_{1,3} - 48pU\mu\omega_0 - 8\omega_0^2\alpha_{1,3} + 64p^3U\mu\omega_0)a_{2,2}^{(2)} + (-48pU\mu + 32p^3U\mu \\
& - 32\mu p^3U\alpha_{1,3} - 32\mu p^2\omega_0 + 48\mu\omega_0 + 32p^2\omega_0\mu\alpha_{1,3} + 32\mu pU\alpha_{1,3} - 32\mu\omega_0\alpha_{1,3})b_{2,2}^{(2)} \\
& + (-32\omega_0\alpha_{1,3} + 32p^2\omega_0\alpha_{1,3} - 32p^2\omega_0 + 48\omega_0)c_{2,2}^{(2)} - 16p^3U\mu\omega_0\alpha_{1,3} - 6pU\mu\omega_0 - 3\mu p^2U^2\alpha_{1,3} \\
& - 3\omega_0^2\alpha_{1,3} + 3\mu\omega_0^2 + 8p^2\omega_0^2\alpha_{1,3} + 8\mu p^2\omega_0^2\alpha_{1,3} + 3p^2U^2\mu + 8\mu p^4U^2\alpha_{1,3} + 3\omega_0^2 \\
& + 6pU\mu\omega_0\alpha_{1,3} - 3\mu\omega_0^2\alpha_{1,3}], \\
a_{3,3}^{(3)} &= -\frac{2}{D_{3,3}}[(-8\omega_0^2 + 8\mu\omega_0^2 + 8p^2U^2\mu - 16pU\mu\omega_0)a_{2,2}^{(2)} + (8pU\mu - 8\mu\omega_0)b_{2,2}^{(2)} - p^2U^2\mu \\
& - \omega_0^2 + 2pU\mu\omega_0 - \mu\omega_0^2 + 8\omega_0c_{2,2}^{(2)}], \\
a_{1,1}^{(3)} &= -(a_{1,3}^{(3)} + a_{3,1}^{(3)} + a_{3,3}^{(3)}), \\
b_{1,1}^{(3)} &= \frac{1}{D_{1,1}}[(48p^2\omega_0^2 + 16p^4U^2\mu - 32\omega_0^2 + 16\mu p^2\omega_0^2 - 32p^3U\omega_0 - 32p^3U\mu\omega_0 + 16\omega_0 pU)a_{0,2}^{(2)} \\
& + (48p^2\omega_0^2 + 16p^4U^2\mu - 32\omega_0^2 + 16\mu p^2\omega_0^2 - 32p^3U\omega_0 - 32p^3U\mu\omega_0 + 16\omega_0 pU)a_{2,0}^{(2)} \\
& + (-16pU\mu\omega_0 + 8p^2U^2\mu - 8\omega_0 pU + 8\mu\omega_0^2 + 8\omega_0^2)a_{2,2}^{(2)} + (-32p^2\omega_0 + 16\mu p^2U \\
& - 16\mu p\omega_0)b_{2,0}^{(2)} + (8pU\mu - 8\mu\omega_0 - 16\omega_0)b_{2,2}^{(2)} + (32p^2\omega_0 - 16\omega_0 p)c_{2,0}^{(2)} + 16\mu p^2U^2a_{1,1}^{(3)} \\
& - 32\omega_0 pUa_{1,1}^{(3)} + 4p^2\omega_0^2 + 2pU\mu\omega_0 + 16\mu\omega_0^2a_{1,1}^{(3)} - 8p^3U\mu\omega_0 + 8p^3U\omega_0 - \omega_0^2 + 4p^4U^2\mu \\
& + 4\mu p^2\omega_0^2 - 32\mu\omega_0 pUa_{1,1}^{(3)} - p^2U^2\mu + 16\omega_0^2a_{1,1}^{(3)} + \omega_0 pU + 16a_{1,1}^{(3)} - \mu\omega_0^2 - 16\mu a_{1,1}^{(3)} + 8\omega_0c_{2,2}^{(2)}], \\
c_{1,1}^{(3)} &= \frac{1}{D_{1,1}}[(48\mu p^2\omega_0^2 + 16p^4U^2\mu + 16p^2\omega_0^2 - 64p^3U\mu\omega_0 - 16p^2U^2\mu - 32\mu\omega_0^2 \\
& + 48pU\mu\omega_0)a_{0,2}^{(2)} + (48\mu p^2\omega_0^2 + 16p^4U^2\mu + 16p^2\omega_0^2 - 64p^3U\mu\omega_0 - 16p^2U^2\mu - 32\mu\omega_0^2 \\
& + 48pU\mu\omega_0)a_{2,0}^{(2)} + (8\omega_0^2 + 8\mu\omega_0^2 - 8pU\mu\omega_0)a_{2,2}^{(2)} + (16\mu + 16p^2U^2\mu - 16 - 16\omega_0^2 \\
& - 16\mu\omega_0^2)a_{1,1}^{(3)} + (-16\mu p^2U + 16\mu p\omega_0 + 32p^3U\mu - 32\mu p^2\omega_0)b_{2,0}^{(2)} + (8pU\mu - 8\mu\omega_0)b_{2,2}^{(2)} \\
& + (32\mu p^2\omega_0 - 32p^3U\mu + 16\omega_0 p)c_{2,0}^{(2)} + (8\omega_0 + 16\mu\omega_0 - 16pU\mu)c_{2,2}^{(2)} + \omega_0^2 + \mu\omega_0^2 - 4p^2\omega_0^2 \\
& - pU\mu\omega_0 - 4\mu p^2\omega_0^2 - 12p^4U^2\mu + 16p^3U\mu\omega_0],
\end{aligned}$$

$$\begin{aligned}
\omega_2 &= \frac{1}{D_{\omega_2}} [(8pU\mu\omega_0 + 4\mu p^2\omega_0^2 + 4p^4U^2\mu + 4\omega_0^2 - 4p^2\omega_0^2 - 4p^2U^2\mu - 4\mu\omega_0^2 \\
&- 8p^3U\mu\omega_0)a_{0,2}^{(2)} + (8pU\mu\omega_0 + 4\mu p^2\omega_0^2 + 4p^4U^2\mu + 4\omega_0^2 - 4p^2\omega_0^2 - 4p^2U^2\mu - 4\mu\omega_0^2 \\
&- 8p^3U\mu\omega_0)a_{2,0}^{(2)} + (-8pU\mu\omega_0 + 4p^2U^2\mu - 4 + 4\omega_0^2 + 4\mu + 4\mu\omega_0^2)a_{1,1}^{(3)} + (8p^3U\mu \\
&- 8\mu p^2\omega_0 - 4\mu p^2U + 4\mu p\omega_0)b_{2,0}^{(2)} + (-2\mu\omega_0 + 2pU\mu)b_{2,2}^{(2)} + (4\omega_0p - 8p^2\omega_0)c_{2,0}^{(2)} \\
&- 3\mu p^2\omega_0^2 - 3p^4U^2\mu - 3p^2\omega_0^2 + 6p^3U\mu\omega_0 - 2\omega_0c_{2,2}^{(2)}], \\
b_{1,3}^{(3)} &= \frac{1}{\alpha_{1,3}D_{1,3}} [(-48\mu\omega_0\alpha_{1,3} + 48pU\mu\omega_0^2\alpha_{1,3} - 64p^3U\omega_0^2 - 48p^2U^2\mu\omega_0\alpha_{1,3} + 32p^3U\alpha_{1,3} \\
&+ 48\omega_0\alpha_{1,3} - 32p^2\omega_0\alpha_{1,3} - 96\omega_0^3 - 48pU\alpha_{1,3} - 16pU\omega_0^2\alpha_{1,3} + 64p^2\omega_0^3 + 32p^2\omega_0\mu\alpha_{1,3} \\
&+ 96pU\omega_0^2 + 48\mu pU\alpha_{1,3} - 16\mu\omega_0^3\alpha_{1,3} + 16\omega_0^3\alpha_{1,3} - 32\mu p^3U\alpha_{1,3} + 16p^3U^3\mu\alpha_{1,3})a_{0,2}^{(2)} \\
&+ (8\omega_0^3\alpha_{1,3} - 32p^3U\alpha_{1,3} - 24\omega_0\alpha_{1,3} + 24pU\mu\omega_0^2\alpha_{1,3} + 24pU\alpha_{1,3} + 48\omega_0^3 \\
&- 32p^2\omega_0\mu\alpha_{1,3} - 48pU\omega_0^2 - 24p^2U^2\mu\omega_0\alpha_{1,3} - 8pU\omega_0^2\alpha_{1,3} + 64p^3U\omega_0^2 - 64p^2\omega_0^3 \\
&+ 24\mu\omega_0\alpha_{1,3} - 8\mu\omega_0^3\alpha_{1,3} + 32\mu p^3U\alpha_{1,3} - 24\mu pU\alpha_{1,3} + 32p^2\omega_0\alpha_{1,3} + 8p^3U^3\mu\alpha_{1,3})a_{2,2}^{(2)} \\
&+ (32\mu\omega_0^2\alpha_{1,3} + 48\omega_0^2 - 32p^2\mu\alpha_{1,3} - 32p^2\omega_0^2 - 64pU\mu\omega_0\alpha_{1,3} - 48\alpha_{1,3} + 64p^3U\mu\omega_0\alpha_{1,3} \\
&+ 32p^2\alpha_{1,3} + 48\mu\alpha_{1,3} - 32\mu p^2\omega_0^2\alpha_{1,3} - 32\mu p^4U^2\alpha_{1,3} + 32\mu p^2U^2\alpha_{1,3})b_{2,2}^{(2)} + (-48\omega_0^2 \\
&+ 32p^2\omega_0^2 - 32p^2\omega_0^2\alpha_{1,3} + 32\omega_0^2\alpha_{1,3} + 48\omega_0pU - 32p^3U\omega_0 - 32pU\omega_0\alpha_{1,3} \\
&+ 32p^3U\omega_0\alpha_{1,3})c_{2,2}^{(2)} + 3\mu\omega_0^3\alpha_{1,3} - 3\mu pU\alpha_{1,3} - 3\omega_0\alpha_{1,3} + 3\omega_0^3\alpha_{1,3} + 8p^3U\omega_0^2\alpha_{1,3} \\
&- 3p^3U^3\mu\alpha_{1,3} + 8p^5U^3\mu\alpha_{1,3} + 9p^2U^2\mu\omega_0\alpha_{1,3} + 24p^3U\mu\omega_0^2\alpha_{1,3} + 3\mu\omega_0\alpha_{1,3} \\
&- 3pU\omega_0^2\alpha_{1,3} - 8p^2\omega_0^3\alpha_{1,3} - 8\mu p^2\omega_0^3\alpha_{1,3} - 24p^4U^2\mu\omega_0\alpha_{1,3} + 3pU\alpha_{1,3} - 9pU\mu\omega_0^2\alpha_{1,3}], \\
c_{1,3}^{(3)} &= \frac{1}{\alpha_{1,3}D_{1,3}} [(64p^4U^2\mu\omega_0 - 96p^2U^2\mu\omega_0 - 32p^2\omega_0\alpha_{1,3} + 64\mu p^2\omega_0^3 - 16\omega_0^3\alpha_{1,3} \\
&+ 32p^2\omega_0\mu\alpha_{1,3} - 96\mu\omega_0^3 + 192pU\mu\omega_0^2 - 32pU\mu\omega_0^2\alpha_{1,3} - 128p^3U\mu\omega_0^2 + 48\omega_0\alpha_{1,3} \\
&- 48\mu\omega_0\alpha_{1,3} + 16p^2U^2\mu\omega_0\alpha_{1,3} + 16\mu\omega_0^3\alpha_{1,3})a_{2,0,2} + (-24\omega_0\alpha_{1,3} - 16pU\mu\omega_0^2\alpha_{1,3} \\
&+ 8\mu\omega_0^3\alpha_{1,3} + 24\mu\omega_0\alpha_{1,3} - 64\mu p^2\omega_0^3 + 48p^2U^2\mu\omega_0 + 128p^3U\mu\omega_0^2 - 32p^2\omega_0\mu\alpha_{1,3} \\
&+ 32p^2\omega_0\alpha_{1,3} - 96pU\mu\omega_0^2 + 48\mu\omega_0^3 - 8\omega_0^3\alpha_{1,3} + 8p^2U^2\mu\omega_0\alpha_{1,3} - 64p^4U^2\mu\omega_0)a_{2,2}^{(2)} \\
&+ (32pU\mu\omega_0\alpha_{1,3} - 32p^3U\mu\omega_0\alpha_{1,3} + 32p^3U\mu\omega_0 + 32\mu p^2\omega_0^2\alpha_{1,3} - 48pU\mu\omega_0 - 32\mu p^2\omega_0^2 \\
&- 32\mu\omega_0^2\alpha_{1,3} + 48\mu\omega_0^2)b_{2,2}^{(2)} + (-32\omega_0^2\alpha_{1,3} - 48\mu\omega_0^2 - 48p^2U^2\mu + 48\alpha_{1,3} - 48\mu\alpha_{1,3} \\
&+ 32p^4U^2\mu + 32p^2\omega_0^2\alpha_{1,3} + 32p^2\mu\alpha_{1,3} + 32\mu p^2\omega_0^2 - 64p^3U\mu\omega_0 + 96pU\mu\omega_0 - 32p^2\alpha_{1,3})c_{2,2}^{(2)} \\
&+ 3\omega_0\alpha_{1,3} - 3\omega_0^3\alpha_{1,3} - 3\mu\omega_0^3\alpha_{1,3} + 8\mu p^2\omega_0^3\alpha_{1,3} - 16p^3U\mu\omega_0^2\alpha_{1,3} + 6pU\mu\omega_0^2\alpha_{1,3} \\
&- 3p^2U^2\mu\omega_0\alpha_{1,3} + 8p^4U^2\mu\omega_0\alpha_{1,3} - 3\mu\omega_0\alpha_{1,3} + 8p^2\omega_0^3\alpha_{1,3}], \\
b_{3,3}^{(3)} &= \frac{1}{D_{3,3}} [(-8\mu\omega_0^3 + 40pU\omega_0^2 + 24pU\mu\omega_0^2 + 8p^3U^3\mu - 8\mu\omega_0 + 8pU\mu - 24p^2U^2\mu\omega_0 \\
&- 8pU + 8\omega_0 - 40\omega_0^3)a_{2,2}^{(2)} + (-64pU\mu\omega_0 - 16 + 32p^2U^2\mu + 32\mu\omega_0^2 + 16\mu + 48\omega_0^2)b_{2,2}^{(2)}
\end{aligned}$$

$$\begin{aligned}
& + (-16\omega_0^2 + 16\omega_0 pU)c_{2,2}^{(2)} - p^3 U^3 \mu - 3pU\mu\omega_0^2 + pU + 3p^2 U^2 \mu\omega_0 - pU\mu - \omega_0 + \mu\omega_0^3 - pU\omega_0^2 + \mu\omega_0 + \omega_0^3], \\
c_{3,3}^{(3)} = & -\frac{1}{D_{3,3}} [(8\mu\omega_0 + 40\mu\omega_0^3 + 8\omega_0^3 - 80pU\mu\omega_0^2 + 40p^2 U^2 \mu\omega_0 - 8\omega_0)a_{2,2}^{(2)} + (-16\mu\omega_0^2 \\
& + 16pU\mu\omega_0)b_{2,2}^{(2)} + (48p^2 U^2 \mu + 16\mu + 48\mu\omega_0^2 - 16 + 32\omega_0^2 - 96pU\mu\omega_0)c_{2,2}^{(2)} \\
& + \mu\omega_0 + p^2 U^2 \mu\omega_0 - \omega_0 - 2pU\mu\omega_0^2 + \omega_0^3 + \mu\omega_0^3], \\
b_{3,1}^{(3)} = & \frac{1}{\alpha_{3,1} D_{3,1}} [-8\mu p^3 U \alpha_{3,1} + 8\mu p^2 \omega_0 \alpha_{3,1} - \mu pU \alpha_{3,1} - \omega_0 \alpha_{3,1} + \mu\omega_0 \alpha_{3,1} - 8p^2 \omega_0 \alpha_{3,1} \\
& + (192\mu\omega_0^2 \alpha_{3,1} + 192\mu p^3 U \omega_0 \alpha_{3,1} - 96\mu p^4 U^2 \alpha_{3,1} + 16\mu \alpha_{3,1} + 192\mu p^2 U^2 \alpha_{3,1} + 288p^2 \omega_0^2 \\
& + 32\mu p^2 \alpha_{3,1} - 96\mu p^2 \omega_0^2 \alpha_{3,1} - 384\mu pU \omega_0 \alpha_{3,1} - 32p^2 \alpha_{3,1} + 144\omega_0^2 - 16\alpha_{3,1})b_{2,2}^{(2)} \\
& + (864p^3 U \omega_0 + 288\omega_0^2 p \alpha_{3,1} - 96p^2 \omega_0^2 \alpha_{3,1} - 864p^2 \omega_0^2 - 288p^2 U \omega_0 \alpha_{3,1} + 96p^3 U \omega_0 \alpha_{3,1})c_{2,0}^{(2)} \\
& + 33\omega_0^3 \alpha_{3,1} + (288p^3 U^2 \mu \alpha_{3,1} + 192\mu p^3 U \omega_0 \alpha_{3,1} + 864p^2 \omega_0^2 + 96\mu p^2 \alpha_{3,1} - 96p^2 \alpha_{3,1} \\
& - 576\omega_0 \mu p^2 U \alpha_{3,1} - 96\mu p^4 U^2 \alpha_{3,1} + 288\mu p \omega_0^2 \alpha_{3,1} - 96\mu p^2 \omega_0^2 \alpha_{3,1})b_{2,0}^{(2)} + (288p^3 U \omega_0 \\
& - 192pU \omega_0 \alpha_{3,1} - 96p^2 \omega_0^2 \alpha_{3,1} + 96p^3 U \omega_0 \alpha_{3,1} + 192\omega_0^2 \alpha_{3,1} - 288p^2 \omega_0^2 + 144\omega_0 pU \\
& - 144\omega_0^2)c_{2,2}^{(2)} + (-24pU \omega_0^2 \alpha_{3,1} - 32p^3 U \alpha_{3,1} + 32p^2 \omega_0 \alpha_{3,1} + 24p^3 U^3 \mu \alpha_{3,1} - 8\mu pU \alpha_{3,1} \\
& - 24\mu \omega_0^3 \alpha_{3,1} + 576p^3 U \omega_0^2 + 8\mu \omega_0 \alpha_{3,1} + 32\mu p^3 U \alpha_{3,1} - 576p^2 \omega_0^3 + 72\mu pU \omega_0^2 \alpha_{3,1} \\
& + 24\omega_0^3 \alpha_{3,1} + 8pU \alpha_{3,1} - 72\omega_0 \mu p^2 U^2 \alpha_{3,1} - 144pU \omega_0^2 - 32\mu p^2 \omega_0 \alpha_{3,1} - 8\omega_0 \alpha_{3,1} \\
& + 144\omega_0^3)a_{2,2}^{(2)} + (-32p^3 U \alpha_{3,1} - 48\mu \omega_0^3 \alpha_{3,1} - 16pU \alpha_{3,1} + 288pU \omega_0^2 + 16\omega_0 \alpha_{3,1} \\
& + 16\mu pU \alpha_{3,1} + 48p^3 U^3 \mu \alpha_{3,1} - 48pU \omega_0^2 \alpha_{3,1} + 48\omega_0^3 \alpha_{3,1} - 144\omega_0 \mu p^2 U^2 \alpha_{3,1} + 32p^2 \omega_0 \alpha_{3,1} \\
& + 144\mu pU \omega_0^2 \alpha_{3,1} + 32\mu p^3 U \alpha_{3,1} + 576p^3 U \omega_0^2 - 16\mu \omega_0 \alpha_{3,1} - 32\mu p^2 \omega_0 \alpha_{3,1} - 288\omega_0^3 \\
& - 576p^2 \omega_0^3)a_{2,0}^{(2)} + 99\omega_0 \mu p^2 U^2 \alpha_{3,1} - 99\mu pU \omega_0^2 \alpha_{3,1} - 33p^3 U^3 \mu \alpha_{3,1} + 72\mu p^3 U \omega_0^2 \alpha_{3,1} \\
& - 33pU \omega_0^2 \alpha_{3,1} - 24\mu p^2 \omega_0^3 \alpha_{3,1} + 24p^5 U^3 \mu \alpha_{3,1} + 24p^3 U \omega_0^2 \alpha_{3,1} - 72\omega_0 \mu p^4 U^2 \alpha_{3,1} \\
& - 24p^2 \omega_0^3 \alpha_{3,1} + 8p^3 U \alpha_{3,1} + 33\mu \omega_0^3 \alpha_{3,1} + pU \alpha_{3,1}], \\
c_{3,1}^{(3)} = & \frac{1}{\alpha_{3,1} D_{3,1}} [(-576p^4 U^2 \mu \omega_0 - 16\mu \omega_0 \alpha_{3,1} - 288\mu \omega_0^3 + 1152p^3 U \mu \omega_0^2 + 48\mu \omega_0^3 \alpha_{3,1} \\
& - 576\mu p^2 \omega_0^3 + 32p^2 \omega_0 \alpha_{3,1} + 576pU \mu \omega_0^2 + 48\omega_0 \mu p^2 U^2 \alpha_{3,1} - 288p^2 U^2 \mu \omega_0 - 96\mu pU \omega_0^2 \alpha_{3,1} \\
& - 32\mu p^2 \omega_0 \alpha_{3,1} - 48\omega_0^3 \alpha_{3,1} + 16\omega_0 \alpha_{3,1})a_{2,0}^{(2)} + (24\omega_0 \mu p^2 U^2 \alpha_{3,1} - 8\omega_0 \alpha_{3,1} - 576p^4 U^2 \mu \omega_0 \\
& + 24\mu \omega_0^3 \alpha_{3,1} + 32p^2 \omega_0 \alpha_{3,1} - 48\mu pU \omega_0^2 \alpha_{3,1} + 1152p^3 U \mu \omega_0^2 + 144\mu \omega_0^3 - 24\omega_0^3 \alpha_{3,1} \\
& - 32\mu p^2 \omega_0 \alpha_{3,1} + 144p^2 U^2 \mu \omega_0 - 288pU \mu \omega_0^2 + 8\mu \omega_0 \alpha_{3,1} - 576\mu p^2 \omega_0^3)a_{2,2}^{(2)} + (-288\mu p \omega_0^2 \alpha_{3,1} \\
& - 96\mu p^3 U \omega_0 \alpha_{3,1} + 864\mu p^2 \omega_0^2 - 864p^3 U \mu \omega_0 + 288\omega_0 \mu p^2 U \alpha_{3,1} + 96\mu p^2 \omega_0^2 \alpha_{3,1})b_{2,0}^{(2)} \\
& + (192\mu pU \omega_0 \alpha_{3,1} + 288\mu p^2 \omega_0^2 - 288p^3 U \mu \omega_0 - 192\mu \omega_0^2 \alpha_{3,1} - 144pU \mu \omega_0 + 96\mu p^2 \omega_0^2 \alpha_{3,1} \\
& - 96\mu p^3 U \omega_0 \alpha_{3,1} + 144\mu \omega_0^2)b_{2,2,2} + (-864p^4 U^2 \mu - 864\mu p^2 \omega_0^2 + 1728p^3 U \mu \omega_0 - 288\omega_0^2 p \alpha_{3,1} \\
& - 96\mu p^2 \alpha_{3,1} + 96p^2 \omega_0^2 \alpha_{3,1} + 96p^2 \alpha_{3,1})c_{2,0}^{(2)} + (32p^2 \alpha_{3,1} - 192\omega_0^2 \alpha_{3,1} - 16\mu \alpha_{3,1} - 32\mu p^2 \alpha_{3,1}
\end{aligned}$$



$$\begin{aligned}
& - 288p^4U^2\mu + 96p^2\omega_0^2\alpha_{3,1} + 288pU\mu\omega_0 + 16\alpha_{3,1} - 144\mu\omega_0^2 - 288\mu p^2\omega_0^2 - 144p^2U^2\mu \\
& + 576p^3U\mu\omega_0c_{2,2}^{(2)} - 48\mu p^3U\omega_0^2\alpha_{3,1} + 24\mu p^2\omega_0^3\alpha_{3,1} + 66\mu pU\omega_0^2\alpha_{3,1} - 8\mu p^2\omega_0\alpha_{3,1} + \omega_0\alpha_{3,1} \\
& - 33\mu\omega_0^3\alpha_{3,1} - 33\omega_0^3\alpha_{3,1} + 24\omega_0\mu p^4U^2\alpha_{3,1} - \mu\omega_0\alpha_{3,1} + 24p^2\omega_0^3\alpha_{3,1} + 8p^2\omega_0\alpha_{3,1} - 33\omega_0\mu p^2U^2\alpha_{3,1},
\end{aligned}$$

$$D_{3,1} = 32(9\omega_0^2 - \alpha_{3,1} + 9\mu\omega_0^2 - 18pU\mu\omega_0 + 9p^2U^2\mu + \mu\alpha_{3,1}),$$

$$D_{3,3} = 32(-1 + \mu + 3\omega_0^2 + 3p^2U^2\mu + 3\mu\omega_0^2 - 6pU\mu\omega_0),$$

$$D_{1,3} = 32(p^2U^2\mu - 2pU\mu\omega_0 + \mu\omega_0^2 - \alpha_{1,3} + \mu\alpha_{1,3} + \omega_0^2),$$

$$D_{1,1} = 32(pU\mu - \omega_0 - \mu\omega_0),$$

$$D_{\omega_2} = 8(pU\mu - \omega_0 - \mu\omega_0).$$

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