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Two-Dimensional Plane Steady-State Thermocapillary Flow

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Abstract—The problem of a two-dimensional steady flow of a fluid in a flat channel with a free boundary when the surface tension coefficient depends linearly on the temperature is considered. On the channel bottom, a fixed temperature distribution is maintained. The temperature in the fluid is distributed in accordance with the quadratic law, which is consistent with the velocity field of the Xiemenz type. The arising boundary-value problem is strongly nonlinear and inverse with respect to the pressure gradient along the channel. The application of the tau-method shows that this problem has three different solutions. In the case of a thermally insulated free boundary, only one solution exists. Typical flow patterns are studied for each solution.

Key words: tau-method, free boundary, thermocapillarity, inverse problem.

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The interest in studying the influence of capillary forces on the equilibrium and motion of a fluid under low gravity is associated with the development of space technologies [1, 2]. To mention just a few, the technological applications of capillary effects include crystal growth, manufacture of composite materials with new properties in weightlessness, and obtaining high-purity metals and glasses as a result of thermocapillary deposition of droplets and bubbles of a foreign phase in space conditions. The temperature dependence of the surface tension coefficient is one of the important factors which determine the diversity of phase interface dynamics in the presence of a nonuniform temperature field.

The authors of [1–3] considered the problem of thermocapillary convection of a weightless fluid in a flat layer with a free thermally insulated surface and a heated bottom using the Navier-Stokes and thermal conduction equations when the surface tension coefficient is a quadratic function of temperature. In the case of a half-space, this problem was investigated in [4] but, unlike [1, 3], a linear temperature distribution was maintained on the free boundary.

The present study aims at finding the solutions of the steady-state problem describing a two-dimensional flow of a viscous heat-conducting fluid in an open flat channel. The flow is induced by the thermocapillary forces applied along the free surface, which cause Marangoni convection. In contrast to [1, 3, 4], the surface tension coefficient is assumed to be a linear function of temperature. Such convection may be predominant in microgravity conditions or in thin-film flows.

1. FORMULATION OF THE PROBLEM

A two-dimensional steady-state flow of a viscous heat-conducting fluid in the absence of external forces is described by the equations:

$$u_1u_{1x} + u_2u_{1y} + \frac{1}{\rho}p_x = \nu(u_{1xx} + u_{1yy}),$$

$$u_1u_{2x} + u_2u_{2y} + \frac{1}{\rho}p_y = \nu(u_{2xx} + u_{2yy}),$$

$$u_{1x} + u_{2y} = 0,$$

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$$u_1\theta_x + u_2\theta_y = \chi(\theta_{xx} + \theta_{yy}). \tag{1.1}$$

Here, $u_1(x, y)$, $u_2(x, y)$ are the velocity vector components; $p(x, y)$ is the pressure; $\theta(x, y)$ is the temperature; $\rho > 0$, $\nu > 0$, and $\chi > 0$ are constant density, kinematic viscosity, and thermal conductivity of the fluid, respectively.

Let $u_1 = w(y)x$, $u_2 = v(y)$, $p = p(x, y)$, $\theta = \theta(x, y)$ be the solution of system (1.1). This representation of the velocities is called the velocity field of the Hiemenz type [5]. The substitution of this solution in the first three of Eqs. (1.1) leads to the relations:

$$\begin{gathered} w + v_y = 0, \\ vw_y + w^2 = f + \nu w_{yy}, \\ \frac{1}{\rho}p = d(y) - \frac{fx^2}{2}, \\ d_y = \nu v_{yy} - vv_y. \end{gathered} \tag{1.2}$$

Here, f is an arbitrary constant.

The last equation in (1.1) for temperature takes the form

$$wx\theta_x + v\theta_y = \chi(\theta_{xx} + \theta_{yy}).$$

Among the solutions of this equation, there is a quadratic one with respect to the variable x:

$$\theta = a(y)x^2 + m(y)x + b(y). \tag{1.3}$$

In what follows, for simplicity, we assume that $m(y) \equiv 0$. This means that the temperature field has an extremum at $x = 0$: a maximum if $a(y) < 0$ or a minimum if $a(y) > 0$ for all $y \in [0, 1]$, including the solid wall $y = 0$. To describe the flow of a viscous heat-conducting fluid in a flat channel with a fixed solid bottom wall $y = 0$ and an upper free surface $y = l = \text{const} > 0$, we will use a solution in the form (1.2)–(1.3). Then, for $0 < y < l$ the unknown functions $w(y)$, $v(y)$, $a(y)$, and $b(y)$ satisfy the equations:

$$\begin{gathered} vw_y + w^2 = \nu w_{yy} + f, \\ w + v_y = 0, \quad vv_y + d_y = \nu v_{yy}, \\ 2wa + va_y = \chi a_{yy}, \quad vb_y = \chi b_{yy} + 2\chi a. \end{gathered} \tag{1.4}$$

It is assumed that the surface tension coefficient σ depends linearly on the temperature:

$$\sigma(\theta) = \sigma^0 - \kappa(\theta - \theta^0).$$

Here, σ^0, κ, and $\theta^0 = \text{const} > 0$. On the free boundary $y = l$, the conditions are satisfied [6]:

$$v(l) = 0, \quad w_y = -2\kappa a(l), \tag{1.5}$$

$$k\theta_y + \gamma(\theta - \theta_{\text{gas}}) = Q. \tag{1.6}$$

Conditions (1.5) follow from the kinematic and dynamic conditions, respectively. In the heat contact condition (1.6), $k > 0$ is the thermal conductivity coefficient, $Q(x)$ is the given heat flux, and $\gamma \geq 0$ is the interphase heat transfer coefficient; in what follows $\gamma = \text{const}$. From the condition for normal stresses, it turns out that the free surface remains flat. This assumption can be realized, for example, due to the action of a sufficiently large capillary pressure (the value of σ^0 is sufficiently large) [7]. In accordance with the representation of temperature (1.3), in condition (1.6), in a general case, it is necessary to assume that

$$\theta_{\text{gas}} = a_1x^2 + a_2, \quad Q = b_1x^2 + b_2.$$

Here, the constants a_k, b_k, $k = 1, 2$ are assumed to be given. Accordingly, the following conditions for $a(y)$ and $b(y)$ are satisfied on the free boundary:

$$ka_y(l) + \gamma a(l) = b_1 + \gamma a_1, \tag{1.7}$$

$$kb_y(l) + \gamma b(l) = b_2 + \gamma a_2. \tag{1.8}$$

The boundary conditions on the solid wall take the form

$$w(0) = 0, \quad v(0) = 0, \quad a(0) = a_{10}, \quad b(0) = b_{10} \tag{1.9}$$

with known constants a_{10} and b_{10}.

It is worth to note the following features of the problem formulated above. The problem is nonlinear and inverse, since the constant f is to be found. Indeed, if we eliminate $v(y)$ from the mass conservation equation, we obtain a problem for the functions $w(y)$ and $a(y)$. The problem for the function $b(y)$ (with the known $v(y)$ and $a(y)$) is separated. The function $d(y)$ is found by integration from the third equation (1.4), accurate to a constant.

Remark 1. If the solution of problem (1.4)–(1.6), (1.9) is sought in the form

$$w = \varepsilon w^{(1)} + \varepsilon^2 w^{(2)} + ..., \quad v = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + ...,$$
$$a = a^{(1)} + \varepsilon a^{(2)} + ..., \quad b = b^{(1)} + \varepsilon b^{(2)} + ..., \quad f = f^{(1)} + \varepsilon^2 f^{(2)} + ...,$$

where ε is a small parameter (Marangoni number), then substituting these expressions into the corresponding equations and boundary conditions and taking the limit as $\varepsilon \to 0$, we obtain a linear problem for $w^{(1)}$, $v^{(1)}$, $a^{(1)}$, $b^{(1)}$, and $f^{(1)}$. The solution of this problem (at small Marangoni numbers) can be interpreted as a creeping two-dimensional motion of a viscous heat-conducting fluid located on a heated substrate. The problem of two-dimensional creeping motion of two viscous heat-conducting liquids with a linear dependence of surface tension on temperature is studied in [8].

2. DERIVATION OF THE SYSTEM OF NONLINEAR ALGEBRAIC EQUATIONS

We will now give the complete formulation of the obtained nonlinear problem for the functions $w(y)$, $a(y)$, and the constant f in dimensionless form, taking into account that from the second of Eqs. (1.4) we have:

$$v(y) = -\int_0^y w(y)dy. \tag{2.1}$$

This problem formulation is as follows:

$$L_1(W, F) \equiv \mathrm{Pr} W_{\xi\xi} + W_\xi \left(\int_0^\xi W(z)dz \right) - W^2 + F = 0, \quad 0 < \xi = y/l < 1, \tag{2.2}$$

$$L_2(W, A) \equiv A_{\xi\xi} + A_\xi \left(\int_0^\xi W(z)dz \right) - 2AW = 0, \quad 0 < \xi < 1, \tag{2.3}$$

$$W(0) = 0, \quad A(0) = 1, \quad W_\xi(1) = -2\mathrm{M}A(1), \tag{2.4}$$

$$A_\xi(1) + \mathrm{Bi}A(1) = 0, \quad \int_0^1 W(z)dz = 0. \tag{2.5}$$

Here, $W(\xi) = w(y)l^2/\chi$, $A(\xi) = a(y)/a_{10}$, $F = fl^4/\chi^2$, $\mathrm{Pr} = \nu/\chi$ is the Prandtl number, $\mathrm{M} = \kappa a_{10} l^3/\chi\mu$ is the Marangoni number (see above), and $\mathrm{Bi} = \gamma l/k$ is the Biot number. The integral condition in (2.4), obtained from the first of Eqs. (1.5) with account of (2.1), is an additional condition for

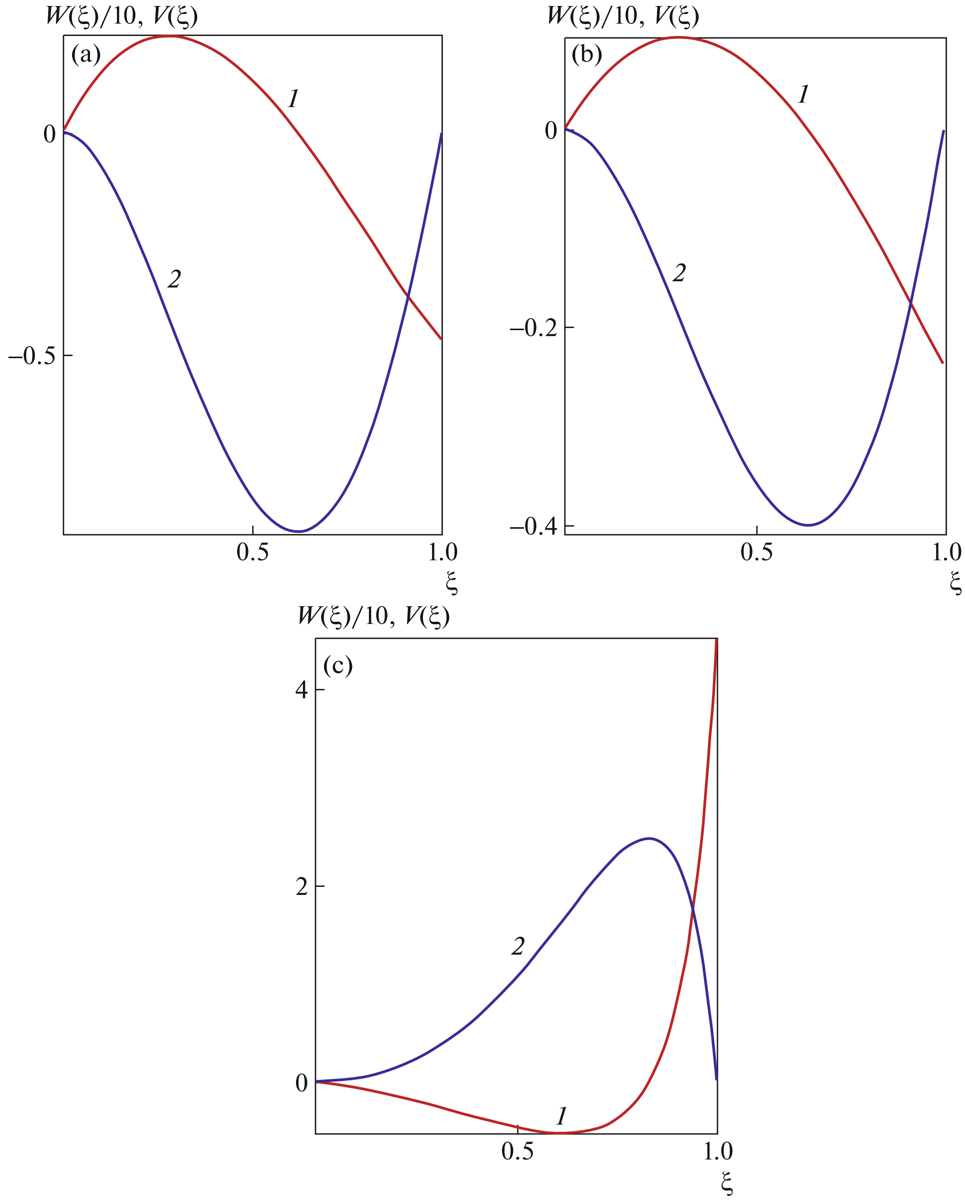

Fig. 1. Profiles of the dimensionless function $W(\xi)$ (*1*) and transverse velocity $V(\xi)$ (*2*) for F_1 (a), F_2 (b), and F_3 (c).

determining the constant F. The linear problem describing the creeping motion of a fluid (see Remark 1) has a unique nontrivial solution:

$$A_0(\xi) = -\frac{\mathrm{Bi}}{1+\mathrm{Bi}}\xi + 1, \quad W_0(\xi) = \frac{F_0}{6\mathrm{Pr}}\left(2\xi - 3\xi^2\right), \quad F_0 = \frac{3\mathrm{MPr}}{1+\mathrm{Bi}}. \tag{2.6}$$

In the papers [1, 3], the problem of fluid flow in a flat channel with a given temperature distribution on the bottom and the thermally insulated free surface ($\mathrm{Bi} = 0$) was considered. As a result of the separation of variables, a nonlinear two-point boundary-value problem was obtained. This problem describes the motion of a fluid in a layer where the constant F plays the role of an eigenvalue and the Prandtl and Marangoni numbers are parameters. The non-uniqueness of the solution of this problem (from one to three solutions) depending on the parameter M was established ($\mathrm{Pr} = 0$, i.e. the limiting case of a perfectly heat-conducting fluid was considered). In the present study, to solve the problem (2.2)–(2.5), we use the tau-method, which is a modification of the Galerkin method [9]. The approximate solution is

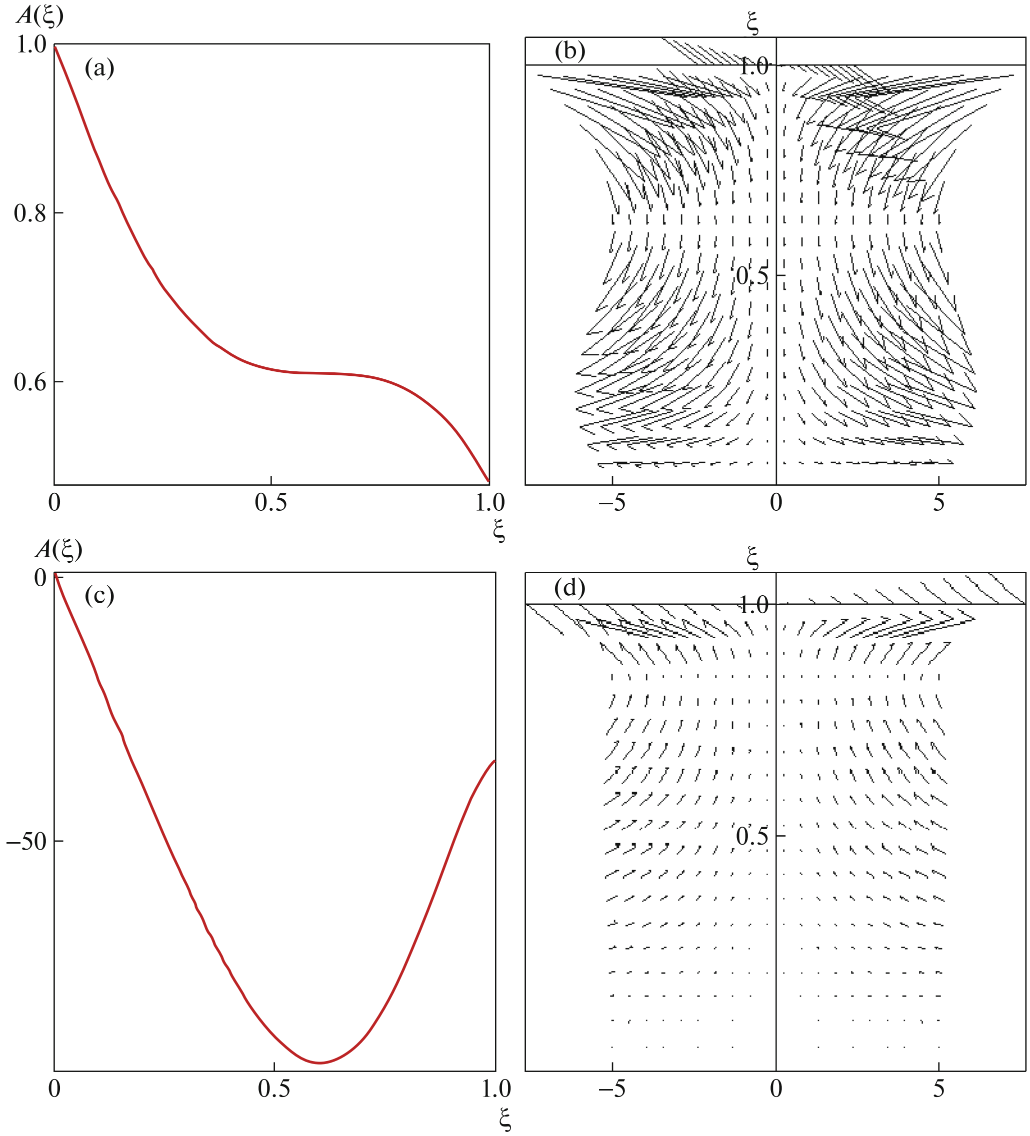

Fig. 2. Profile of the dimensionless function $A(\xi)$ and the streamlines in the layer for F_1 (a, b) and F_3 (c, d).

sought in the form of sums

$$W_n(\xi) = \sum_{k=0}^{n+1} W^k R_k(\xi), \quad A_n(\xi) = \sum_{k=0}^{n+1} A^k R_k(\xi). \tag{2.7}$$

Here, $R_k(\xi)$ are shifted Legendre polynomials. The unknown coefficients W^k, A^k and the constant F are found from the system of Galerkin approximations

$$\int_0^1 L_1(W_n, F) R_m(\xi) d\xi = 0, \quad \int_0^1 L_2(W_n, A_n) R_m(\xi) d\xi = 0, \quad m = 0, ..., n-1 \tag{2.8}$$

and transformed boundary conditions (2.4), (2.5)

$$\sum_{k=0}^{n+1} (-1)^k W^k = 0, \quad \sum_{k=0}^{n+1} (-1)^k A^k = 1, \quad \sum_{k=0}^{n+1} W^k R_k{}'(1) = -2\mathrm{M} \sum_{k=0}^{n+1} A^k,$$

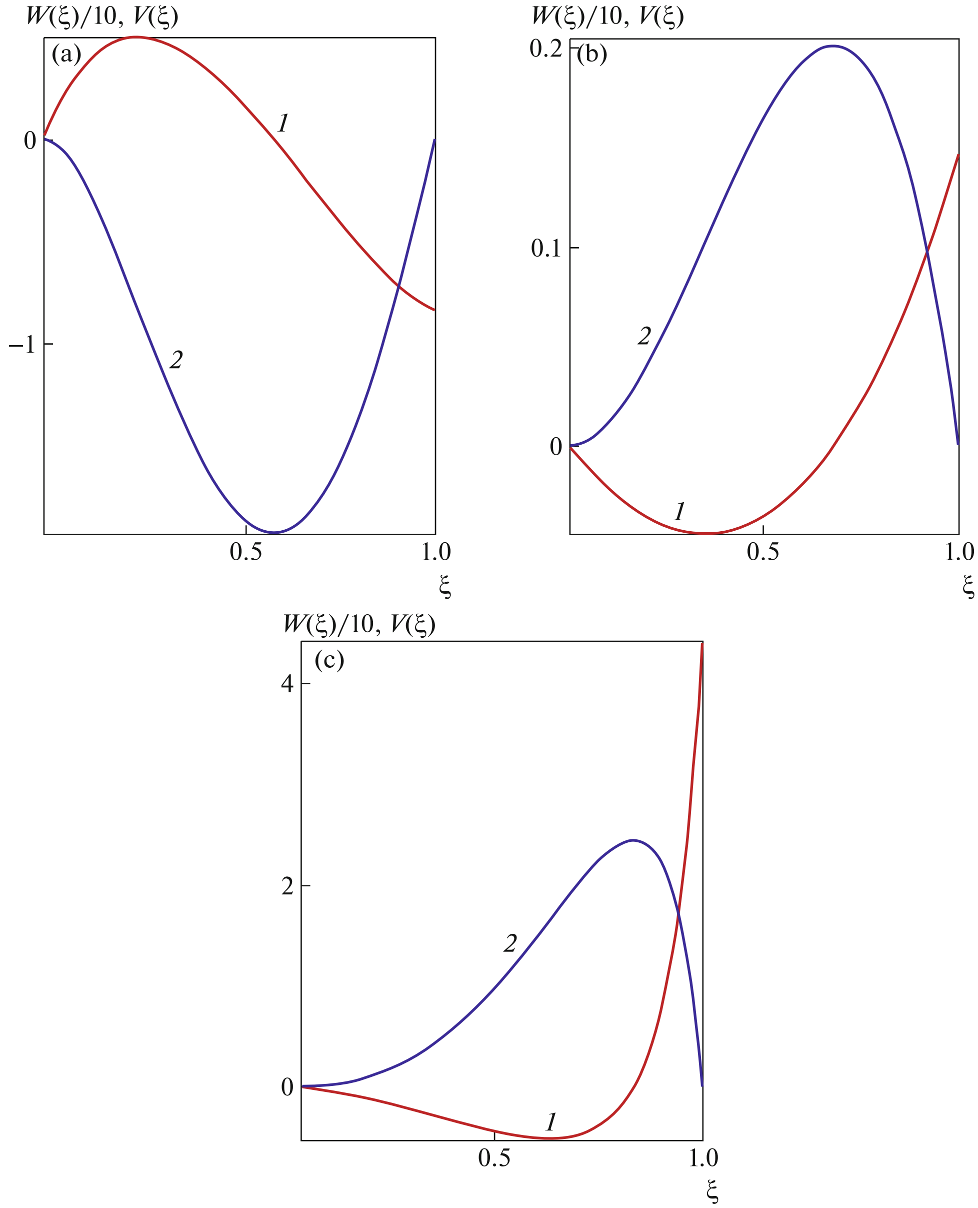

Fig. 3. Profiles of the dimensionless function $W(\xi)$ (*1*) and transverse velocity $V(\xi)$ (*2*) for F_1 (a), F_2 (b), and F_3 (c), $\mathrm{M} = -10$.

$$\sum_{k=0}^{n+1} A^k {R_k}'(1) + \mathrm{Bi} \sum_{k=0}^{n+1} A^k = 0, \quad W^0 = 0. \tag{2.9}$$

The last of Eqs. (2.9) is obtained from the integral condition (2.5), taking into account the orthogonality of the Legendre polynomials on the interval $[0, 1]$ with weight 1 [10]. Thus, Eqs. (2.8) and (2.9) form a closed system of nonlinear algebraic equations for the coefficients W^k, A^k, and the constant F.

3. NUMERICAL RESULTS

The calculations were carried out for $\mathrm{Pr} = 0.2$, $\mathrm{Bi} = 2$, $\mathrm{M} = 10$ ($a_{10} > 0$, i.e., the temperature at the point $x = 0$, $y = 0$ was minimal) and $n = 17$. Three different values of the dimensionless constant F were found: $F_1 = 14.1397$, $F_2 = 4.5359$, and $F_3 = 4.4877$. The difference between the values obtained for $n = 16$ and 17 is of the order 10^{-11}, 10^{-14}, and 10^{-6} for F_1, F_2, and F_3, respectively. This indicates a

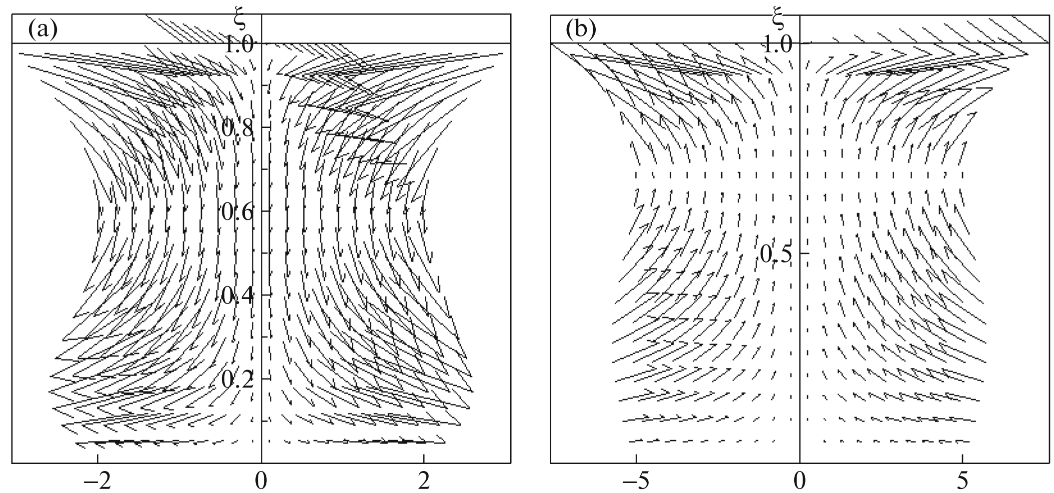

Fig. 4. Streamlines in the layer for F_1 (a) and F_2 (b), $\mathrm{M} = -10$.

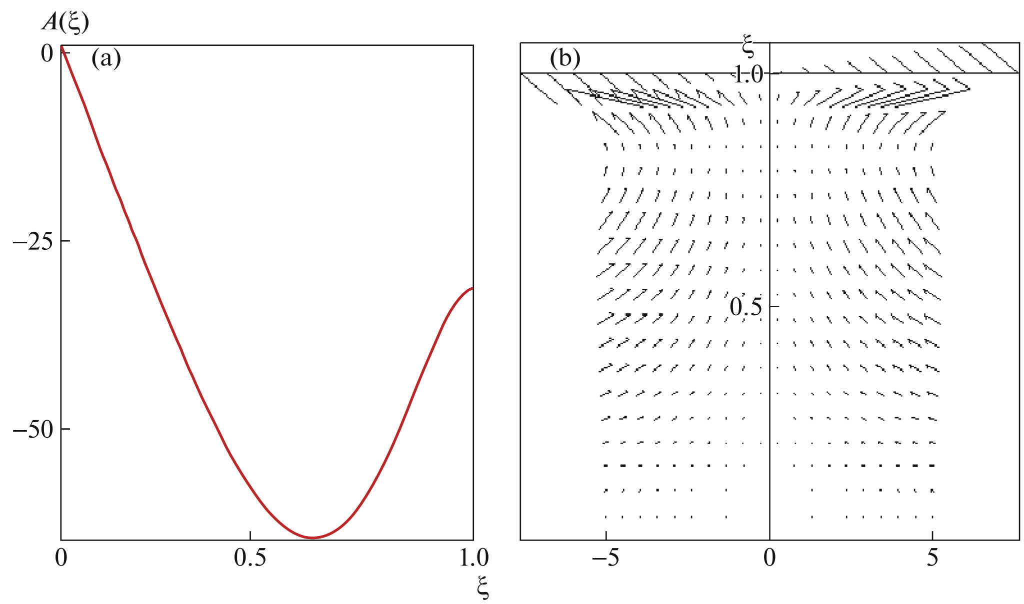

Fig. 5. Profile of the dimensionless function $A(\xi)$ and the streamlines in the layer for $F = 3.97$ and $\mathrm{M} = -10$.

good convergence of the τ-method in solving this boundary-value problem. It is also worth noting that for $\mathrm{M} \ll 1$ the solutions tend to the unique solution of the linear problem (2.6) describing the creeping motion in the layer. For example, for $\mathrm{M} = 0.01$ we found that $|F_0 - F_{1,2,3}| \approx 10^{-6}$. Figure 1 shows the profiles of the dimensionless function $W(\xi)$ and the transverse velocity $V(\xi)$ (2.1) for the values F_1, F_2, and F_3, respectively. The profiles for F_1 and F_2 are similar, but it should be noted that the flow corresponding to the parameter F_1 is more intense, thus $\max\limits_{\xi\in[0,1]} |W(\xi, F_1)| = 4.65$, $\max\limits_{\xi\in[0,1]} |V(\xi, F_1)| = 0.9$, and $\max\limits_{\xi\in[0,1]} |W(\xi, F_2)| = 2.37$, $\max\limits_{\xi\in[0,1]} |V(\xi, F_2)| = 0.4$. Figure 2 shows the profile of the function $A(\xi)$ and the velocity field for F_1 and F_3. In the first case, the function $A(\xi)$ on the free boundary $\xi = 1$ is positive, hence the temperature at $x = 0$ is minimal and increases in the direction of the x-axis. Since the fluid travels in the direction of larger surface tension, near the free surface a recirculated flow zone shown in Fig. 2a arises. In the second case, $A(1) < 0$ and the temperature at $x = 0$ attains a maximum. Accordingly, near the free surface the fluid travels towards the x-axis (Fig. 2b). It is clear that in both cases a more intense motion is formed near the free surface $\xi = 1$.

In the case when $\mathrm{M} = -10$ ($a_{10} < 0$ and the temperature at $x = 0$, $y = 0$ is maximal), for the parameter F we obtained the values $F_1 = 50.08$, $F_2 = -1.3368$, and $F_3 = 4.271$. In Fig. 3, we present the profiles of the dimensionless function $W(\xi)$ and the transverse velocity $V(\xi)$ for the values F_1, F_2, and F_3, respectively. The profiles for $F = 4.271$ are similar to those for $\mathrm{M} = 10$, $F = 4.4877$ (see Fig. 1c); therewith $\max\limits_{\xi\in[0,1]} |W(\xi, F = 4.4877)| = 43.962$, $\max\limits_{\xi\in[0,1]} |V(\xi, F = 4.4877)| = 2.44$, and $\max\limits_{\xi\in[0,1]} |W(\xi, F = 4.271)| = 45.174$, $\max\limits_{\xi\in[0,1]} |V(\xi, F = 4.271)| = 2.476$.

Figure 4 shows the streamlines in the layer for F_1 and F_2. It is clear that the flow corresponding to the parameter Z_1 is the most intense one. In both cases, the most intense motion is formed near the free boundary $\xi = 1$.

In the case of a thermally isolated free boundary ($\mathrm{Bi} = 0$), we obtained the single value of the dimensionless constant $F = 3.97$. For small Marangoni numbers, this solution also tends to the unique solution of the linear problem (2.6). Figure 5 shows the profile of the variable function $A(\xi)$ and the streamlines in the layer. Since $A(1) < 0$, near the free surface the fluid moves in the direction of the x-axis. It is clear that the most intense flow is formed near $\xi = 1$, i.e., the free boundary.

We should also comment on the influence of the governing parameters on the intensity of arising flows: with increase in the Marangoni number M, the flow velocity increases, and with increase in the Prandtl number Pr the velocity decreases.

Remark 2. To evaluate the accuracy of the solutions, one can use the following relations:

$$W(1)W'(1) - \int_0^1 (W')^2 d\xi - \frac{3}{2\mathrm{Pr}} \int_0^1 W^3 d\xi = 0$$

$$\mathrm{Pr}\left[W'(0) - W'(1)\right] + 2\int_0^1 W^2 d\xi - F = 0. \qquad (3.1)$$

The first relation in (3.1) is obtained by the multiplication of Eq. (2.2) by $W(\xi)$ and the integration over $\xi \in [0.1]$ with account of the first condition (2.4) and the integral condition (2.5). The second relation follows from the integration of Eq. (2.2) over the domain of definition. Thus, substituting the solutions obtained for all the cases considered above in equalities (3.1), we obtained that these equalities are satisfied to an accuracy of about 10^{-10} and 10^{-50}, respectively.

SUMMARY

The problem of two-dimensional steady-state fluid flow in a flat channel with a free boundary is studied in the case when along the free boundary the surface tension coefficient is linearly dependent on temperature and on the bottom a given temperature distribution is maintained. The non-uniqueness of the solution of this problem is established: for $\mathrm{Bi} \neq 0$ three different solutions are found, and for $\mathrm{Bi} = 0$ only one solution exists. All solutions found for small Marangoni numbers tend to a unique nontrivial solution of a linear problem describing the creeping motion of a fluid in an open channel. For each solution, the typical flow patterns are constructed.

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