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Fluid Kinetic Energy Asymptotic Expansion for Two Variable Radii Moving Spherical Bubbles at Small Separation Distance

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Abstract—Two spherical bubbles with changing radii are considered to be moving in ideal fluid along their center-line. The exact expression for the fluid kinetic energy is obtained. The Stokes stream function is expanded in Gegenbauer polynomials in bispherical coordinates. This expansion is used to obtain the exact series for the fluid kinetic energy quadratic form coefficients. The new series are confirmed to be correct by comparison with the known ones. The main advantage of the new kinetic energy form is the possibility to obtain asymptotic expansions at small separation distance between the bubbles. These expansions are obtained and their convergence is analyzed. The results of this work can be used to describe the bubbles approach before the contact and their coalescence in acoustic field.

Keywords: bubble interaction, Bjerknes force, Stokes stream function, fluid kinetic energy, axial symmetry, asymptotic expansion

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INTRODUCTION

The problem of spherical gas bubbles interaction in a pulsating pressure field is the research subject of a large number of both theoretical and experimental studies, starting with Bjerknes' studies in the 19th century [1] and ending with the recent studies [2–5]. Bjerknes established that the interaction force between pulsating spheres located at large distances is inversely proportional to the square of the distance between them. This dependence was confirmed experimentally [6, 7].

The dynamics of spheres of variable radii at a large distance was studied analytically in [8–10]. The refinement of Bjerknes' results is related to obtaining an expansion for the hydrodynamic interaction force in inverse powers of the distance between the spheres' centers. The kinetic energy was found with accuracy up to terms of the order of r^{-3} [11, 12] and r^{-4} [13], and the solution itself was accurate up to r^{-5} [14] and to r^{-6} [3].

However, it was theoretically [3, 15] and experimentally [4, 16–19] shown that, when approaching the contact, this dependence is not applicable and should be found from the solution of the problem of two pulsating spheres interaction in the exact formulation.

The problem of the gas bubbles in the wave's acoustic field is conveniently studied by the method of generalized Lagrange coordinates. The main component of the Lagrange function is the kinetic energy. For spherical bubbles, the problem of calculating the kinetic energy as a function of the sphere's radii, the distance between the sphere's centers, and the rates of change of the radii and centers arises.

Two of the methods used to construct the exact solution of this problem are considered to be the most effective: (i) the reflection method, with the help of which Hicks constructed the exact solution for the motion of two spheres of constant radii [20]. Using the same method, an exact solution was obtained for spheres of variable radii [21–23], although attempts were made earlier [24–26]. In the case of constant radii, kinetic energy is the quadratic form of the two velocities of the spheres' centers and has three coefficients found by Hicks. In the case of variable radii, the quadratic form contains ten coefficients [21, 23], including three Hicks coefficients.

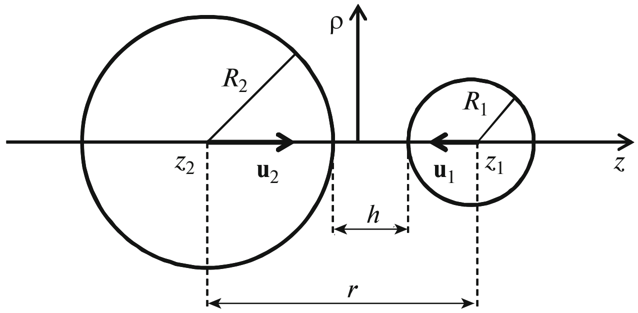

Fig. 1. Problem statement.

(ii) The second method represents the solution in bispherical coordinates [27]. Such research direction was carried out in the case of constant radii [28–30] and in the case of variable radii [8, 31].

It should be noted that, from the series of kinetic energy coefficients obtained by the first method, their three-term expansions were found near the contact of the spheres both in the case of spheres of constant radii [32] and in the case of spheres of variable radii [33]. For the series obtained by the second method in the case of spheres of constant radius [28, 29], an algorithm was developed for obtaining an expansion up to any order in a small gap [34]. In case of variable radii spheres, there was no such expansion foud in literature. This study aims to bridge this gap.

In this study, the exact expression in the bispherical coordinates of kinetic energy is given for the case of variable radii. The derivation of asymptotic expansions near the contact is also given. The study is performed in three stages: (i) the construction of an exact solution of the boundary value problem for the stream function; (ii) calculation of the kinetic energy; and (iii) construction of asymptotic expansions for the kinetic energy coefficients.

1. STREAM FUNCTION

A potential axisymmetric flow of an incompressible ideal fluid with density ρ_l is considered in a region bounded internally by two spheres. The fluid's flow is caused by two spheres of radii R_1, R_2 varying at rates $\dot{R}_1$, $\dot{R}_2$. The sphere centers have coordinates z_1, z_2 ($z_1 > z_2$) on the z axis and move at velocities $u_1 = -\dot{z}_1$, $u_2 = \dot{z}_2$ directed towards each other (Fig. 1). The distance between the spheres' centers is $r = z_1 - z_2$, and the distance between the spheres' surfaces (gap) is $h = r - R_1 - R_2$.

The fluid's velocity components v_ρ, v_θ, and v_z in the cylindrical coordinate system ρ, θ, and z ($x = \rho\cos\theta$, $y = \rho\sin\theta$) are expressed in terms of the stream function ψ:

$$v_\rho = \frac{1}{\rho}\frac{\partial\psi}{\partial z}, \quad v_\theta = 0, \quad v_z = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}. \tag{1.1}$$

The equation for the stream function ψ follows from the condition of the velocity field potentiality and has the form [35]

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial z}\right) = 0. \tag{1.2}$$

The stream function must satisfy the boundary conditions following from the condition that the normal velocities of the fluid v_n and sphere surface w_n are equal:

$$(\mathbf{v}_i, \mathbf{n}) = (\mathbf{w}_i, \mathbf{n}), \quad i = 1, 2. \tag{1.3}$$

To solve the boundary value problem (1.2) and (1.3), it is convenient to pass to the bispherical coordinates ξ, ζ, θ ($x = \rho\cos\theta$, $y = \rho\sin\theta$):

$$\rho = c\frac{\sin\zeta}{\cosh\xi - \cos\zeta} \quad z = c\frac{\sinh\xi}{\cosh\xi - \cos\zeta}. \tag{1.4}$$

Then, the surfaces of the spheres of radii R_1 and R_2 are given by the following equations

$$\xi = \tau_1 = \text{const}, \quad \xi = -\tau_2 = \text{const}, \quad \zeta \in [0, \pi], \quad \theta \in [0, 2\pi], \tag{1.5}$$

where $R_i \sinh \tau_i = c$ and $r = R_1 \cosh\tau_1 + R_2 \cosh\tau_2$.

Thus, it is possible to determine the spheres' surface using parameters τ_1, τ_2, and c, which are expressed through R_1, R_2 and a small gap h as follows:

$$\tau_i = \sqrt{2hp}/R_i + O(h^{3/2}), \quad c = \sqrt{2hp} + O(h^{3/2}), \quad p = R_1R_2/(R_1 + R_2). \tag{1.6}$$

The stream function will be sought in the following form [27, 29]:

$$\psi(\xi, \zeta) = (\cosh \xi - \cos \zeta)^{-1/2} \sum_{n=0}^{\infty} U_n(\xi) C_n^{-1/2}(\cos \zeta),$$
$$U_n(\xi) = \frac{\alpha_n \sinh((n-1/2)(\xi + \tau_2)) + \beta_n \sinh((n-1/2)(\tau_1 - \xi))}{\sinh((n-1/2)\tau)}, \quad \tau = \tau_1 + \tau_2, \tag{1.7}$$

where $C_n^{-1/2}(\mu)$ are the Gegenbauer polynomials [36] and $\mu = \cos \zeta$.

Coefficients α_n and β_n are found from the boundary conditions (1.3) on the spheres' surfaces at $\xi = \tau_1$ and $\xi = -\tau_2$:

$$(\mathbf{v}_i, \mathbf{n}) = \frac{1}{\rho} \frac{\partial \psi}{\partial \zeta} \frac{\cosh \tau_i - \cos \zeta}{c} = (-1)^i \left(u_i \left(-\frac{\cosh \tau_i \cos \zeta - 1}{\cosh \tau_i - \cos \zeta} \right) + \dot{R}_i \right) = (\mathbf{w}_i, \mathbf{n}). \tag{1.8}$$

After integrating equation (1.8) over ζ, we obtain that

$$\psi(\tau_1, \zeta) = u_1 c^2 \frac{1}{2} \frac{1-\mu^2}{(\cosh\tau_1 - \mu)^2} + \dot{R}_1 c^2 \frac{1}{\cosh\tau_1 - \mu} + \left(\frac{\dot{R}_2 c^2}{\sinh^2 \tau_2} - \frac{\dot{R}_1 c^2 \cosh\tau_1}{\sinh^2 \tau_1} \right), \quad \mu = \cos \zeta. \tag{1.9}$$

The expression for $\psi(-\tau_2, \zeta)$ is obtained by permuting index 1 by 2 and replacing the sign of the expression for $\psi(\tau_1, \zeta)$. Using identities [29] at $\tau > 0$ and $-1 \le \mu \le 1$,

$$(\cosh \tau - \mu)^{1/2} = \sum_{n=0}^{\infty} \frac{C_n^{-1/2}(\mu)}{\sqrt{2}} e^{-(n-1/2)\tau},$$

$$\frac{1}{(\cosh \tau - \mu)^{1/2}} = \sum_{n=0}^{\infty} C_n^{-1/2}(\mu) \frac{-(n-1/2)\sqrt{2}}{\sinh \tau} e^{-(n-1/2)\tau},$$

$$\frac{1}{2} \frac{1-\mu^2}{(\cosh \tau - \mu)^{3/2}} = \sum_{n=0}^{\infty} C_n^{-1/2}(\mu) \sqrt{2} n(n-1) e^{-(n-1/2)\tau},$$

an expression for α_n is found:

$$\alpha_n = \left(u_1 2n(n-1) - \dot{R}_1 \frac{2n-1}{\sinh \tau_1} + \left(\frac{\dot{R}_2}{\sinh^2 \tau_2} - \frac{\dot{R}_1 \cosh \tau_1}{\sinh^2 \tau_1} \right) \right) \frac{c^2}{\sqrt{2}} e^{-(n-1/2)\tau_1}; \tag{1.10}$$

the expression for β_n is obtained by permuting index 1 by 2 and replacing the sign of α_n.

It should be noted that, in the case of spheres with constant radii movement ($\dot{R}_i = 0$), the summation in expansion (1.7) begins at $n = 2$ [29] since the coefficients α_n defined by formula (1.10) are zero at $n = 0$ and 1. However, the set of Gegenbauer polynomials with $n \ge 2$ does not form a complete basis. The latter circumstance plays an important role in the case of spheres with variable radii movement. At the beginning of the summation at $n = 2$, it is necessary to first isolate some function [8] and find the remainder in the form of an expansion in the Gegenbauer polynomials with $n \ge 2$. However, the expansion in the complete basis (1.7) begins at $n = 0$, which eliminates the need for a separate function. One also has this advantage when solving the problem of the viscous interaction of two spheres of variable radii.

The expression for the stream function (1.7), up to a constant, is identical with the expression obtained in [8]. However, several misprints were found in the paper mentioned above. The choice of a constant is not fundamental but it affects the kinetic energy coefficients, the form of which is given in the next section.

2. KINETIC ENERGY

The kinetic energy of a fluid is expressed through the integral of v^2 over the region outside two spheres:

$$T = \frac{\rho_l}{2}\iiint v^2 dV. \tag{2.1}$$

Using Green's formula, integral (2.1) can be found by the following formula [29]:

$$\frac{T}{\pi\rho_l} = \oint \psi\frac{1}{\rho}\psi'_\zeta d\xi - \psi\frac{1}{\rho}\psi'_\xi d\zeta = \int\limits_{\xi=-\tau_2}^{\tau_1} \psi\frac{1}{\rho}\psi'_\zeta\Bigg|_{\zeta=0}^{\zeta=\pi} d\xi + \int\limits_{\zeta=0}^{\pi} \psi\frac{1}{\rho}\psi'_\xi\Bigg|_{\xi=-\tau_2}^{\xi=\tau_1} d\zeta. \tag{2.2}$$

It should be noted that, in the case of constant radii spheres, the stream function is zero on the axis of symmetry [29] and the first integral does not make a contribution. Using the expression for the stream function (1.7), we obtain the kinetic energy in the form of a quadratic form in velocity:

$$T = 2\pi\rho_l\left\{\sum_{j=1}^{2}\left(A_j u_j^2 + D_j\dot{R}_j^2 + \sum_{i=1}^{2} C_{ij}\dot{R}_j u_i\right) + 2(Bu_1u_2 + E\dot{R}_1\dot{R}_2)\right\},$$

$$A_1 = \frac{R_1^3}{6} + c^3\sum_{n=2}^{\infty}\frac{(2n-1)^2-1}{2Q_n(\tau_1)(Q_n(\tau)-1)}, \quad B = \frac{c^3}{2}\sum_{n=2}^{\infty}\frac{(2n-1)^2-1}{Q_n(\tau)-1},$$

$$C_{11} = 2c^3\sum_{n=2}^{\infty}\frac{S_n(\tau_1)}{Q_n(\tau_1)(Q_n(\tau)-1)}, \quad C_{12} = 2c^3\sum_{n=2}^{\infty}\frac{S_n(\tau_2)}{Q_n(\tau)-1}, \tag{2.3}$$

$$D_1 = R_1^3 + c^3\sum_{n=2}^{\infty}\frac{2}{(2n-1)^2-1}\frac{S_n^2(\tau_1)}{Q_n(\tau_1)(Q_n(\tau)-1)},$$

$$E = \frac{(R_1R_2)^2}{r} + c^3\sum_{n=2}^{\infty}\frac{2}{(2n-1)^2-1}\frac{S_n(\tau_1)S_n(\tau_2)}{Q_n(\tau)(Q_n(\tau)-1)},$$

where $Q_n(x) = \exp((2n-1)x)$, and the notation $S_n(x) = (Q_n(x) - (2n-1)\sinh x - \cosh x)/\sinh^2 x$ is introduced, while the coefficients A_2, C_{21}, C_{22} and D_2 are obtained by permuting 1 by 2 in the formulas for A_1, C_{12}, C_{11}, and D_1. Furthermore, series (2.3) can be expressed in terms of the initial parameters R_1, R_2, and r, by substituting

$$c = \sqrt{(r^2 - R_1^2 - R_2^2)^2 - 4R_1^2R_2^2}/(2r), \quad e^{-\tau_1} = b - \sqrt{b^2-1}, \tag{2.4}$$

where $b = (r^2 + R_1^2 - R_2^2)/(2rR_1)$. Similarly, we can express the exponentials $e^{-\tau_2}$ and $e^{-\tau}$.

If the kinetic energy coefficients (2.3) are presented in the form of exponential series, then they completely coincide with series [20] and [15, 21, 23], which proves the reliability of the performed calculations.

The kinetic energy was found in the form of a series of inverse powers of r [3]. Comparison with the exact solution shows the coincidence of the terms to r^{-6}; however, the next members do not match in terms of order.

When considering the interaction of two bubbles of variable radii with fixed centers [37, 38], the kinetic energy coefficients turned out to be different from the exact ones (2.3), since the problem was solved under the assumption that the velocity potential on the sphere's surface was constant. Such an inaccuracy does not affect the main asymptotic behavior of the Bjerknes force at large distances between the spheres.

In a slightly different form, the kinetic energy coefficients can be obtained from the expressions for hydrodynamic forces [8]. Given the small misprints, they are consistent with expressions (2.3).

It should be noted that series [15, 20–23] are more convenient for obtaining expansions at large distances $r \gg R_1, R_2$. Near the contact, only a three-term expansion was obtained from these series at small $h \ll R_1, R_2$ [32, 33]. The following expansion terms are conveniently obtained from series (2.3). The algorithm for obtaining them is presented in the next section.

3. ASYMPTOTIC EXPANSION

3.1. Asymptotic Expansion near the Contact

To obtain the asymptotic expansion of the fluid kinetic energy near the contact, we use the method described in [34]. In this study, the method is presented for spheres of constant radii, i.e., for coefficients A_1, A_2, and B. The development of this method is proposed for the case of variable radii, i.e., for the remaining seven coefficients.

The coefficient A_1 is written in the form

$$A_1 = \frac{R_1^3}{6} + c^3 \sum_{n=2}^{\infty} \frac{e^{-(1+\lambda_1)t}}{1-e^{-t}} \frac{(2n-1)^2-1}{2}, \quad t = (2n-1)\tau, \quad \lambda_1 = \tau_1/\tau. \tag{3.1}$$

Substituting the Mellin transform under the sum sign and summing over n, we have

$$A_1 = \frac{R_1^3}{6} + \frac{c^3}{2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \tau^{-s}\Gamma(s) Z(s)\zeta(s,1+\lambda_1)\,ds, \quad \sigma > 3, \tag{3.2}$$

where

$$Z(s) = \sum_{n=2}^{\infty}(2n-1)^{-s}((2n-1)^2-1) = \zeta(s-2)(1-2^{-(s-2)}) - \zeta(s)(1-2^{-s}),$$

$\zeta(s,a)$ is the Hurwitz zeta function, and $\zeta(s) = \zeta(s,1)$ is the Riemann zeta function [36].

We calculate integral (3.2) using the residue theorem. For this purpose, it is necessary to find the poles of the integrand. They are located at points 3, 1, –1, ... and determine the orders of the terms of the asymptotic expansion. The residue at the first point determines the main term of the expansion; the residue at the second point, the next one; etc. Given the values of the residues at points 3, 1, –1, ..., $-2l+1$, we obtain the following expansion:

$$\begin{aligned} A_1 = \frac{R_1^3}{6} + \frac{c^3}{2}\Bigg(&\frac{\zeta(3,1+\lambda_1)}{\tau^3} + \frac{1}{2\tau}\left(\ln\frac{\tau}{2} + \psi(1+\lambda_1) + \frac{1}{6}\right) \\ &- \sum_{k=1}^{l}\frac{\tau^{2k-1}}{(2k-1)!}Z(-2k+1)\zeta(-2k+1,1+\lambda_1)\Bigg) + r_{A_1}^{2l-1}, \end{aligned} \tag{3.3}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function [36], while the remainder term has the form

$$r_{A_1}^m = \frac{c^3}{2}\frac{1}{2\pi i}\int_{\sigma_m-i\infty}^{\sigma_m+i\infty}\tau^{-s}\Gamma(s)Z(s)\zeta(s,1+\lambda_1)\,ds, \quad -m-1 < \sigma_m < -m. \tag{3.4}$$

Similarly for coefficient B, we obtain [34]:

$$B = \frac{c^3}{4\tau}\left(\frac{2\zeta(3)}{\tau^2} + \ln\frac{\tau}{2} + \left(\psi(1) + \frac{1}{6}\right) - 2\sum_{k=1}^{n}\frac{\tau^{2k}}{(2k-1)!}Z(-2k+1)\zeta(-2k+1)\right) + r_B^{2l-1}, \tag{3.5}$$

where r_B^{2l-1} is determined similarly to $r_{A_1}^{2l-1}$.

We give the asymptotic expansions of the remaining coefficients:

$$\begin{aligned} C_{1j} = \frac{2c^3}{\sinh^2\tau_j}\Bigg(&\frac{2\tau - \zeta(2,1+\lambda_1(2-j))}{2\tau^2}\sinh\tau_j + \sum_{k=0}^{m}\frac{(-\tau)^k}{k!}Z_{C_{1j}}(-k,\lambda_1) \\ &+ (-1)^{j-1}\frac{\gamma\cosh^{j-1}\tau_j + \cosh^{2-j}\tau_j\,\psi(\lambda_1+2-j)}{2\tau} + \frac{\cosh\tau_j - 1}{\tau}\left(1 + \frac{1}{2}\ln\frac{\tau}{2}\right)\Bigg) + r_{C_{1j}}^m, \end{aligned} \tag{3.6}$$

$$j = 1, 2,$$

where $\gamma \approx 0.577216$ is the Euler constant and

$$\begin{aligned} Z_{C_{1j}}(s,\lambda_j) = (\zeta(s,\lambda_1^{j-1}) - \zeta(s,(1+\lambda_1)^{2-j})\cosh\tau_j)(\zeta(s)(1-2^{-s})-1) \\ - \zeta(s,(1+\lambda_1)^{2-j})\sinh\tau_j(\zeta(s-1)(1-2^{-(s-1)})-1), \\ j = 1,2. \end{aligned} \tag{3.7}$$

For coefficients D_1 and E, the derivation of the asymptotic expansion is much more complicated:

$$D_1 = R_1^3 + \frac{2c^3}{\sinh^4\tau_1}\sum_{k=-1}^{n}\underset{s=-k}{\text{res}}(\tau^{-s}\Gamma(s)Z_{D_1}(s,\lambda_1)) + r_{D_1}^m, \tag{3.8}$$

$$E = \frac{(R_1R_2)^2}{r} + \frac{2c^3}{\sinh^2\tau_1\sinh^2\tau_2}\sum_{k=-1}^{n}\underset{s=-k}{\text{res}}(\tau^{-s}\Gamma(s)Z_E(s,\lambda_1,\lambda_2)) + r_E^m, \tag{3.9}$$

where

$$\begin{aligned} Z_{D_1}(s,\lambda_1) = \sinh^2\tau_1\zeta(s,1+\lambda_1)H(s-2) + (\sinh(2\tau_1)\zeta(s,1+\lambda_1) - 2\sinh\tau_1\zeta(s))H(s-1) \\ + (\zeta(s,1-\lambda_1) - 2\cosh\tau_1\zeta(s) + \cosh^2\tau_1\zeta(s,1+\lambda_1))H(s), \\ Z_E(s,\lambda_1,\lambda_2) = \sinh\tau_1\sinh\tau_2\zeta(s,2)H(s-2) \\ - (\sinh\tau_1\zeta(s,1+\lambda_1) + \sinh\tau_2\zeta(s,1+\lambda_2) - \sinh\tau\zeta(s,2))H(s-1) \\ + (\zeta(s) - \cosh\tau_1\zeta(s,1+\lambda_1) - \cosh\tau_2\zeta(s,1+\lambda_2) + \cosh\tau_1\cosh\tau_2\zeta(s,2))H(s), \end{aligned}$$

and the function $H(s)$ has the form

$$H(s) = \sum_{n=2}^{\infty}\frac{(2n-1)^{-s}}{(2n-1)^2-1}.$$

Given the recurrence ratio

$$H(s) = H(s+2) + \zeta(s+2)(1-2^{-(s+2)}) - 1,$$

and also that $H(0) = 1/4$ and $H(1) = 3/4 - \ln 2$, the asymptotic expansion for coefficients D_1 and E after deducting the residue finally takes the form

$$\begin{aligned} D_1 = R_1^3 + \frac{2c^3}{\sinh^4\tau_1}\left\{-\frac{\cosh\tau_1 - 1}{2\tau}\Big(\psi(1+\lambda_1) - \sinh\tau_1 + 2 + \cosh\tau_1(\psi(1+\lambda_1)\right. \\ \left. + \ln\tau - 1 + 3\ln 2) + \ln\left(\frac{\tau}{2}\right)\Big) + \sum_{k=0}^{m}\frac{(-\tau)^k}{k!}W_{D_1}(k,\lambda_1)\right\} + r_{D_1}^m, \end{aligned} \tag{3.10}$$

$$\begin{aligned} E = \frac{(R_1R_2)^2}{r} + \frac{c^3}{\tau\sinh^2\tau_1\sinh^2\tau_2}\left\{\sinh\tau_1\sinh\tau_2\left(-\ln(2\tau) - \frac{3}{2} + \gamma\right) + \frac{1}{2}(\sinh\tau - \sinh\tau_1\right. \\ \left. - \sinh\tau_2) + \left(\frac{3}{2} - 2\ln 2\right)(\cosh\tau_1 - 1)(\cosh\tau_2 - 1) - 2\sum_{k=0}^{m}\frac{(-\tau)^{k+1}}{k!}W_E(k,\lambda_1,\lambda_2)\right\} + r_E^m. \end{aligned} \tag{3.11}$$

In this case, the functions W_{D_1} and W_E are defined as follows:

$$\begin{aligned} W_{D_1}(k,\lambda_1) = \sum_{j=-1}^{1}(3j^2 - 2)\zeta(-k,1+j\lambda_1)\cosh^{j+1}\tau_1 G(k) \\ + \sinh\tau_1[2(\cosh\tau_1\zeta(-k,1+\lambda_1) - \zeta(-k))G(k+1) + \sinh\tau_1\zeta(-k,1+\lambda_1)G(k+2)] \\ + \frac{1}{4}[F_k(0,1-\lambda_1) + F_k(2\tau_1,1+\lambda_1) - 2F_k(\tau_1,1)], \\ W_E(k,\lambda_1,\lambda_2) = (\sinh\tau_1\sinh\tau_2 G(k+2) + \sinh\tau G(k+1) + \cosh\tau_1\cosh\tau_2 G(k))\zeta(-k,2) \end{aligned}$$

$$-\sum_{j=1}^{2}\left(\sinh\tau_j\zeta(-k,1+\lambda_j)G(k+1)+\left(\cosh\tau_j\zeta(-k,1+\lambda_j)-\frac{1}{2}\zeta(-k)\right)G(k)\right)$$
$$+\frac{1}{4}[F_k(0,1)+F_k(\tau,2)-F_k(\tau_1,1+\lambda_1)-F_k(\tau_2,1+\lambda_2)],$$

where

$$F_k(x,y)=\{\zeta^{(1,0)}(-k,y)+L(k)\zeta(-k,y)\}\{(1+(-1)^k)\sinh x+(1-(-1)^k)\cosh x\},$$
$$G(k)=\frac{1}{2}\left\{\frac{1}{2}-k+(1-(-1)^k)\left(-\ln 2+\sum_{j=1}^{(k-1)/2}\zeta(1-2j)(1-2^{-(1-2j)})\right)\right\},$$
$$L(k)=\sum_{j=1}^{k} j^{-1}-\ln\frac{\tau}{2}.$$

As noted earlier, the coefficients A_2, C_{22}, C_{21}, and D_2 are found by permuting the indices.

3.2. Asymptotic Expansion Comparison

For the case of constant radii spheres, the asymptotic expansion of the kinetic energy [34] is in agreement with the three-term expansion [32]. The asymptotic expansions of kinetic energy obtained above, for the case of spheres of variable radii with a three-term expansion [33] are also in agreement.

3.3. Residual Member Assessment

The expansion of $X=\sum_{n=0}^{m}X_n(\varepsilon)+R_X^m(\varepsilon)$ [36] is in the sense of the Poincare asymptotic in parameter ε if $\lim\limits_{\varepsilon\to 0}\left|R_X^m/X_m\right|=0$. For definiteness, we consider the expansion of A_1 (3.3). We present it in the form

$$A_1=x_{A_1}^0+\sum_{k=1}^{l}x_{A_1}^{2k-1}+r_{A_1}^{2l-1}, \tag{3.12}$$

where

$$x_{A_1}^{2k-1}=-\frac{R_1^3\sinh^3\tau_1}{2}\frac{\tau^{2k-1}}{(2k-1)!}Z(-2k+1)\zeta(-2k+1,1+\lambda_1).$$

We prove that $\lim\limits_{\tau\to 0}\left|r_{A_1}^{2l-1}/x_{A_1}^{2l-1}\right|=0$. For the expression:

$$\left|r_{A_1}^{2l-1}\right|=\frac{R_1^3\sinh^3\tau_1}{2}\frac{1}{2\pi}\tau^{-\sigma_{2l-1}}\int_{-\infty}^{\infty}\left|\Gamma(\sigma_{2l-1}+it)Z(\sigma_{2l-1}+it)\zeta(\sigma_{2l-1}+it,1+\lambda_1)\right|dt, \tag{3.13}$$
$$-2l<\sigma_{2l-1}<-2l+1,$$

the following estimate was obtained [34]:

$$\left|r_{A_1}^{2l-1}\right|=O(\tau^{2l-1+3}).$$

Then, given that

$$\left|x_{A_1}^{2l-1}\right|=O(\tau^{2l-1+3}),$$

a stronger statement can be proved:

$$\left|r_{A_1}^{2l-1}\right|=o(\tau^{2l-1+3}). \tag{3.14}$$

For this purpose, we note that

$$\left|r_{A_1}^{2l-1}\right|=\left|x_{A_1}^{2l+1}+r_{A_1}^{2l+1}\right|\le\left|x_{A_1}^{2l+1}\right|+\left|r_{A_1}^{2l+1}\right|=O(\tau^{2l+1+3}),$$

then

$$\lim_{\tau\to 0}\left|\frac{r_{A_1}^{2l-1}}{x_{A_1}^{2l-1}}\right|=0,$$

and thus it is proved that the decomposition of A_1 is asymptotic in the sense of Poincare. In this case, the series diverges for any $\tau > 0$, and it is necessary to limit ourselves to a finite number of members of the series for calculations.

As noted earlier [39], it is advisable to limit the summation of the asymptotic series at $m = \eta$, where η is found from the condition $d\left|x_{A_1}^m\right|/dm\Big|_\eta \sim 0$.

We determine the asymptotics of $x_{A_1}^m$ at large values of m. Given the identicality

$$\zeta(-m, a+1) = \zeta(-m, a) + a^m$$

and the Hurwitz formula [36]

$$\zeta(1-m, a) = 2\frac{(m-1)!}{(2\pi)^m}\sum_{n=1}^{\infty}\frac{\cos\left(2\pi na - \frac{1}{2}\pi m\right)}{n^m}, \quad 1 \geq a \geq 0, \quad m \geq 1,$$

for large m, at $0 \leq a \leq 2$, function $\zeta(-m, a)$ can be approximated as

$$\zeta(-m, a) \approx 2\frac{m!}{(2\pi)^{m+1}}\cos\left(2\pi a - \frac{\pi}{2}(m+1)\right) \sim 2\frac{m!}{(2\pi)^{m+1}},$$

and the $x_{A_1}^m$ asymptotics can be estimated as

$$\frac{x_{A_1}^m}{p^3} \sim 4\tau\sqrt{2\pi(m+2)}\left(\frac{m+2}{2\pi^2 e}\tau\right)^{m+2}. \tag{3.15}$$

From the condition $d\left|x_{A_1}^m\right|/dm\Big|_\eta \sim 0$, we obtain that $\eta \sim 2\pi^2/\tau$. Near the contact, taking into account formula (1.6), we obtain that $\eta \sim 2\pi^2/\sqrt{2h/p}$. The η values for all other coefficients are calculated similarly. With this choice of η, the error is of order $e^{-\eta}$. This estimate is confirmed by numerous numerical calculations.

3.4. Expansion over h near Contact

For practical purposes, it is more convenient to switch from parameter $\tau = \tau_1 + \tau_2$ to gap h. Then, the expansion of the kinetic energy coefficients takes the form

$$X = f_X(h) + g_X(h)\ln\left(\frac{h}{2p}\right), \quad X = \{A_i, B, C_{ij}, D_i, E\}. \tag{3.16}$$

It is necessary to find six pairs of functions $f_X(h)$ and $g_X(h)$ for the coefficients A_1, B, C_{11}, C_{12}, D_1, and E or 12 independent functions in total. For the remaining coefficients, the functions $f_X(h)$ and $g_X(h)$ are obtained by permuting the indices. Note that the number of independent functions can be reduced to 10. For this purpose, we prove that the functions $g_X(h)$ for coefficients A_1 and B coincide and the functions $g_X(h)$ for coefficients C_{11}, C_{21} also coincide. Indeed, formulas (3.3) and (3.5) imply that the coefficients g_{A_1}, and g_B are obtained from the term $c^3/(4\tau)\ln(\tau/2)$, in which argument τ needs to be expressed through h. Coefficients $g_{C_{11}}$, and $g_{C_{21}}$ are obtained similarly from the term $c^3(\cosh\tau_1 - 1)/(\tau\sinh^2\tau_1)\ln(\tau/2)$. The functions $f_X(h)$ and $g_X(h)$ can be expanded in degrees of h. Sufficient accuracy is achieved by cubic polynomials of the form

$$\begin{aligned} f_X(h) &= f_X^0 + f_X^1 h + f_X^2 h^2 + f_X^3 h^3 + O(h^4), \\ g_X(h) &= g_X^1 h + g_X^2 h^2 + g_X^3 h^3 + O(h^4). \end{aligned} \tag{3.17}$$

It can be shown that, up to a permutation of the indices, the logarithmic singularity is determined by four polynomials. Their first three coefficients are shown in Table 1.

Polynomials $f_X(h)$ are bulkier. Therefore, it is more convenient to give numerical values of the coefficients of polynomials $f_X(h)$ at the specified ratio of radii. They are given in Tables 2–4, respectively, for ratios of the radii of $R_2/R_1 = \{1, 3, 10\}$.

Table 1. Analytical view of g_X^1, g_X^2, g_X^3 for coefficients of kinetic energy, $\alpha_i = R_i/(R_1 + R_2)$

X	g_X^1	g_X^2	g_X^3
A_1, B	$\frac{p^2}{4}$	$\frac{5p}{24}(\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2)$	$\frac{1}{360}(13\alpha_1^4 - 98\alpha_1^3\alpha_2 + 183\alpha_1^2\alpha_2^2 - 98\alpha_1\alpha_2^3 + 13\alpha_2^4)$
C_{11}, C_{21}	$\frac{p^2}{2}$	$\frac{p}{12}(5\alpha_1^2 - 5\alpha_1\alpha_2 + 2\alpha_2^2)$	$\frac{1}{180}(13\alpha_1^4 - 98\alpha_1^3\alpha_2 + 123\alpha_1^2\alpha_2^2 - 38\alpha_1\alpha_2^3 - 2\alpha_2^4)$
D_1	$\frac{p^2}{4}$	$\frac{p}{24}(5\alpha_1^2 - 5\alpha_1\alpha_2 - \alpha_2^2)$	$\frac{11}{720} + \frac{1}{48}\alpha_1^2(\alpha_1^2 - 16\alpha_1\alpha_2 + 4\alpha_2^2)$
E	$\frac{p^2}{4}$	$\frac{p}{24}(2\alpha_1^2 - 5\alpha_1\alpha_2 + 2\alpha_2^2)$	$\frac{11}{720} - \frac{1}{48}(\alpha_1^2 - \alpha_1\alpha_2 + \alpha_2^2)(\alpha_1^2 + 9\alpha_1\alpha_2 + \alpha_2^2)$

Table 2. Numerical values of $f_X^0, f_X^1, f_X^2, f_X^3$ for coefficients of kinetic energy at $R_2/R_1 = 1$

X	f_X^0/R_1^3	f_X^1/R_1^3	f_X^2/R_1^3	f_X^3/R_1^3
A_1	0.19257	0.03834	–0.05783	–0.0064
B	0.07513	–0.01375	–0.03339	–0.00841
A_2	0.19257	0.03834	–0.05783	–0.0064
C_{11}	0.07315	0.02403	–0.09609	0.00345
C_{12}	0.28191	–0.12419	–0.04413	–0.00127
C_{21}	0.28191	–0.12419	–0.04413	–0.00127
C_{22}	0.07315	0.02403	–0.09609	0.00345
D_1	1.05634	–0.02539	–0.02837	0.00549
E	0.52088	–0.20799	0.02508	–0.00506
D_2	1.05634	–0.02539	–0.02837	0.00549

The convergence of the approximations of coefficients A_1 and D_1 by polynomials of the first (1), second (2), and third degree (3) to exact dependences (thick line) is shown in Figs. 2a and 3a, and the convergence for their derivatives is shown in Figs. 2b and 3b. As can be seen in these figures, a significant increase in accuracy is observed as the degree of the polynomial increases.

3.5. Hydrodynamic Force

The hydrodynamic force acting on the sphere for an arbitrary distance between them is determined by the Lagrange formula

$$F_1 = -\frac{d}{dt}\frac{\partial T}{\partial \dot{z}_1} + \frac{\partial T}{\partial z_1} = \frac{d}{dt}\frac{\partial T}{\partial u_1} + \frac{\partial T}{\partial r} = \frac{d}{dt}\frac{\partial T}{\partial u_1} + \frac{\partial T}{\partial h}. \tag{3.18}$$

Using this formula and the asymptotic expansions of the coefficients of the kinetic energy, the expansion of the force near the contact can be obtained with any degree of accuracy in h. The main asymptotics of the hydrodynamic force is

$$\frac{F_1}{2\pi\rho_l} = -\frac{d}{dt}\left(\frac{1}{2}p^2 h \ln\left(\frac{h}{2p}\right)\dot{h}\right) + \frac{1}{4}p^2 \ln\left(\frac{h}{2p}\right)\dot{h}^2 + O(h^0), \tag{3.19}$$

where $p = R_1R_2/(R_1 + R_2)$, $h = r - R_1 - R_2$, and $\dot{h} = -(u_1 + u_2 + \dot{R}_1 + \dot{R}_2)$. The asymptotic expression (3.19) (which coincides with the asymptotic behavior found previously by the thin-layer method [40]) contains a logarithmic feature that is difficult to obtain if the kinetic energy is represented as a finite expansion in inverse powers of distance r between the centers of the bubbles.

Table 3. Numerical values of $f_X^0, f_X^1, f_X^2, f_X^3$ for coefficients of kinetic energy at $R_2/R_1 = 3$

X	f_X^0/R_1^3	f_X^1/R_1^3	f_X^2/R_1^3	f_X^3/R_1^3
A_1	0.22593	0.15456	–0.12096	–0.00727
B	0.25356	0.03244	–0.08352	–0.00943
A_2	4.64004	0.05236	–0.0598	–0.02441
C_{11}	0.18871	0.14903	–0.22995	0.04843
C_{12}	0.65753	0.07208	–0.20445	–0.00445
C_{21}	1.81240	–0.59585	0.00243	–0.01
C_{22}	0.3404	0.06923	–0.13721	–0.03191
D_1	1.18701	–0.07682	–0.03699	0.01731
E	2.32151	–0.45781	–0.00579	0.00787
D_2	27.21124	0.019	–0.08045	–0.00683

Table 4. Numerical values of $f_X^0, f_X^1, f_X^2, f_X^3$ for coefficients of kinetic energy at $R_2/R_1 = 10$

X	f_X^0/R_1^3	f_X^1/R_1^3	f_X^2/R_1^3	f_X^3/R_1^3
A_1	0.25175	0.29173	–0.11741	–0.04509
B	0.45156	0.20392	–0.1117	–0.04079
A_2	167.02413	0.16238	–0.09517	–0.03972
C_{11}	0.28907	0.33643	–0.31326	0.03108
C_{12}	0.98412	0.45188	–0.23408	–0.08365
C_{21}	8.51296	–1.12663	–0.13156	0.01537
C_{22}	0.77253	0.34336	–0.20516	–0.07911
D_1	1.35988	–0.10779	–0.0565	0.02402
E	9.22154	–0.64576	–0.09002	0.012
D_2	1000.41854	0.18447	–0.10969	–0.03978

In the particular case of $u_2 = \dot{R}_2 = 0$, $R_2 \to \infty$ (a sphere near the wall), formula (3.19) takes form [31]

$$\frac{F_1}{2\pi\rho_l} = -\frac{1}{4} R_1^2 \ln\left(\frac{h}{R_1}\right)(u_1 + \dot{R}_1)^2 + O(h^0). \tag{3.20}$$

When considering a sphere expanding according to the law $R = \beta t^{1/2}$ that is in contact with the plane, it was previously obtained [41] that the force of attraction to the plane is $F = 0.29\pi\beta^4\rho_l$. This result is consistent with the force of $F = 0.288954\pi\beta^4\rho_l$ found from the asymptotic expansions of this study.

Thus, the obtained expansions for the forces of interaction of two spheres of variable radii generalize all the previously known results.

CONCLUSIONS

The exact solution of the boundary value problem for the stream function is obtained in the case of two spheres of variable radii. It generalizes the solution for hard spheres. Based on the found stream function, a new kind of the fluid's kinetic energy, in which the coefficients of the quadratic form are represented by series, is derived. The identicality of the new series with the previously obtained series [20] and [21–23] is shown. The advantage of the new rows is that they can be re-expanded in the gap between the spheres

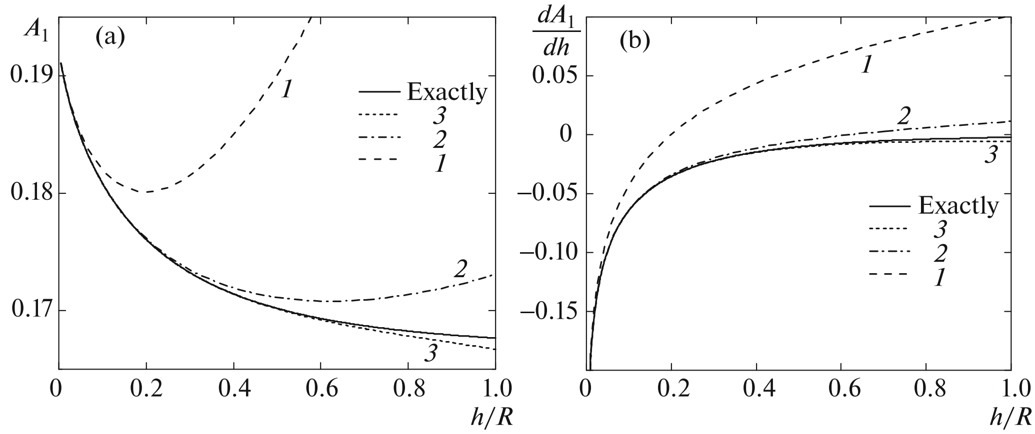

Fig. 2. Convergence of the approximations of (a) coefficient A_1 and (b) derivative dA_1/dh by polynomials of the (*1*) first, (*2*) second, and (*3*) third degree to exact dependences (thick line).

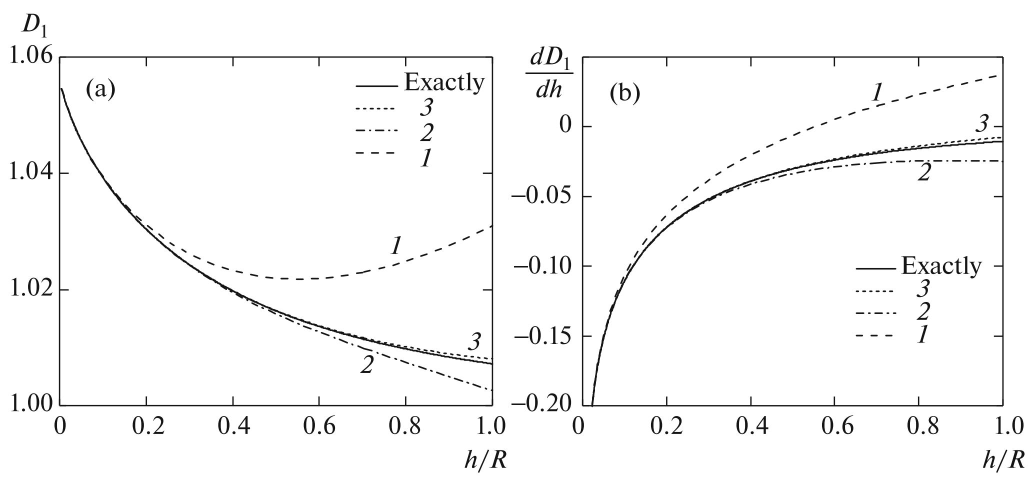

Fig. 3. Convergence of the approximations of (a) coefficient D_1 and (b) derivative dD_1/dh by the polynomials of the (*1*) first, (*2*) second, and (*3*) third degree to exact dependences (thick line).

instead of the commonly used distance between the centers of the bubbles. Using a new form of kinetic energy, asymptotic expansions of the kinetic energy coefficients near the contact are found. The remaining term of the expansion is proved to be exponentially small. The found asymptotic expressions generalize all the results so far known. They are necessary for describing the dynamics of spherical bubbles near the contact and for analyzing the possibility of their coalescence (for example, upon acoustic exposure to them).

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