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On the Orientation of Convective Rolls in an Inclined Rectangular Channel

D. E. Pivovarov

Institute of Mechanics, Moscow State University Michurinskii pr. 1, Moscow, 119192 Russia
E-mail: pivovarov@imec.msu.ru

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Abstract—An analysis of the linear stability of convective flow in a channel of rectangular cross-section inclined to the horizon is presented. The behavior of three-dimensional monotonic disturbances is considered for different values of the channel width and fluid properties. The main flow is obtained in an analytical form. Two angles of inclination, at which the convective roll changes, are determined. A strong dependence of the smaller inclination angle on the channel width and its weak dependence on the medium characteristics and a weak dependence of the greater inclination angle on the channel width are established.

Keywords: free thermal convection, inclined channel, stability, secondary flows

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In hydrodynamics the term "convective rolls" is applied to the chain of pairwise parallel vortices rotating in opposite directions. They are one of the possible solutions of the Rayleigh—Bénard problem [1]. This solution can be observable in the case in which a horizontal layer of a fluid at rest heated from below is confined by solid walls, whose temperature difference is higher than a certain critical value. Precisely this value determines the boundary of the equilibrium position stability. At the boundary conditions of other forms the value of the critical temperature can vary and even the problem solution itself can become different in nature.

An analysis of the equilibrium position stability within the framework of the linear theory makes it possible to determine the frequency of normal disturbances but not their direction. Because of this, in an unbounded horizontal layer the convective rolls are oriented in a random fashion. In the experiments, where the layer dimensions are always finite and the fluid can occupy a region of rectangular planform, a tendency of the convective roll arrangement parallel to the short side can be traced [2]. Numerical calculations [3] lead to the same conclusion. In the case of infinite rolls the factor determining their orientation is the angle of the layer inclination to the horizon [4]. At a nonzero inclination angle any equilibrium position is impossible and the main flow represents a laminar upward-downward motion. A large body of theoretical calculations and laboratory and numerical experiments, recently reviewed in [5], show that the convective rolls are arranged along the main flow. These rolls are usually referred to as longitudinal.

To analyze the conjugate effect of the above-mentioned factors on the roll orientation it is necessary to consider inclined closed layers. This issue is only poorly touched upon in the literature. In the experimental study [6] devoted to convection in a narrow slot it was shown that the presence of side walls makes impossible the longitudinal roll development in an inclined position. They are replaced by a chain of rolls perpendicular to the main flow streamlines. These rolls are referred to as transverse. The possibility of the formation of longitudinal rolls was not considered in that study. In [7] a rectangular slot was considered instead of a slot; here, both longitudinal and transverse rolls can be formed. The investigation was limited by the determination of the boundaries of stability with respect to the transverse and longitudinal disturbances within the framework of the linear theory for a fixed box geometry. The critical flow parameters were determined using an iteration procedure, since under the conditions of a closed problem they control the representation of the flow itself. The authors showed that at small inclination angles the flow loses the stability against small disturbances directed along the walls, while at large angles it loses the stability with respect to those aligned with the flow.

It is well known that near the vertical position of the layer the flow loses the stability against longitudinal disturbances [8]. Thus, generally the domain of the observed secondary flow can be subdivided into

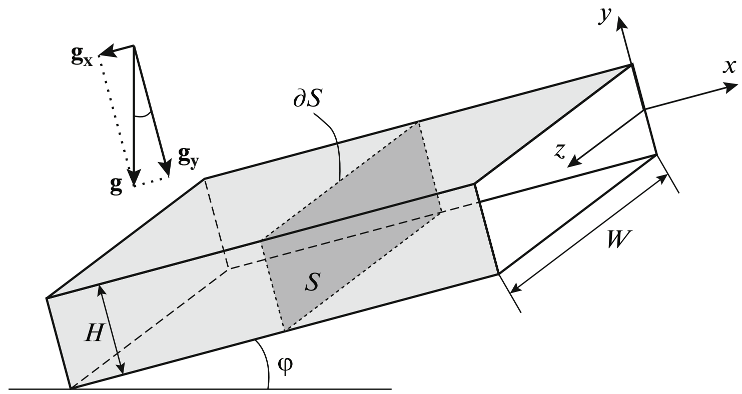

Fig. 1. Inclined rectangular channel.

three regions, whose boundaries Tare given by two inclination angles, at which the roll orientation changes. The smaller angle φ_1 determines the boundary of the interaction between the two factors described above. As noted above, the data on the value of this angle are almost lacking in the literature. The greater angle φ_2 determines the boundary between the thermal and hydrodynamic instabilities. Here, a strong dependence on the thermal properties of the medium is known, so that this angle can even become zero in the case of liquid metals [9]. The relationship between these angles obeys the inequality $\varphi_1 \le \varphi_2$.

In this study, the mutual influence of these factors on the value of the angle φ_1 and the effect of the channel width on the angle φ_2 are considered. Since the endwall effect is unimportant for this interaction, the investigation is performed for an infinitely long channel of rectangular cross-section in the presence of end partitions closing the system.

1. FORMULATION OF THE PROBLEM

We will consider the problem of thermogravitational convection in a channel (Fig. 1). The channel is inclined by an angle φ to the horizon and has a rectangular cross-section S measuring $H \times W$. There is a constant temperature difference Θ between the wall spaced apart at a distance H. In the horizontal position the lower wall is hotter. Two other walls are thermally insulated. It is assumed that the endwalls of the channel are infinitely far. This leads to a circulatory flow of the fluid in the channel. The fluid flows upward along the hotter wall and downward along the colder one. The flow intensity is determined by the angle of inclination: the greater the angle of inclination the higher the velocity. In the horizontal position the fluid is at rest. The temperature difference determines the flow stability.

This flow can be described by the dimensionless system of equations of thermal convection in the Boussinesq approximation

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\mathrm{Ra}}{\mathrm{Pr}} \mathbf{u} \nabla \mathbf{u} = -\nabla p + \nabla^2 \mathbf{u} + T\gamma, \tag{1.1}$$

$$\frac{\partial T}{\partial t} + \frac{\mathrm{Ra}}{\mathrm{Pr}} \mathbf{u} \nabla T = \frac{\nabla^2 T}{\mathrm{Pr}}, \tag{1.2}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.3}$$

where $\mathbf{u}, p, T$ are the dimensionless velocity vector and pressure and temperature deviations from their mean values. The equations are written in a rectangular coordinate system which can be conveniently fitted to the channel. The channel inclination is determined by the direction of the gravity unit vector $\gamma = (-\sin\varphi, -\cos\varphi, 0)$. The non-dimensionalizing procedure is conventional for the problem of flow stability in inclined layers, when time is normalized by the characteristic value of the momentum transport by viscosity ν between the heated walls [10]. The problem includes two geometric and two physical parameters. These are the angle φ of inclination of the channel relative to the horizontal line, the dimensionless channel width W/H, the Rayleigh number $\mathrm{Ra} = g\beta\Theta H^3/\nu\chi$, where β is the thermal expansion coefficient, and the Prandtl number $\mathrm{Pr} = \nu/\chi$, where χ is the temperature diffusion coefficient.

The values of the functions T and $\mathbf{u} = (u, v, w)$ at the solid walls are determined by the boundary conditions

$$\mathbf{u} = 0, \quad \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = \pm W/2H, \tag{1.4}$$

$$\mathbf{u} = 0, \quad T = \mp 0.5 \quad \text{at} \quad y = \pm 0.5. \tag{1.5}$$

The flow closure condition is given by zero fluid flow rate through a channel cross-section

$$\iint_S u\,dy\,dz = 0. \tag{1.6}$$

2. STABILITY ANALYSIS

The basic fluid flow in an inclined channel can be calculated from system (1.1)–(1.6) under three assumptions, namely, the flow steadiness and unidimensionality and the conductive regime of heat transfer. The experiments have shown that the last assumption is valid in the case of vertical layers and, therefore, channels of infinite length. The system obtained under these assumptions turns out to be linear

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + T \sin\varphi, \tag{2.1}$$

$$\frac{\partial p}{\partial y} = T \cos\varphi, \tag{2.2}$$

$$\frac{\partial p}{\partial z} = 0, \tag{2.3}$$

$$\frac{\partial^2 T}{\partial y^2} = 0. \tag{2.4}$$

The exact solution of system (2.1)–(2.4) can be obtained using the Fourier method in the form of a series with the known analytical expressions for its coefficients

$$u_0 = \frac{2H}{W}\sum_{k=1}^{\infty}\frac{\sin\mu_k z}{\mu_k^3}\left(\frac{\sinh\mu_k y}{\sinh\mu_k/2} - 2y\right), \quad \left(\mu_k = \frac{H(2k-1)\pi}{W}\right), \tag{2.5}$$

$$p_0 = \mathrm{const} - \frac{1}{2}y^2\cos\varphi, \tag{2.6}$$

$$T_0 = -y. \tag{2.7}$$

An analysis of the stability of flow (2.5)–(2.6) is made within the framework of the linear theory. The equations for the disturbance amplitudes are derived (see [11]) by substituting into Eqs. (1.1)–(1.5) the sum of the basic stationary solution $\mathbf{u}_0, T_0, p_0$ and the normal disturbances

$$(\tilde{\mathbf{u}}, \tilde{T}, \tilde{p}) \propto (\hat{\mathbf{u}}, \hat{T}, \hat{p})(y, z)\exp\lambda t + ikx, \tag{2.8}$$

where the hatted quantities mean the disturbance amplitudes.

It is known [12] that the basic flow instability in an inclined layer is monotonic in nature at the Prandtl numbers $\mathrm{Pr} < 12.7$. Assuming that this is valid for disturbances (2.8) we obtain that the decay rate λ is real and the stability boundary can be determined at $\lambda = 0$. In this case, the amplitude equations represent a generalized eigenvalue problem for the Rayleigh number Ra

$$\frac{\mathrm{Ra}}{\mathrm{Pr}}(\mathbf{u}_0\nabla\hat{\mathbf{u}} + \hat{\mathbf{u}}\nabla\mathbf{u}_0) + \nabla\hat{p} + \nabla^2\hat{\mathbf{u}} = \hat{T}\gamma, \tag{2.9}$$

$$\mathrm{Ra}(\mathbf{u}_0\nabla\hat{T} + \hat{\mathbf{u}}\nabla T_0) - \nabla^2\hat{T} = 0, \tag{2.10}$$

$$\nabla\cdot\hat{\mathbf{u}} = 0, \tag{2.11}$$

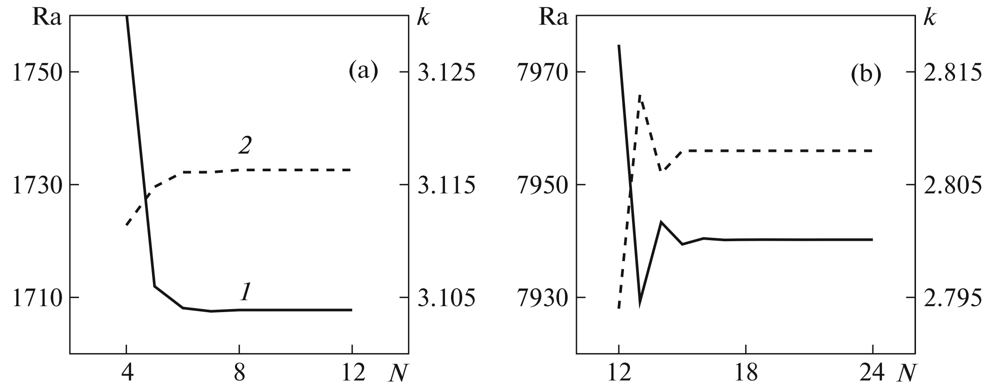

Fig. 2. Critical Rayleigh number Ra (*1*) and wavenumber k (*2*) for different numbers of collocation points normal to the isothermal surfaces for the horizontal (a) and vertical (b) layers.

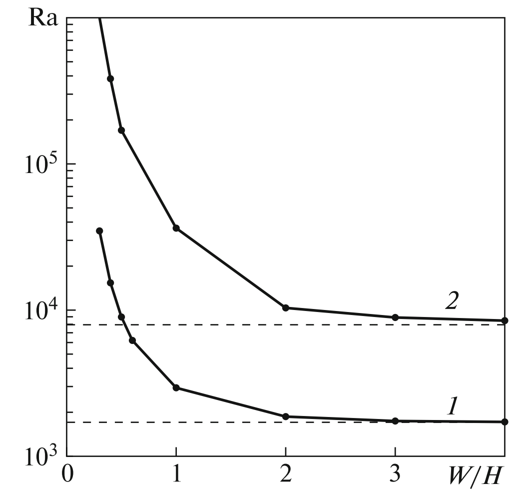

Fig. 3. Dependence of the critical Rayleigh number Ra on the width of the horizontal (*1*) and vertical (*2*) channels.

$$\nabla = ik + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \nabla^2 = -k^2 + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

with the homogeneous boundary conditions

$$\hat{\mathbf{u}} = 0, \quad \frac{\partial \hat{T}}{\partial z} = 0 \quad \text{at} \quad z = \pm W/2H, \tag{2.12}$$

$$\hat{\mathbf{u}} = 0, \quad \hat{T} = 0 \quad \text{at} \quad y = \pm 0.5. \tag{2.13}$$

The eigenvalue problem (2.9)–(2.13) is solved using the method devised by the author in [13]. The problem solution is approximated by a Lagrange polynomial constructed in the zeros of the Chebyshev polynomial and satisfying the boundary conditions for the disturbances. The polynomial coefficients are determined using the method of collocations in the zeros of the Chebyshev polynomial. This procedure was developed for analyzing the stability of flows in enclosures but can be also applied to partially unbounded regions, such as layers and channels.

The method [13] was tested against the problems of flow stability in vertical and horizontal layers. Figures 2a and 2b, presents the dependence of the critical Rayleigh number Ra and the wavenumber k on the number of approximation points between the isothermal surfaces. The method convergence is clearly visible. The critical values for the horizontal layer are in agreement with the data of book [14]. It was noticed that the number of the points ensuring the convergence is greater in the vertical than in the hori-

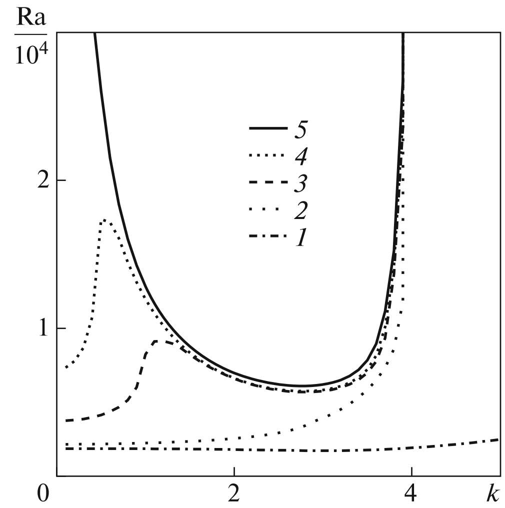

Fig. 4. Neutral curves for different values of the angle φ at $\Pr = 0.71$; *1* to *5* correspond to $\varphi = 0, 30, 60, 75, 90°$.

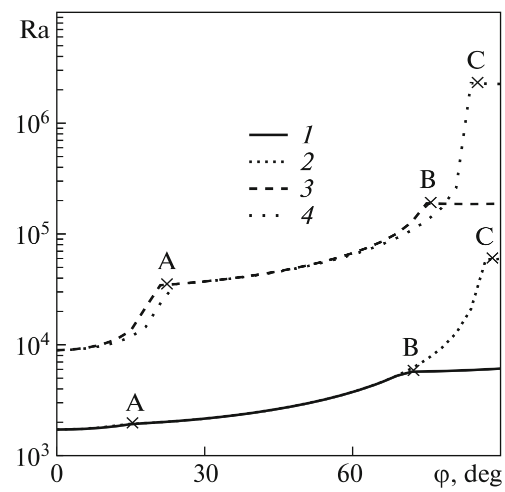

Fig. 5. Critical curves for the horizontal and transverse disturbances at two values of the ratio W/H and the Prandtl number Pr: *1*, $\Pr = 0.71$, $W/H = 4$; *2*, $\Pr = 6.7$, $W/H = 4$; *3*, $\Pr = 0.71$, $W/H = 0.5$; and *4*, $\Pr = 6.7$, $W/H = 0.5$.

zontal layer. In all subsequent calculations the chosen number of the points corresponded to the convergence in the vertical layer.

The critical Rayleigh numbers in the horizontal and vertical channels must be somewhat greater due to the effect of the side walls. This is confirmed by the results of calculations presented in Fig. 3. With increase in the channel width the critical Rayleigh number asymptotically reaches the value calculated for the layer. The corresponding asymptotes are plotted as broken lines.

3. RESULTS

The eigenvalue problem (2.9)–(2.13) for the Rayleigh numbers was solved in the ranges of the inclination angle from 0 to 90°, the channel width from 0.5 to 4, and the Prandtl number from 0.2 to 6.7.

In analyzing the flow stability with respect to the normal disturbances the main characteristic is the neutral curve separating the parameter ranges in which the flow is stable or unstable.

As distinct from an inclined layer, where it is necessary to analyze the neutral curves wit respect to longitudinal and transverse disturbances, in the problem under consideration the disturbances are periodic along one of the directions, namely, along the channel. In Fig. 4 the neutral curves are plotted for different inclination angles. The point of minimum on a neutral curve determines the critical Rayleigh number, at which the flow loses stability, and the corresponding critical wavenumber k. Depending on the angle φ, two local minima can be observable, one of which is in the region of longwave disturbances. If the global minimum occurs at $k = 0$, then it is the longitudinal disturbances that are critical; otherwise, it is the transverse disturbances. It can be noticed that both types of the critical disturbances are possible. Thus, in Fig. 4 it can be seen that for the longitudinal disturbances a minimum is observable at $\varphi = 60°$, while for the transverse ones it is observable at $\varphi = 75°$.

To determine the point, where the direction of the critical modes is changed, we have plotted the dependence of the critical Rayleigh number on the angle of inclination of the channel (Fig. 5). The curves plotted in the figure relate to different channel widths and Prandtl numbers. All the plots presented have two bends denoted by letters in the figure. The bends A correspond to the angle φ_1 and the bends B and Crelate to the angle φ_2. For these angles the systems of both longitudinal and transverse rolls canaries. According to the data obtained from the neutral curves, the basic flow loses stability with respect to the transverse disturbances to the left of φ_1 and to the right of φ_2 and to the longitudinal disturbances in between these angles. Thus, generally the critical value mentioned in the Introduction is piecewise-smooth in shape and has two discontinuity points of the derivative (see curve *3*).

In the limiting case of an infinitely wide channel [9], that is, a layer, the left bend is absent, while the position of the right bend is determined by the Prandtl number. This is also observable in the case of wide channels. The angular distance between point B on curve *1* and point C on curve *2* is approximately 15°. In the case of narrow channels the first bend remains almost fixed with variation in Pr. Point A on curves *1* and *2* means the same bend which is slightly displaced to the right in comparison with curves *3* and *4*. The latter curves correspond to narrow channels.

With decrease in the channel width the common plot is displaced upward, into the zone of large Rayleigh numbers (curves *3* and *4*). This is attributable to the locking effect of the side walls decelerating the flow due to the no-slip conditions. In this case, the left point A on curve *1* is displaced to the right by about 8°, while point B remains almost fixed (2° displacement). Thus, the channel narrowing leads to an enlargement of the range of inclination angles, in which transverse rolls can be observable, due to an increase in φ_1 as the channel becomes narrower.

At the same Prandtl number Pr = 6.7 it was noticed that in narrow channels point C is displaced to the left, while at Pr = 0.71 point B is displaced to the right. Thus, the dependence of the angle φ_2 on the channel width changes in nature for different Prandtl numbers.

It is noticed that, as distinct from the monotonic asymptotics of the Rayleigh number, the wavenumber is higher in narrow channels, reaches a minimum at $W/H \sim 3$, and then increases up to a value corresponding to the horizontal layer.

SUMMARY

The analytical solution of the problem of thermal convection in an inclined channel is constructed in the case of the conductive heat transfer mode. The linear stability of the solution obtained is analyzed. It is shown that the loss of stability can occur under the action of spatial disturbances, periodic either along of or transverse to the channel.

Two angles of inclination, at which the direction of the critical disturbances changes, are determined. It is established that with variation in the channel width the value of the smaller angle φ_1 considerably changes, whereas the greater angle φ_2 varies only slightly. The situation is opposite with variation in the Prandtl number: weak variation in the angle φ_1 and strong variation in the angle φ_2.

The results presented indicate that an investigation of convective rolls in the plane formulation of the problem in the presence of an angle of inclination is not adequate to the actual flow pattern but can be performed only with account for the effect of side walls.

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DECLARATION OF CONFLICTING INTERESTS

The Author declares no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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