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On Vortex Generation by a Rotating Cylinder

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Abstract—The problem of evolution of an axisymmetric vortex generated by an infinitely elongated cylinder rotating around its axis in a compressible viscous fluid is considered. The asymptotic solution is constructed for large times. The conditions under which the velocity circulation at long distances is higher than in the incompressible fluid case are determined.

Keywords: vortex, rotating cylinder, compressibility, heat flux, asymptotic solution

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1. INTRODUCTION

Solutions of viscous gas equations, which describe rather simple flows, help one to understand the physical features of complex flows. A study of the problem of the point vortex diffusion in a viscous incompressible fluid can be found in almost every hydrodynamics textbook. At the same time, it is known that the field of the point vortex circumferential velocity

$$w = \frac{\Gamma_0}{r}, \tag{1.1}$$

where $2\pi\Gamma_0$ is the vortex circulation and r is the distance from the vortex, is the exact solution of the Euler equations as well as the Navier–Stokes equations. In the second case, a constant energy supply is required to maintain the velocity field (1.1) [1]. In addition, the fluid moving with velocity (1.1) has an infinite kinetic energy. The velocity field (1.1) is induced in a three-dimensional flow by an infinite rectilinear vortex filament with intensity $2\pi\Gamma_0$. Let us separate a fluid layer with the unit length along the vortex filament axis and calculate its kinetic energy

$$\int \frac{w^2}{2} dm = \int_0^\infty \frac{\Gamma_0^2}{2r^2} \rho 2\pi r dr = \pi\Gamma_0^2\rho\int_0^\infty \frac{dr}{r}. \tag{1.2}$$

Here, m is the fluid mass and ρ is its density.

Integral (1.2) has a logarithmic singularity at both zero and infinity. Consequently, the question arises as to how this velocity field could be created. It is impossible to create field (1.1) in the whole space. Usually, not one, but, for example, two vortices with opposite intensities are formed in physically realized flows. Then, the velocity field has more rapidly decaying asymptotic representation at long distances. Thus, the singularity is eliminated at $r \to \infty$. The singularity is eliminated at $r \to 0$ because the resulting vortex is not infinitely thin but distributed.

We can propose a way to create field (1.1) in a bounded region of the space. This can be arranged in incompressible viscous fluid using an infinite cylinder rotating around its axis [2, 3].

Previously, in [3, 4], nonstationary and limiting stationary flows generated by a rotating cylinder with a specified flowrate of fluid through its surface were studied. Problems concerning the incoming flow interaction with the rotating cylinder were studied in many publications, for example, in [4–6]. The stability of a moving cylinder in a circulating flow was studied in [7].

In this paper, the problem of an infinite cylinder rotation in compressible gas with a temperature-dependent viscosity is considered. It is shown that the problem solution for the whole space can be obtained only within the nonstationary formulation. The flow does not reach a stationary state anywhere, except a bounded region near the cylinder surface; this was not taken into account earlier in [8], where an attempt was made to obtain a stationary solution in the whole space.

2. FORMULATION OF THE PROBLEM

Let a circular cylinder with radius r_* and infinite length be placed in a viscous perfect gas at rest with temperature $T = T_0$, density $\rho = \rho_0$, and dynamic viscosity $\mu = \mu_0$ and thermal conductivity $\lambda = \lambda_0$. At the time $t = 0$, the cylinder starts to rotate around its axis with angular velocity w_*/r_*, which is maintained constant. The perturbed gas state is studied at $t > 0$ provided that the gas temperature on the cylinder surface is also kept constant $T = T_*$.

It is assumed that in cylindrical coordinate system (x, r, θ), the nonstationary flow is laminar and depends only on coordinate r. Thus, the possible flow instability is neglected. The equations and boundary conditions that determine the gas state can be written in the following form [9]:

$$\rho\left(\frac{\partial \Gamma}{\partial t} + v\Gamma'\right) = \mu\left(\Gamma'' - \frac{\Gamma'}{r}\right) + \mu'\left(\Gamma' - \frac{2\Gamma}{r}\right),$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + vT'\right) = \frac{\partial p}{\partial t} + vp' + \frac{c_p}{\mathrm{Pr}}\left[\mu\left(T'' + \frac{T'}{r}\right) + \mu' T'\right]$$

$$+ \mu\left[\frac{1}{r^2}\left(\Gamma' - \frac{2\Gamma}{r}\right)^2 + \frac{4}{3}\left(v'^2 - \frac{vv'}{r} + \frac{v^2}{r^2}\right)\right], \tag{2.1}$$

$$\rho\left(\frac{\partial v}{\partial t} + vv' - \frac{\Gamma^2}{r^3}\right) = -p' + \frac{4}{3}\mu\left(v'' + \frac{v'}{r} - \frac{v}{r^2}\right) + \frac{2}{3}\mu'\left(2v' - \frac{v}{r}\right),$$

$$\frac{\partial \rho}{\partial t} + \frac{(r\rho v)'}{r} = 0, \quad p = R\rho T,$$

$$\Gamma = 0, \quad T = T_0, \quad \rho = \rho_0, \quad v = 0 \quad (t = 0,\ r > r_*),$$

$$\Gamma = \Gamma_* = w_* r_*, \quad T = T_*, \quad v = 0 \quad (t > 0,\ r = r_*), \tag{2.2}$$

$$\Gamma \to 0, \quad T \to T_0, \quad \rho \to \rho_0, \quad v \to 0, \quad \mu \to \mu_0 \quad (t > 0,\ r \to \infty),$$

where $2\pi\Gamma$ is the circulation of the azimuthal component of velocity w; v is the velocity radial component; p is the gas pressure; $\mathrm{Pr} = \mu c_p/\lambda \sim O(1)$ is the Prandtl number; c_p is the specific heat capacity at constant pressure; and R is the universal gas constant, $()' \equiv \partial/\partial r$. The Prandtl number Pr and c_p are assumed constant.

According to Eqs. (2.1), the radial scale of the region where the viscous perturbations occur is proportional to $\sqrt{\nu_0 t}$, where $\nu_0 = \mu_0/\rho_0$. The nondimensional parameter $\sqrt{\nu_0 t}/r_*$, the ratio of characteristic independent spatial scales of the problem, can vary within a wide range at $t > 0$. In this paper, the problem of constructing an asymptotic solution of Eqs. (2.1), (2.2) is formulated for relatively large times, when $\sqrt{\nu_0 t}/r_* = \varepsilon^{-1} \gg 1$. For definiteness, the case of the viscosity linear dependence on gas temperature $\mu(T)/\mu_0 = T/T_0$ is considered.

In order to solve the problem, the entire flow region can be divided into three asymptotic subregions. The spatial scales r_* and $\sqrt{\nu_0 t}$ correspond to the internal and external regions of viscous perturbations (regions G_1 and G_2, respectively). It is shown below that the external limit of the solution in region G_2 corresponds to the zero azimuthal velocity and finite (but small at large Reynolds numbers) flowrate caused by the time-dependent radial velocity. When the speed of propagation of gas perturbations is limited, the flowrate should vanish at long distances from the cylinder, thus a third asymptotic region G_3 emerges, whose sizes are determined by acoustic perturbations. The spatial scale of region G_3 is $a_0 t$, where a_0 is the speed of sound in unperturbed gas, and is the largest of the three scales: its ratio to the size of region G_2 is

$a_0 t\big/\sqrt{\nu_0 t} = \mathrm{Re}/(\mathrm{M}_1\,\varepsilon) \gg 1$, where $\mathrm{Re} = w_* r_*/\nu_0 \gg 1$, $\mathrm{M}_1 = w_*/a_0$, $a_0 = \sqrt{(æ-1)\,c_p T_0}$, and æ is the adiabatic parameter.

The purpose of this article was to determine the flow characteristics in regions G_1 and G_2. In each of these regions, instead of independent variables r and t, we introduce new independent variables: $\eta = r/r_*$, ε in region G_1 and $\tau = r\big/\sqrt{\nu_0 t}$, ε in region G_2.

We can write the dependent variables in Eqs. (2.1), (2.2) in the nondimensional form

$$F = \frac{T^2}{T_0^2},\quad \gamma = \frac{\Gamma}{\Gamma_*},\quad \bar\rho = \frac{\rho}{\rho_0},\quad \bar p = \frac{p}{p_0},\quad \bar v = \frac{t}{r}v,\quad \bar\mu = \frac{\mu}{\mu_0}$$

Below we omit overlines above the nondimensional quantities. Equations (2.1), (2.2) can be rewritten in nondimensional form. In region G_1 $(1 \le \eta < \infty)$, we have

$$\gamma'' - \frac{\gamma'}{\eta} + \frac{F'}{2F}\left(\gamma' - \frac{2\gamma}{\eta}\right) = \varepsilon^2\frac{\rho}{\sqrt F}\left(-\frac{\varepsilon}{2}\frac{\partial\gamma}{\partial\varepsilon} + \eta v\gamma'\right),$$

$$F'' + \frac{F'}{\eta} + 2\frac{(æ-1)\,\mathrm{Pr}\,\mathrm{M}_1^2}{\eta^2}\sqrt F\left(\gamma' - \frac{2\gamma}{\eta}\right)^2 = \varepsilon^2\,\mathrm{Pr}\left[\frac{\rho}{\sqrt F}\left(-\frac{\varepsilon}{2}\frac{\partial F}{\partial\varepsilon} + \eta v F'\right)\right.$$

$$\left. - 2\frac{(æ-1)}{æ}\left(-\frac{\varepsilon}{2}\frac{\partial p}{\partial\varepsilon} + \eta v p'\right) - \frac{8}{3}\frac{æ(æ-1)}{\mathrm{Re}^2}\mathrm{M}_1^2\,\varepsilon^2\sqrt F\left(\eta v'(v + \eta v') + v^2\right)\right],$$

$$\frac{1}{\rho}p' - \frac{æ\,\mathrm{M}_1^2}{\eta^3}\gamma^2 = \frac{æ}{\mathrm{Re}^2}\mathrm{M}_1^2\,\varepsilon^2\left\{\frac{\sqrt F}{\rho}\left[\frac{4}{3}(\eta v'' + 3v') + \frac{F'}{3F}(2\eta v' + v)\right]\right. \tag{2.3}$$

$$\left. + \varepsilon^2\eta\left[\frac{\varepsilon}{2}\frac{\partial v}{\partial\varepsilon} + v - v(v + \eta v')\right]\right\},$$

$$(\rho\eta^2 v)' = \frac{\varepsilon}{2}\eta\frac{\partial\rho}{\partial\varepsilon},\qquad p = \rho\sqrt F,$$

$$\gamma = 1,\quad F = F_* = \frac{T_*^2}{T_0^2},\quad v = 0\quad (\eta = 1). \tag{2.4}$$

In region G_2 $(0 < \tau < \infty)$:

$$\gamma'' - \frac{\gamma'}{\tau} + \frac{F'}{2F}\left(\gamma' - \frac{2\gamma}{\tau}\right) + \frac{\rho}{\sqrt F}\left(\frac12 - v\right)\tau\gamma' = -\frac{\rho}{\sqrt F}\frac{\varepsilon}{2}\frac{\partial\gamma}{\partial\varepsilon},$$

$$F'' + \frac{F'}{\tau} + \frac{\rho\,\mathrm{Pr}}{\sqrt F}\left(\frac12 - v\right)\tau F' = -\mathrm{Pr}\left\{\frac{\rho}{\sqrt F}\frac{\varepsilon}{2}\frac{\partial F}{\partial\varepsilon} - 2\frac{(æ-1)}{æ}\left(\frac{\varepsilon}{2}\frac{\partial p}{\partial\varepsilon} + \left(\frac12 - v\right)\tau p'\right)\right.$$

$$\left. + 2\frac{(æ-1)\varepsilon^2\mathrm{M}_1^2}{\tau^2}\sqrt F\left(\gamma' - \frac{2\gamma}{\tau}\right)^2 + \frac83\frac{(æ-1)}{\mathrm{Re}^2}\mathrm{M}_1^2\,\varepsilon^2\sqrt F(\tau v'(v + \tau v') + v^2)\right\},$$

$$\frac{1}{\rho}p' = æ\,\mathrm{M}_1^2\,\varepsilon^2\left\{\frac{\gamma^2}{\tau^3} + \frac{1}{\mathrm{Re}^2}\left[\tau\left(\frac{\varepsilon}{2}\frac{\partial v}{\partial\varepsilon} + \tau v'\left(\frac12 - v\right) + v - v^2\right)\right.\right. \tag{2.5}$$

$$\left.\left. + \frac{\sqrt F}{\rho}\left(\frac43(\tau v'' + 3v') + \frac{F'}{3F}(2\tau v' + v)\right)\right]\right\},$$

$$(\rho\tau^2 v)' = \frac{\varepsilon}{2}\tau\frac{\partial\rho}{\partial\varepsilon} + \frac{\tau^2}{2}\rho',\qquad p = \rho\sqrt F,$$

$$\gamma \to 0,\quad F \to 1,\quad p \to 1,\quad v \to 0\quad (\tau \to \infty). \tag{2.6}$$

Here,

$$(\,)' = \begin{cases}\partial/\partial\eta & \text{in } G_1\\ \partial/\partial\tau & \text{in } G_2.\end{cases}$$

It follows from Eqs. (2.3)–(2.6) that at $\varepsilon \to 0$ and $M_1 \sim F_* \sim O(1)$, the required functions in regions G_1 and G_2 are of the order of $O(1)$ or less. Obviously, the boundary conditions (2.4), (2.6) are not sufficient to unambiguously define these functions in each of the regions G_1 and G_2. It is necessary to require the asymptotic solution in region G_1 at $\eta \to \infty$ matching with the asymptotic solution in region G_2 at $\tau \to 0$.

3. ASYMPTOTIC SOLUTION OF THE EQUATIONS IN REGION G_2

It follows from the third equation in (2.5) that, in the order of $O(\varepsilon^2)$, the static pressure in region G_2 is constant $p = 1$ at $\tau \sim O(1)$ and has the singularity $p \sim \varepsilon^2/\tau^2$ at $\tau \to 0$. We obtain from Eqs. (2.5) that, within the terms of the order of $O(\varepsilon^2)$, the gas density and the velocity radial component can be expressed through the function F:

$$\rho = \frac{1}{\sqrt{F}}, \quad v = \frac{c(\varepsilon)}{\tau^2} + \frac{F'}{2\tau \Pr}. \tag{3.1}$$

Let the dependence $c(\varepsilon)$ remain unknown for the time being. This quantity should be determined from the condition $v(\tau,\varepsilon) \le O(1)$ at $\tau \le O(1)$, following from the conditions of matching with the solution in region G_1.

In the main approximation, function $F(\tau,\varepsilon)$ satisfies the equation

$$F'' + \frac{1}{\tau}F'\left(1 + \frac{1}{F}\tau^2\left(\frac{1}{2} - v\right)\Pr\right) + \frac{\varepsilon \Pr}{2F}\frac{\partial F}{\partial \varepsilon} = 0. \tag{3.2}$$

The asymptotic behavior of the solution of Eq. (3.2) at $\tau \to 0$ can be written in the form:

$$F \sim \alpha_0 + \alpha \ln \tau. \tag{3.3}$$

From the conditions that function $F(\tau,\varepsilon)$ should match the solution in region G_1 at $\tau \to 0$, and its value should maintain the order of $O(1)$ at $\tau \sim O(\varepsilon)$, it follows

$$\alpha_0(M_1, F_*, \Pr) \sim O(1), \quad \alpha = \frac{\alpha_1(M_1, F_*, \Pr)}{\ln \varepsilon}, \quad \alpha_1 \sim O(1). \tag{3.4}$$

Taking relations (3.3), (3.4) into account we present functions $F(\tau,\varepsilon)$ and $\gamma(\tau,\varepsilon)$ as a power series in the small parameter $1/\ln \varepsilon$:

$$F(\tau,\varepsilon) = F_0(\tau) + \frac{\alpha_1}{\ln \varepsilon}F_1(\tau) + \frac{1}{\ln^2 \varepsilon}F_2(\tau) + O\left(\frac{1}{\ln^3 \varepsilon}\right),$$
$$\gamma(\tau,\varepsilon) = \gamma_0(\tau) + \frac{1}{\ln \varepsilon}\gamma_1(\tau) + O\left(\frac{1}{\ln^2 \varepsilon}\right). \tag{3.5}$$

Thus, in the main approximation, the solution for temperature and circulation depends only on the self-similar variable τ. The function F_0 meets the nonlinear equation

$$F_0'' + \frac{1}{\tau}F_0'\left(1 + \frac{1}{F_0}\tau^2\left(\frac{1}{2} - v\right)\Pr\right) = 0 \tag{3.6}$$

and the boundary conditions

$$F_0(0) = \alpha_0, \quad F_0'(0) = 0, \quad F_0(\infty) = 1, \tag{3.7}$$

The only solution for $F_0(\tau)$ that meets (3.6), (3.7), is

$$F_0(\tau) = \alpha_0 = 1. \tag{3.8}$$

According to expressions (3.1),

$$\rho(\tau,\varepsilon) = 1 - \frac{\alpha_1}{2\ln \varepsilon}F_1(\tau) + O(\ln^{-2}\varepsilon), \quad v(\tau,\varepsilon) = \frac{1}{\ln \varepsilon}v_1(\tau) + O(\ln^{-2}\varepsilon).$$

Functions F_1, γ_0 meet the linear differential equations

$$F_1'' + \frac{F_1'}{\tau}\left(1 + \frac{\Pr}{2}\tau^2\right) = 0, \quad \gamma_0'' - \frac{\gamma_0'}{\tau}\left(1 - \frac{\tau^2}{2}\right) = 0 \tag{3.9}$$

and the boundary conditions

$$F_1(\infty) = 0, \quad \gamma_0(\infty) = 0. \tag{3.10}$$

The solution of Eqs. (3.9) that meet conditions (3.3)–(3.5) and (3.10) is:

$$F_1 = \int\limits_\infty^x \frac{e^{-x^2/4}}{x} dx, \quad \gamma_0 = Ae^{-\tau^2/4}, \tag{3.11}$$

where $x = \tau\sqrt{\Pr}$. In the main approximation, the circulation distribution differs from the incompressible fluid case by an unknown constant $A(\mathrm{M}_1, F_*, \Pr)$ that is determined by the condition of asymptotic matching of solution (3.11) and the solution in region G_1.

From representations (3.1) and (3.5), solution (3.11), and condition $v \le O(1)$, at $\tau \sim O(\varepsilon)$ (region of overlapping with the solution in region G_1), it follows:

$$v_1(\tau) = -\frac{\alpha_1}{2\Pr\tau^2}(1 - \tau F_1').$$

For the next approximation of the functions determining the temperature and circulation, we have:

$$\begin{aligned} F_2'' + \frac{F_2'}{\tau}\left(1 + \frac{\Pr}{2}\tau^2\right) &= \frac{\alpha_1 \Pr}{2}[F_1 + \tau F_1'(\alpha_1 F_1 + 2v_1)], \\ \gamma_1'' - \frac{\gamma_1'}{\tau}\left(1 - \frac{\tau^2}{2}\right) &= -\frac{\alpha_1 F_1'}{2}\left(\gamma_0' - \frac{2\gamma_0}{\tau}\right) + \frac{\tau\gamma_0'}{2}(\alpha_1 F_1 + 2v_1). \end{aligned} \tag{3.12}$$

The boundary conditions for Eqs. (3.12) are:

$$F_2(\infty) = 0, \quad \gamma_1(\infty) = 0. \tag{3.13}$$

The solution of Eqs. (3.12) with boundary conditions (3.13) includes two more constants c_1 and c_2 yet unknown.

$$\begin{aligned} F_2 = \alpha_1 \Bigg[F_1 \ln x - 2\int\limits_\infty^x \frac{e^{-x^2/4}}{x}\ln x dx + \frac{\alpha_1}{4}\Bigg(4\int\limits_\infty^x \frac{e^{-x^2/2}}{x}dx - \frac{1}{2}\int\limits_\infty^x xe^{-x^2/4}\ln^2 x dx \\ - e^{-x^2/4}\ln^2 x - 2F_1 e^{-x^2/4} + F_1^2\Bigg)\Bigg] + c_2 F_1, \\ \gamma_1 = \frac{\alpha_1 A}{2}\Bigg\{\left(1 + \Pr + \frac{1}{\Pr} + \frac{\tau^2}{2}\right)e^{-\tau^2/4}\int\limits_\infty^\tau \frac{e^{-\tau^2\Pr/4}}{\tau}d\tau - \left(2 + \Pr + \frac{1}{\Pr}\right)\int\limits_\infty^\tau \frac{e^{-\tau^2(1+\Pr)/4}}{\tau}d\tau \\ + \frac{1}{\Pr}\left[\int\limits_\infty^\tau \frac{e^{-\tau^2/4}}{\tau}d\tau + e^{-\tau^2/4}\left(e^{-\tau^2\Pr/4} - \ln\tau\right)\right]\Bigg\} + c_1\gamma_0. \end{aligned} \tag{3.14}$$

In the first two approximations at $\tau \to \infty$, functions F, γ, ρ differ from their limiting values (2.6) by exponentially small values. The nondimensional radial component of velocity $v \to 0$ decays by the power law $v \sim \alpha_1/(\tau^2 \ln\varepsilon)$. This behavior of $v(\tau)$ corresponds to the fact that with time, at $\tau \to \infty$, the expansion of region G_2 induces a field of radial velocities equivalent to the source (or sink), where the ratio between the source intensity and the vortex circulation unlimitedly decreases with the increasing the Reynolds number that is time-dependent and small compared to the characteristic vortex circulation Γ_*

$$Q = \lim_{\tau\to\infty} 2\pi r \frac{r}{t} v = -\frac{\pi\alpha_1\nu_0}{\Pr\ln\varepsilon}. \tag{3.15}$$

In region G_3, there is the problem of perturbation propagation from the source that "is turned on" at time $t = 0$ and takes the form of (3.15) at large times. Since $\varepsilon(t)$ decreases with time, the source intensity decreases. It follows from asymptotic (3.15) and Eqs. (2.1) that radial velocity v and perturbations of temperature $\tilde{T} = T - 1$, density $\tilde{\rho} = \rho - 1$, and pressure $\tilde{p} = p - 1$ are of the order of $O(\mathrm{M}_1^2 \varepsilon^2 / \mathrm{Re}^2 \ln\varepsilon)$ and meet the linear Euler equations, which can be transformed into the following form:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv)}{\partial r}\right) - \frac{1}{a_0^2}\frac{\partial^2 v}{\partial t^2} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{p}}{\partial r}\right) - \frac{1}{a_0^2}\frac{\partial^2 \tilde{p}}{\partial t^2} = 0, \quad \tilde{\rho} = \frac{1}{æ}\tilde{p}, \quad \tilde{T} = \frac{æ - 1}{æ}\tilde{p}.$$

At $r \geq a_0 t$, the flow characteristics correspond to the unperturbed gas state. In contrast to the main approximation of the equations in regions G_1 (4.12) and G_2 (3.5), the equations in region G_3 are not reduced to ordinary differential equations, and the solution is not self-similar.

It follows from representations (3.5), solutions (3.8) and (3.11) that on the scale $\tau \sim O(\varepsilon)$, i.e., in region G_1, the first two approximations become the terms of the same order of smallness. In order to carry out the procedure of matching the solution in region G_2 and the solution in region G_1, we describe the behavior of functions F, γ at $\tau \to 0$ (the inner limit of the outer expansion) as follows:

$$\begin{aligned} F &= 1 + \frac{\alpha_1}{\ln\varepsilon}\left(\ln\tau + c_F + \frac{1}{2}\ln \mathrm{Pr}\right) \\ &+ \frac{\alpha_1}{\ln^2\varepsilon}\left\{\left[\frac{c_2}{\alpha_1} + c_F + \frac{\alpha_1}{2}(1 + c_F)\right]\ln\tau + O(1)\right\} + O(\ln^{-3}\varepsilon), \\ \gamma &= A\left\{1 + \frac{\alpha_1}{2\ln\varepsilon}\left[-\ln\tau + \frac{2c_1}{\alpha_1} + \frac{1 - \mathrm{Pr}}{\mathrm{Pr}}c_F + \frac{1}{2}\left(1 + \mathrm{Pr} + \frac{1}{\mathrm{Pr}}\right)\ln\frac{\mathrm{Pr}}{1 + \mathrm{Pr}} \right.\right. \\ &\left.\left. - \frac{1}{2}\ln(1 + \mathrm{Pr}) + \frac{1}{\mathrm{Pr}}\right]\right\} O(\ln^{-2}\varepsilon), \end{aligned} \tag{3.16}$$

where $c_F = -\ln 2 + c/2$, $c = 0.5772$ is the Euler constant.

The first relation from (3.16) rewritten in variables η ($\tau = \varepsilon\eta$) defines the boundary conditions for function F in region G_1 at $\eta \to \infty$ (the outer limit of the inner expansion). In the order of $O(1/\ln\varepsilon)$, we can write:

$$F = 1 + \alpha_1 + \frac{\alpha_1}{\ln\varepsilon}\left(\ln\eta + 2c_F + \frac{1}{2}\ln \mathrm{Pr} + \frac{\alpha_1}{2}(1 + c_F) + \frac{c_2}{\alpha_1}\right) + O(\ln^{-2}\varepsilon). \tag{3.17}$$

Due to the logarithmic nature of the singularity in the problem solution at $\tau \to 0$ (3.16), a situation arises when for matching the solutions in variables of region G_1 (3.17), accurate to $O(\ln^{-n}\varepsilon)$, it is necessary to construct the solution (3.5), (3.8), (3.11), (3.14) in region G_2 accurate to $O(\ln^{-(n+1)}\varepsilon)$.

4. THE ASYMPTOTIC SOLUTION OF THE EQUATIONS IN REGION G_1

We seek the solution of Eqs. (2.3) neglecting the terms of the order of $O(\varepsilon^2)$. Then, we can write the following expressions for functions F and γ:

$$\begin{aligned} &F'' + \frac{F'}{\eta} + 2\frac{(æ - 1)\,\mathrm{Pr}\,\mathrm{M}_1^2}{\eta^2}\sqrt{F}\left(\gamma' - \frac{2\gamma}{\eta}\right)^2 = 0, \\ &\gamma'' - \frac{\gamma'}{\eta} + \frac{F'}{2F}\left(\gamma' - \frac{2\gamma}{\eta}\right) = 0, \quad \gamma = 1, \quad F = F_* \quad (\eta = 1). \end{aligned} \tag{4.1}$$

It is easy to verify that circulation $\gamma(\eta)$ without singularity at $\eta \to \infty$ meeting the condition $\gamma(1) = 1$ can be expressed through function $F(\eta)$:

$$\gamma(\eta) = b\eta^2 \int_\eta^\infty \frac{d\eta}{\eta^3\sqrt{F}}, \quad b = \left(\int_1^\infty \frac{d\eta}{\eta^3\sqrt{F}}\right)^{-1}. \tag{4.2}$$

The substitution of representation (4.2) in the second equation from (4.1) leads to the identity. It follows from Eqs. (2.3), with account of Eqs. (4.1), that $p(\eta)$ and $\rho(\eta)$ are also expressed through the integrals of function $F(\eta)$. We focus on the solution of the first equation from (4.1), since other hydrodynamic functions can be found by the type of function F. Substituting representations (4.2) in the first equation from (4.1) and taking the boundary condition and the solution matching condition into account, we have

$$F'' + \frac{F'}{\eta} + 2\frac{(æ - 1)b^2 \,\mathrm{Pr}\, M_1^2}{\eta^4\sqrt{F}} = 0, \quad F(1) = F_*, \tag{4.3}$$

$$\lim_{\eta\to\infty} F(\eta) = 1 + \alpha_1 + \frac{\alpha_1}{\ln\varepsilon}\left(\ln\eta + 2c_F + \frac{1}{2}\ln \mathrm{Pr} + \frac{\alpha_1}{2}(1 + c_F) + \frac{c_2}{\alpha_1}\right) + O(\ln^{-2}\varepsilon). \tag{4.4}$$

We introduce a new dependent variable $y(\eta)$

$$F(\eta) = [2(æ - 1)b^2 \,\mathrm{Pr}\, \mathrm{M}_1^2]^{2/3} y(\eta) = F_* \frac{y(\eta)}{y(1)}. \tag{4.5}$$

Let us write the equations and boundary conditions, which are met by function $y(\eta)$ accurate to $O(1/\ln^2\varepsilon)$. From problem (4.3), taking substitution (4.5) into account, we come to the equation

$$y'' + \frac{y'}{\eta} + \frac{1}{\eta^4\sqrt{y}} = 0. \tag{4.6}$$

Using the definition of value b and substitution (4.5), we obtain

$$y(1) = \lambda\left(\int_1^\infty \frac{d\eta}{\eta^3\sqrt{y}}\right)^4, \tag{4.7}$$

where $\lambda = F_* \big/ [2(æ - 1)\,\mathrm{Pr}\,\mathrm{M}_1^2]^2 = 1\big/[2(æ - 1)\,\mathrm{Pr}\,\mathrm{M}_*^2]^2$, $\mathrm{M}_* = w_*/a_*$, a_* is the speed of sound in gas near the cylinder surface. From asymptotic (4.4), we obtain

$$\lim_{\eta\to\infty} y(\eta) = \frac{y(1)}{F_*}\left[1 + \alpha_1 + \frac{\alpha_1}{\ln\varepsilon}\left(\ln\eta + 2c_F + \frac{1}{2}\ln \mathrm{Pr} + \frac{\alpha_1}{2}(1 + c_F) + \frac{c_2}{\alpha_1}\right)\right]. \tag{4.8}$$

Equation (4.6) can be rewritten in the form

$$(y'\eta)' + \frac{1}{\eta^3\sqrt{y}} = 0. \tag{4.9}$$

Thus, it follows

$$\int_1^\infty \frac{d\eta}{\eta^3\sqrt{y}} = y'(1) - \lim_{\eta\to\infty} \eta y'(\eta).$$

Consequently, for problem (4.6)–(4.8), the following boundary condition can be used instead of relation (4.7):

$$y(1) = \lambda(y'(1) - \lim_{\eta\to\infty} \eta y'(\eta))^4. \tag{4.10}$$

The asymptotic behavior of the solution of Eq. (4.6) at $\eta \to \infty$ can be described as follows:

$$y(\eta) \sim q_0 + q_1 \ln\eta. \tag{4.11}$$

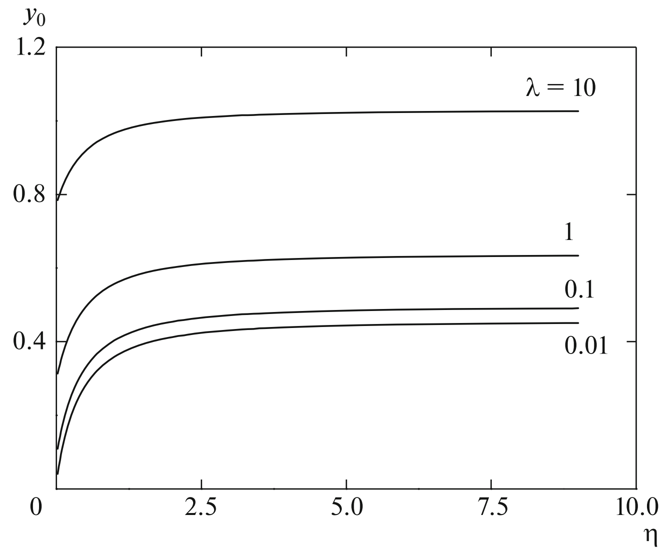

Fig. 1.

Let us look for the solution of Eqs. (4.6), (4.8), (4.10) in the form of the asymptotic series

$$y(\eta) = y_0(\eta) + \frac{1}{\ln \varepsilon} y_1(\eta) + O\left(\frac{1}{\ln^2 \varepsilon}\right). \tag{4.12}$$

From solutions (4.8), (4.11), (4.12), it follows that $y_0(\eta)$ contains no logarithmic term at $\eta \to \infty$. The equation and the boundary conditions for function $y_0(\eta)$ can be written as

$$y_0'' + \frac{y_0'}{\eta} + \frac{1}{\eta^4 \sqrt{y_0}} = 0, \tag{4.13}$$

$$y_0(1) = \lambda (y_0'(1))^4, \tag{4.14}$$

$$\lim_{\eta \to \infty} \eta y_0' = 0. \tag{4.15}$$

For the given parameters of the flow near the cylinder, parameter λ is fixed. Then, if we set an arbitrary value of $y_0'(1)$ in addition to boundary conditions (4.14), (4.15), system (4.13)–(4.15) will be overdetermined. Varying $y_0'(1)$ and solving the Cauchy problem, we can find its value, such that system (4.13)–(4.15) is solvable. The examples of solutions are given in Fig. 1 (from bottom to top): at $\lambda = 0.01$ and $y_0'(1) = 1.1731$; $\lambda = 0.1$ and $y_0'(1) = 0.9754$; $\lambda = 1$ and $y_0'(1) = 0.7396$; and $\lambda = 10$ and $y_0'(1) = 0.5275$.

It follows from (4.4), (4.5) that parameter α_1 is determined by the main approximation in region G_1

$$1 + \alpha_1(F_*, \lambda) = F_* \frac{y_0(\infty)}{y_0(1)}. \tag{4.16}$$

When $F_* \sim O(1)$ and $\lambda \sim O(1)$, parameter α_1 is also of the order of $O(1)$, and can take both positive and negative values. The interaction between the temperature and vortex fields in region G_1 leads to the conversion of the gas rotational motion energy into thermal energy. The relative temperature of the gas $T(\eta)/T_*$ increases near the cylinder surface. The dependence $F_* = \tilde{F}_*(\lambda)$ shown in Fig. 2 corresponds to $\alpha_1(F_*, \lambda) = 0$. If $F_* > \tilde{F}_*(\lambda)$, the heat flux passes from region G_1 to region G_2; otherwise, the heat flux changes its direction.

According to boundary condition (4.8), function $y_1(\eta)$ should have a logarithmic singularity at $\eta \to \infty$. Due to this singularity, both terms in asymptotic (4.12) become of the same order when passing from region G_1 to region G_2. Solution matching becomes possible if both terms of the asymptotic expansion are considered in representation (4.12).

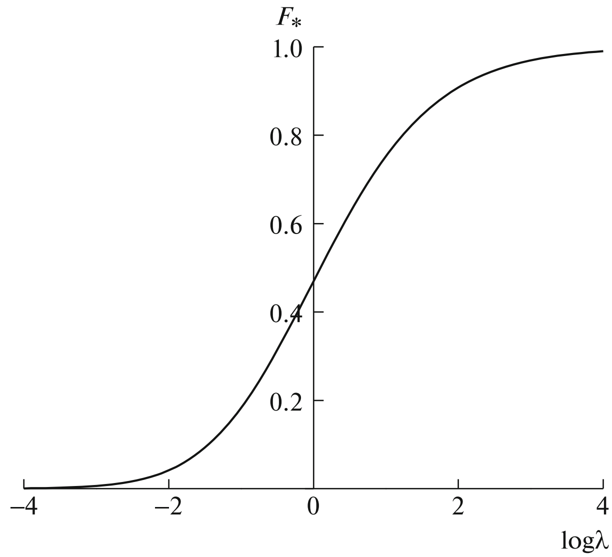

Fig. 2.

The equation and boundary conditions for function $y_1(\eta)$ can be written as follows:

$$y_1'' + \frac{y_1'}{\eta} - \frac{y_1}{2\eta^4 y_0^{3/2}} = 0, \tag{4.17}$$

$$y_1(1) = 4\frac{y_0(1)}{y_0'(1)}\left(y_1'(1) - \frac{\alpha_1 y_0(1)}{F_*}\right), \tag{4.18}$$

$$\lim_{\eta\to\infty} \eta y_1' = \frac{\alpha_1 y_0(1)}{F_*}. \tag{4.19}$$

For an arbitrary value of $y_1'(1)$, system (4.17)–(4.19) is also overdetermined. By varying $y_1'(1)$, we can find its value, at which condition (4.19) is met.

Within the terms of the order of $O(\ln^{-2}\varepsilon)$, the solution for $F(\eta,\lambda)$ in region G_1 can be considered known. At $\eta \to \infty$, this solution can be written in the following form

$$F(\eta) = 1 + \alpha_1(F_*,\lambda) + \frac{1}{\ln\varepsilon}(\alpha_1(F_*,\lambda)\ln\eta + h(F_*,\lambda)), \tag{4.20}$$

where

$$h = (1+\alpha_1)\left[\frac{1}{y_0(\infty)}\lim_{\eta\to\infty}\left(y_1(\eta) - \frac{\alpha_1 y_0(1)}{F_*}\ln\eta\right) - \frac{y_1(1)}{y_0(1)}\right]. \tag{4.21}$$

Using solutions (4.4), (4.20), (4.21) it is possible to find an unknown coefficient c_2 in the asymptotic representation for function F in region G_2.

Using the known functions $y_0(\eta)$ and $y_1(\eta)$, one can express from representations (4.2) and equation (4.9), accurate to $O(\ln^{-2}\varepsilon)$, the solution for the circulation

$$\gamma(\eta) = \frac{\eta^2\left(\eta y'(\eta) - \dfrac{1}{\ln\varepsilon}\dfrac{\alpha_1 y_0(1)}{F_*}\right)}{y'(1) - \dfrac{1}{\ln\varepsilon}\dfrac{\alpha_1 y_0(1)}{F_*}}. \tag{4.22}$$

Integrating expression (4.2) by parts twice and considering (4.20), we obtain at $\eta \to \infty$

$$\gamma(\eta) = \frac{b}{2}\left\{\frac{1}{\sqrt{F}} - \frac{1}{\ln\varepsilon}\frac{\alpha_1}{4F^{3/2}} + O\left(\frac{1}{\ln^2\varepsilon}\right)\right\}. \tag{4.23}$$

Thus, the outer limit of the inner expansion is represented by the expression

$$\gamma(\eta) = \frac{(æ-1)\,\mathrm{Pr}\,\mathrm{M}_1^2}{(y_0'(1))^3}\lim_{\eta\to\infty}\left[k_\eta^{-1/2} - \frac{1}{\ln\varepsilon}\left(\frac{\alpha_1}{4}k_\eta^{-3/2} + \frac{3}{4}\frac{y_1(1)}{y_0(1)}k_\eta^{-1/2}\right)\right], \tag{4.24}$$

where $k_\eta = 1 + \alpha_1 + (\alpha_1 \ln\eta + h)/\ln\varepsilon$.

Relation (4.24) rewritten in variables τ determines the inner limit of the outer expansion for the function γ in region G_2

$$\gamma(\eta) = \frac{(æ-1)\,\mathrm{Pr}\,\mathrm{M}_1^2}{(y_0'(1))^3}\lim_{\tau\to 0}\left[k_\tau^{-1/2} - \frac{1}{\ln\varepsilon}\left(\frac{\alpha_1}{4}k_\tau^{-3/2} + \frac{3}{4}\frac{y_1(1)}{y_0(1)}k_\tau^{-1/2}\right)\right], \tag{4.25}$$

where $k_\tau = 1 + (\alpha_1 \ln\tau + h)/\ln\varepsilon$.

Expanding (4.25) in series in powers of $1/\ln\varepsilon$, we obtain

$$\gamma(\eta) = \frac{(æ-1)\,\mathrm{Pr}\,\mathrm{M}_1^2}{(y_0'(1))^3}\left[1 - \frac{1}{\ln\varepsilon}\left(\frac{\alpha_1}{2}\ln\tau + \frac{2h+\alpha_1}{4} + \frac{3}{4}\frac{y_1(1)}{y_0(1)}\right) + O(\ln^{-2}\varepsilon)\right]. \tag{4.26}$$

The comparison of asymptotic (4.26) with (3.16) allows determining unknown constants A and c_1 that are included in the representation in region G_2. For constant A, we obtain the relation

$$A = \frac{(æ-1)\,\mathrm{Pr}\,\mathrm{M}_1^2}{(y_0'(1))^3}. \tag{4.27}$$

Expression (4.27) can be rewritten in the form

$$A = \left(2\sqrt{\frac{y_0(1)}{F_*}}\int_1^\infty \frac{d\eta}{\eta^3\sqrt{y_0(\eta)}}\right)^{-1}. \tag{4.28}$$

It follows from the equations and boundary conditions (4.13)–(4.15) that $y_0'(\eta) > 0$. Replacing $y_0(\eta)$ in the integrand of (4.28) with a smaller value $y_0(1)$, we come to the inequality

$$A > \left(\frac{2}{\sqrt{F_*}}\int_1^\infty \frac{d\eta}{\eta^3}\right)^{-1} = \sqrt{F_*}. \tag{4.29}$$

In the case of $F_* \geq 1$, the gas temperature in the whole space is higher than the temperature of the unperturbed gas ($F \geq 1$), since there is nothing against the gas heating due to the conversion of rotational motion energy into heat. In this case, it follows from (4.29) that $A > 1$. It can be shown that the same result can be obtained for an arbitrary monotonically increasing dependence $\mu(T)$. This allows us to formulate the *theorem on the circulation jump*: if $F_* \geq 1$ and $d\mu/dT > 0$, the circulation in compressible gas in region $r \sim \sqrt{\nu_0 t}$ exceeds the circulation in incompressible fluid at the same r and t (comparison is presented in Fig. 5, see below). The name of the theorem is associated with the fact that the inner limit of the outer solution ($\tau \to 0$) for circulation $\gamma_0(\tau)$ does not coincide with the value of circulation on the cylinder surface.

Using intermediate asymptotics of temperature (3.17) and circulation (4.23), one can calculate the kinetic energy of the flow in the main approximation

$$\begin{aligned} E &= \pi\int_0^\infty \frac{\rho\Gamma^2}{r}dr \approx \pi\rho_0\Gamma_*^2 A^2\int_1^{\varepsilon^{-1}}\left(1 + \alpha_1 + \alpha_1\frac{\ln\eta}{\ln\varepsilon}\right)^{-3/2} d\ln\eta \\ &= \frac{2A^2}{\alpha_1}\left(1 - \frac{1}{\sqrt{1+\alpha_1}}\right)E_0, \end{aligned} \tag{4.30}$$

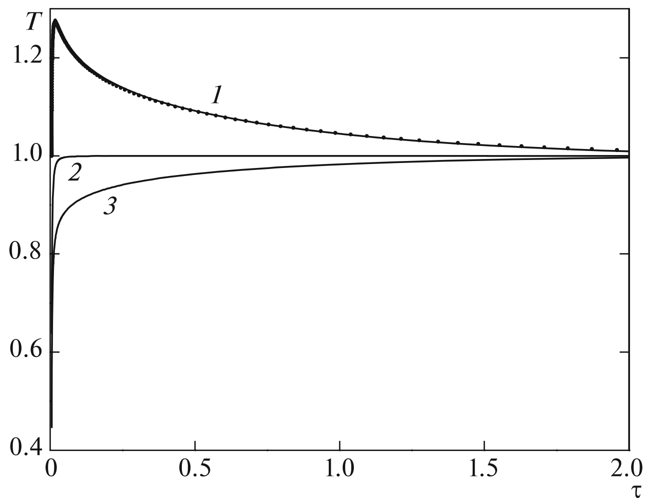

Fig. 3.

where $E_0 = \pi\rho_0\Gamma_*^2 |\ln\varepsilon|$ is the kinetic energy for the incompressible fluid case at the same time. Almost all the energy is the energy of the flow in the intermediate region $r_* \ll r \ll \sqrt{\nu_0 t}$, and over time, it increases unlimitedly with the increase in the size of this region. In the case of $M_1 = F_* = \Pr = 1$, the numerical solution gives the constant values $A \approx 1.19$ and $\alpha_1 = 0.88$. The kinetic energy of the compressible gas flow is less than that of the incompressible fluid: in the limit of $\varepsilon \to 0$, we find from (4.30) that $E/E_0 \approx 0.87$.

5. THE COMPOSITE SOLUTION

Within the terms $O(\ln^{-2}\varepsilon)$, the uniformly valid solution for functions F and γ in the region $G_1 \cap G_2$ can be presented in the form [10]

$$\hat{F}(\tau) = \frac{F^{(1)}(\tau)F^{(2)}(\tau)}{F^{(2)}_{\text{lim}}(\tau)}, \quad \hat{\gamma}(\tau) = \frac{\gamma^{(1)}(\tau)\gamma^{(2)}(\tau)}{\gamma^{(2)}_{\text{lim}}(\tau)},$$

where $F^{(1)}(\tau)$ and $\gamma^{(1)}(\tau)$ are solutions (4.5), (4.12), (4.22) in region G_1 rewritten in variable τ; $F^{(2)}(\tau)$, and $\gamma^{(2)}(\tau)$ are solutions (3.5) in region G_2; and $F^{(2)}_{\text{lim}}(\tau)$ and $\gamma^{(2)}_{\text{lim}}(\tau)$ are the valid limits of outer expansions (3.16).

In Figs. 3 and 4, the behavior of dependences $\hat{T}(\tau) = \sqrt{\hat{F}(\tau)}$ and $\hat{\gamma}(\tau)$ corresponding to the case of $M_1 = \Pr = 1$, $1/\ln\varepsilon = -0.189$, and three different values of F_* are presented: curves *1* correspond to the values of parameters $F_* = 1$ and $\alpha_1 > 0$; curves *2* correspond to $F_* = 0.407$ and $\alpha_1 = 0$; curves *3* correspond to $F_* = 0.2$ and $\alpha_1 < 0$; dashed line shows the numerical solution of the Navier–Stokes equations at $F_* = 1$ (see Section 8). In the case where $\alpha_1 > 0$, the temperature reaches its maximum in region G_1; at $\alpha_1 \le 0$ the temperature increases monotonically with the increase in distance from the cylinder surface. The rotational energy transition into heat causes the gas temperature increase in region G_1, while in region G_2, the gas temperature varies mainly due to the heat diffusion. The increase in the gas temperature causes the increase in dynamic viscosity ($\mu \sim T$) and produces the viscosity gradient $(\partial\mu/\partial r \sim \partial T/\partial r)$, which in the case of $\alpha_1 > 0$ leads to a nonmonotonic behaviour of the rotational motion circulation along the coordinate r. As the distance from the cylinder surface grows, the vorticity

$$\omega = \frac{1}{\eta}\frac{\partial\gamma}{\partial\eta}$$

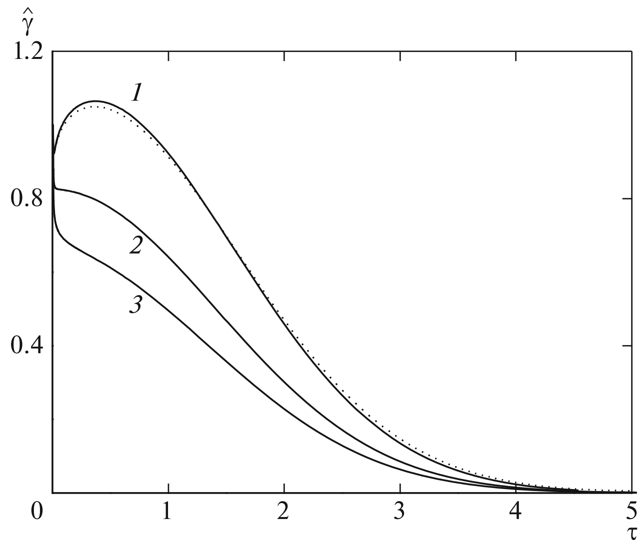

Fig. 4.

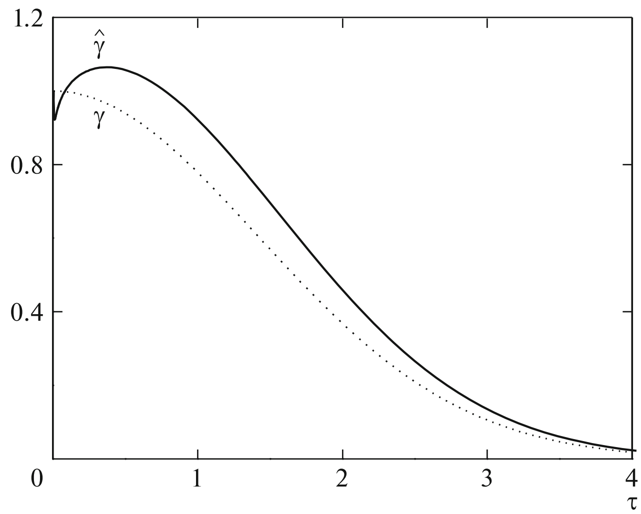

Fig. 5.

changes its sign twice. There is a region where a positive vorticity is generated. In the case corresponding to Fig. 3 at $F_* = 1$, it includes $2.3\varepsilon < \tau < 0.36$. From the second equation of system (4.1), the following relation can be obtained

$$\frac{\partial \omega}{\partial \eta} = \frac{b}{2\eta^2 F^{3/2}} \frac{\partial F}{\partial \eta},$$

from which it follows that the vorticity extremum in region G_1 is reached at the same point ($\tau \approx 3.4\varepsilon$, Fig. 3) as the temperature extremum. The rotational velocity circulation over the interval $0.084–0.099 <$ $\tau < 0.72–0.77$ exceeds the circulation near the cylinder surface. This effect is not observed in incompressible fluid, in which the circulation varies monotonically along the coordinate r, while within region G_1, the circulation is constant. In Fig. 5, function $\hat{\gamma}(\tau)$ for compressible gas at $M_1 = F_* = \Pr = 1$, $1/\ln \varepsilon = -0.189$ (solid curve) is compared with function $\gamma(\tau) = e^{-\tau^2/4}$ for an incompressible fluid (dashed curve).

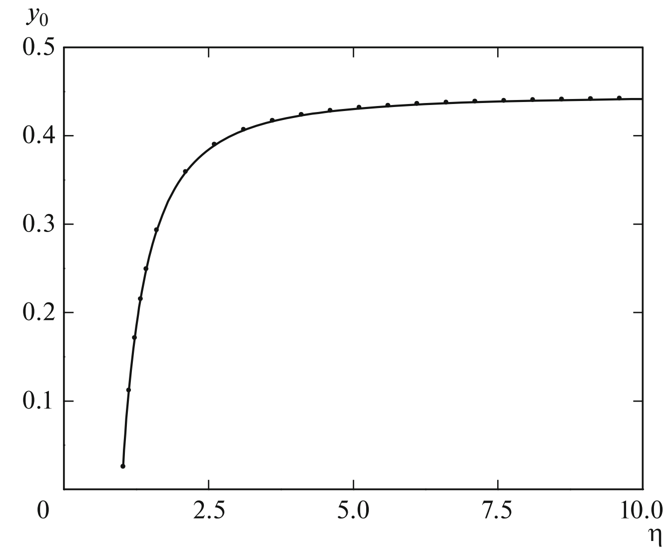

Fig. 6.

6. THE PROBLEM SOLUTION FOR SMALL NUMBERS OF M_1

When $\lambda \to \infty$, which corresponds to small numbers of M_1, the solution for F can be obtained in analytical form. It follows from relation (4.3) and the boundary condition on the cylinder surface that accurate $O(\lambda^{-1})$, the following relation is true in region G_1 :

$$F = F_*\left[1 + \frac{1}{\sqrt{\lambda}}\left(1 - \frac{1}{\eta^2}\right) + d\ln\eta\right]. \tag{6.1}$$

In region G_2, the relation

$$F = 1 + \frac{\alpha_1}{\ln\varepsilon}\ln\tau \tag{6.2}$$

is still true.

The asymptotic matching of solutions (6.1) and (6.2) determines the unknown constants α_1 and d

$$\alpha_1 = \left(1 + \frac{1}{\sqrt{\lambda}}\right)F_* - 1, \quad d = \frac{\alpha_1}{F_*\ln\varepsilon}.$$

The distribution of the circulation is determined in the same way. It follows from (4.27) that in the main approximation

$$A = \sqrt{F_*}.$$

7. THE PROBLEM SOLUTION FOR LARGE NUMBERS OF M_1

In the previous paragraphs, the asymptotic solution of the problem at $M_1 \sim O(1)$ is obtained under the condition $\varepsilon \to 0$. In this case, it is also true that $1/\ln\varepsilon \to 0$. At the same time, at feasible large times (small ε), value $1/\ln\varepsilon$ can no more be negligibly small, for example, at $\varepsilon = 10^{-4}$: $1/\ln\varepsilon \approx -0.1$.

The analysis of the results at $M_1 \sim O(1)$ shows that when M_1 increases, the coefficient α_1 also grows. Therefore, when M_1 becomes large, where coefficient $\alpha_1/\ln\varepsilon$ can no more be a small value. Let us define the conditions under which $\alpha_1/\ln\varepsilon$ is of the order of $O(1)$. For this purpose, let us consider the solution at $\lambda \ll 1$.

Functions $y_0(1)$ and $y_0'(1)$ are bound by relation (4.14) and thus have different orders of smallness in parameter λ. We can solve problem (4.13)–(4.15) numerically at small λ to determine the orders of these functions. In Fig. 6, the solutions of these equations at $\lambda = 10^{-6}$ (solid curve) and $\lambda = 10^{-3}$ (dots) are presented. It is shown that in these cases, dependences $y_0(\eta,\lambda)$ are close to each other. In the limit of $\lambda \to 0$, we denote the value of $y_0(\eta)$ as $y_{00}(\eta)$. The values that are obtained from the numerical solution are of the order of $O(1)$: $y'_{00}(1) = d_1 \approx 1.3641$ and $y_{00}(\infty) = d_2 = 0.4450$. Substituting these values in formula (4.16) we obtain

$$\frac{\alpha_1}{\ln\varepsilon} \sim \frac{F_*}{\ln\varepsilon}\frac{y_{00}(\infty)}{y_{00}(1)} = \frac{1}{\lambda\ln\varepsilon}\frac{F_* d_2}{d_1^4} = \frac{1}{\ln\varepsilon}\frac{d_2}{d_1^4}\left[2(æ-1)\Pr \mathrm{M}_1^2\right]^2 \sim \frac{\mathrm{M}_1^4}{\ln\varepsilon}.$$

Thus, the relations obtained in Sections 2–4, are true at $\mathrm{M}_1^4/\ln\varepsilon \ll 1$. Let us construct a solution at $\mathrm{M}_1^4/\ln\varepsilon \sim O(1)$.

Similar to the case of small perturbations, in region G_2, static pressure $p = 1$ at $\tau \sim O(1)$ within the terms of order of $O(M_1^2\varepsilon^2)$ and has a coordinate singularity $p \sim \mathrm{M}_1^2\varepsilon^2/\tau^2$ at $\tau \to 0$. The density, radial component of velocity, and function F obey relations (3.1) and (3.2). The asymptotic behavior of the solution of Eq. (3.2) at $\tau \to 0$ can be expressed as

$$F \sim \alpha_0 + \frac{\mathrm{M}_1^4}{\ln\varepsilon}\tilde{\alpha}_1 \ln\tau. \tag{7.1}$$

Here, $\alpha_0 \geq O(1)$, $\tilde{\alpha}_1 \sim O(1)$. Since functions F and v according to (7.1) depend on τ and $1/\ln\varepsilon$, the equations for their determination in region G_2 in the main approximation can be written in the following form:

$$F'' + \frac{1}{\tau}F'\left(1 + \frac{1}{F}\tau^2\left(\frac{1}{2} - v\right)\Pr\right) = 0, \quad v = -\frac{1}{2\Pr\tau^2}\left(\frac{\mathrm{M}_1^4\tilde{\alpha}_1}{\ln\varepsilon} - \tau F'\right). \tag{7.2}$$

The unknown coefficients α_0 and $\tilde{\alpha}_1$ determine the solution of the boundary value problem for Eqs. (7.2) and should be chosen based on the condition $F = 1$ at $\tau \to \infty$ and the matching with the solution in region G_1.

The outer limit of the inner expansion for function y in region G_1 is determined from relation (7.1). When $\eta \to \infty$,

$$y(\eta) \sim \frac{y(1)}{F_*}\left(\alpha_0 + \mathrm{M}_1^4\tilde{\alpha}_1 + \frac{\mathrm{M}_1^4}{\ln\varepsilon}\tilde{\alpha}_1\ln\eta\right). \tag{7.3}$$

Expression (7.3) shows that function $y(\eta)$ can be represented in the form of the following asymptotic series

$$y(\eta) = y_0(\eta) + \frac{\mathrm{M}_1^4}{\ln\varepsilon}y_1(\eta).$$

In order to define $y_0(\eta)$ we can formulate a problem that coincides with (4.13)–(4.15) (Fig. 1). The matching condition can be written in the following form

$$\alpha_0 + \mathrm{M}_1^4\tilde{\alpha}_1 = \frac{F_* y_{00}(\infty)}{(y'_{00}(1))^4}\lambda^{-1} = \frac{F_* d_2}{d_1^4}\lambda^{-1}. \tag{7.4}$$

The boundary value problem (7.2) can be solved by varying the initial conditions of the Cauchy problem for Eqs. (7.2). The initial condition is the value of F set by function (7.1). In this case, coefficients α_0 and $\tilde{\alpha}_1$ are bound by relation (7.4). Thus, varying only the value of α_0 and determining $\tilde{\alpha}_1$ by means of (7.4), we find the value of α_0, at which $F = 1$ at $\tau \to \infty$.

After the coefficients α_0 and $\tilde{\alpha}_1$ are found, we solve the problem of determining function $y_1(\eta)$ in region G_1. The algorithm for deriving the function is described in Section 4.

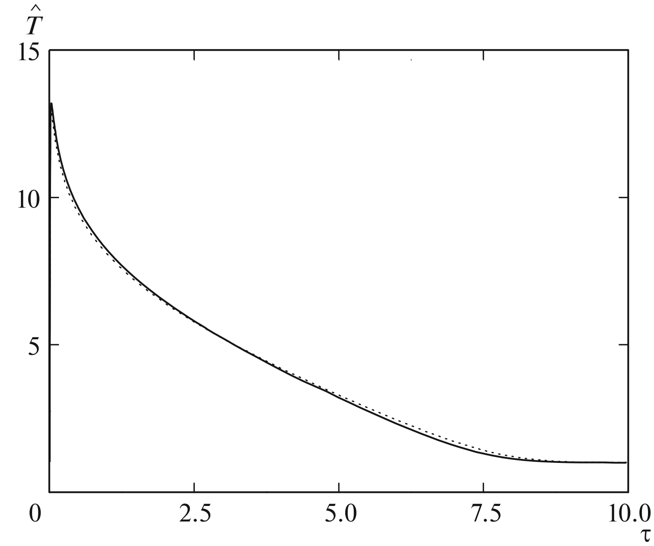

Fig. 7.

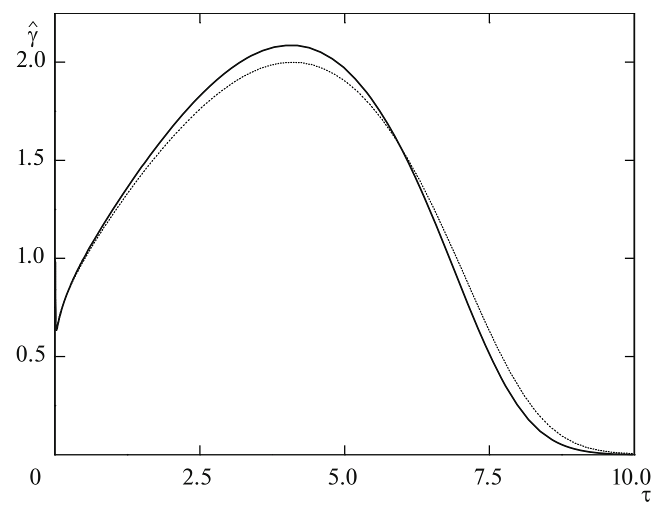

Fig. 8.

Determining the temperature field makes it possible to solve the problem of finding the circulation distribution in region G_2. In the main approximation, function γ should obey the equation

$$\gamma'' - \frac{\gamma'}{\tau} + \frac{F'}{2F}\left(\gamma' - \frac{2\gamma}{\tau}\right) + \frac{1}{F}\left(\frac{1}{2} - \nu\right)\tau\gamma' = 0$$

with the boundary conditions $\gamma = A$ at $\tau \to 0$ and $\gamma = 0$ at $\tau \to \infty$.

In region G_1, expression (4.2) is still true for finding γ. The value of A is chosen using the matching condition

$$A = \frac{(æ - 1)\,\mathrm{Pr}\,\mathrm{M}_1^2}{d_1^3\sqrt{\alpha_0}}.$$

In Figs. 7 and 8, the behavior of dependences $\hat{T}(\tau) = \sqrt{\hat{F}(\tau)}$ and $\hat{\gamma}(\tau)$ (Section 5), corresponding to the case of $M_1 = 7$, $F_* = \mathrm{Pr} = 1$ and $1/\ln\varepsilon = -0.217$ (solid curves), and of the numerical solution of the Navier–Stokes equations at the same values of parameters (dashed curves) is presented.

8. THE NUMERICAL SOLUTION OF THE NAVIER–STOKES EQUATIONS

The asymptotic solution of the problem is compared with the numerical solution of the Navier–Stokes equations (2.1). These equations are calculated by the finite volume method of the second order of accuracy with respect to the space with upwind differences and the first order of accuracy with respect to the time in the circular region $r_* \le r \le 10^3 r_*$ on a radially symmetric grid with 24 000 cells. The number of cells is $N_\theta = 120$ along the circle $r = \text{const}$ and $N_r = 200$ along the radius; the size of cells in the radial direction is 10^{-3} near the cylinder surface $r = r_*$, and 20 on the external boundary of the calculated region $r = 10^3 r_*$. The boundary condition of the unperturbed flow is imposed on the external boundary of the calculated region; the error due to the presence of a radial flow with nonzero v is acceptable.

All calculations are performed at the nondimensional parameters $F_* = 1$, $\mathrm{Pr} = 1$, and $\mathrm{Re} = 100$. The greatest time period of the calculations corresponds to the values: $\varepsilon = 0.005$, $1/|\ln\varepsilon| \approx 0.189$ at $\mathrm{M}_1 = 1$ and $\varepsilon = 0.01$, $1/|\ln\varepsilon| \approx 0.217$ at $\mathrm{M}_1 = 7$. Further calculations during the foreseeable time do not allow any significantly decrease in the value of small parameter $1/\ln\varepsilon$ in expansions (3.5) and (4.12). The time step Δt is such that $r_*/\sqrt{\nu_0 \Delta t} \approx 61.3$ at $\mathrm{M}_1 = 1$ and $r_*/\sqrt{\nu_0 \Delta t} \approx 12.5$ at $\mathrm{M}_1 = 7$.

The circulation and temperature distributions obtained from the numerical solution of the Navier–Stokes equations confirm the asymptotic solution and are shown in Figs. 3, 4, 7, and 8.

CONCLUSIONS

The vortex diffusion problem and the inverse problem of a vortex generation with a rotating cylinder in incompressible fluid are well known. These problems are characterized by the growth of the spatial size of the viscous region according to the law $\sqrt{\nu_0 t}$. The radial distribution of the circulation is a monotonic function that grows in the vortex diffusion problem and declines in the rotating cylinder problem.

The problem of a vortex generation in a compressible gas with the temperature-dependent viscosity is similar to one for the case of incompressible fluid by the size of the viscous region, but differs radically from it by nonmonotonic distribution of circulation within this region. As shown in Eqs. (2.1), the circulation distribution is influenced not only by the viscosity but also by the viscosity gradient. Due to this influence, as opposed to the incompressible fluid, in which the entire vorticity of the rotating fluid is of the same sign, there are such cases in compressible gas, when the vorticity changes the sign twice along the radius. When ν_0 and t are the same and the cylinder temperature is not lower than the unperturbed gas temperature, the incompressible fluid particles close to the cylinder rotate faster than the compressible gas particles; while in the far region, the situation is the opposite (Fig. 5). In the case of a compressible gas with constant viscosity, the circulation distribution is close to the case of incompressible fluid.

The solutions are obtained using the method of matched asymptotic expansions at large times, when the size of the viscous region is much larger than the cylinder radius. We note that in the main approximation, the solution for the temperature and circulation has a discontinuity at the junction of regions G_1 and G_2, i.e., the outer limit of the inner expansion is not equal to the inner limit of the outer expansion; this equality is provided only in the next approximations.

The temperature and circulation in region G_1 obey stationary equations; however, the boundary condition that ensures the solution in region G_1 matching the solution in region G_2 is nonstationary. Therefore, the solution in region G_1 is parametrically dependent on time and reaches the steady state in the limit of $t \to \infty$.

The small parameter $1/\ln\varepsilon$ of the problem slowly decreases with time. The problem can be linearized at $\mathrm{M}_1 \sim O(1)$ in the far region. However, if $\mathrm{M}_1 \gg 1$, it can be required to solve a nonlinear problem of ordinary differential equations in the far region, which is possible only applying the numerical method. For large numbers of M_1, the maximum temperature and coefficient A that is responsible for circulation behavior in the far region are proportional to M_1^2. In Fig. 7, it is shown that in the case of $\mathrm{M}_1 = 7$, the maximum temperature is 13 times higher than the surface temperature of the cylinder. According to Fig. 8, the maximum circulation exceeds the circulation on the cylinder surface only twice; it can be shown that it reaches the order of M_1^2 only at exponentially large times when the problem is linear ($\mathrm{M}_1^4/\ln\varepsilon \ll 1$) in the far region.

All of the main results have been confirmed by the numerical calculations of Eqs. (2.1) for nonstationary axisymmetric flows.

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