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Problem of Deep Bed Filtration in a Porous Medium with the Initial Deposit

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Abstract—The macroscopic model of long-term deep-bed filtration flow of a monodisperse suspension through a porous medium with size-exclusion particle-capture mechanism and without retained-particle mobilization is considered. It is assumed that the pore accessibility and the fractional particle flux depend on the deposit concentration and at the initial time the porous medium contains a nonuniformly distributed deposit. The aim of the study is to find the analytical solution in the neighborhood of a mobile curvilinear boundary, namely, of the suspended-particle concentration front. The property of having fixed sign is proved for the solution. The exact solution of the filtration problem on the curvilinear front is found in explicit form. The sufficient condition of existence of the solution on the concentration front is obtained. An asymptotic solution is constructed in the neighborhood of the front. The time interval of applicability of asymptotics is determined from the numerical solution.

Keywords: deep bed filtration, porous medium, suspended and retained particles, concentration front, analytical solution.

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The investigation of filtration of a suspension in a porous medium is an important problem for many fields of science and technology. The filtration process is accompanied by formation of a deposit in pores which changes the structure and properties of the porous medium [1]. This leads to decrease in the well productivity in oil and gas recovery, industrial filter clogging, deterioration of portable and sewage water purification, bacteria and viruses transport through aquifers, and soil salinization [2–7].

Certain particles precipitate on the porous medium frame during particle transport by fluid flow. The mechanical interaction, diffusion, viscosity, and the electrostatic and gravity forces can play a significant role in the particle retention depending on the properties of suspension and porous medium [8–11]. If the particle and pore sizes are of the same order, then in many cases the size-exclusion particle-capture mechanism is predominant. Solid particles pass freely through the pores of large diameters and are blocked in the pores whose diameters are less than the particle dimension [12]. Complex topology of porous channels and variable dimensions of the pore holes lead to the fact that particles may be captured in the places of pore narrowing far from the filter inlet. In the case of long-term deep-bed filtration the deposit is formed over the entire porous medium but not only in its surface layer [13–15]. With the injection of a suspension of constant concentration, certain particles are transported by fluid flow throughout the porous medium, while the other are captured in narrow pores and form the deposit (Fig. 1).

The traditional mathematical model of one-dimensional deep-bed filtration of incompressible monodisperse suspension in the porous medium with the size-exclusion particle-capture mechanism relates the suspended and retained particle concentrations by a system of two partial differential equations. The mass balance equation for the suspended and retained particles is an analog of the continuity equation; the kinetic equation determines growth in the deposit concentration [16]. More

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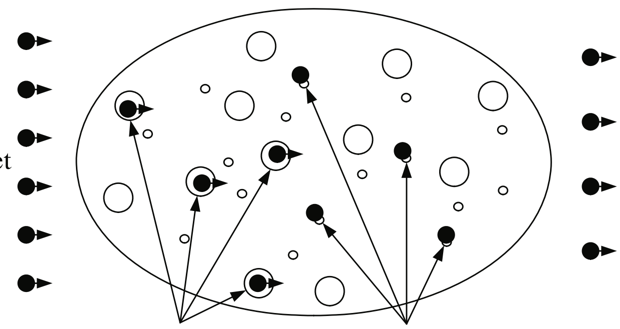

Fig. 1. Cross-section of a porous medium with suspended and retained particles.

complex filtration models constructed for particles and pores of various dimensions on the basis of balance between the suspended and deposited particles were described in [17–22].

The exact solutions of the problem of filtration with pores completely accessible for the suspended particles were obtained in [23, 24]; in this case the characteristics of the system are straight lines.

Introduction of the pore accessibility function and the particle flow through accessible pores (fractional flux factor) leads to curvilinear characteristics. The exact solutions become much more complex [25, 26]. The asymptotic solutions, in which the distance from the concentration front is used as a small parameter, cover the time ranges of the filtration processes occurring in the laboratory and field studies [18].

The models of filtration with a clean porous frame assume that at the initial instant there are no suspended and retained particles in the porous medium [23–26]. In the present study the more generic case is considered, namely, the porous medium contains a deposit and pure water before the beginning of filtration. The model considers suspension filtration in alternating the suspension flow and the reverse pure water flow. The deposit is accumulated in the porous medium during filtration of forward suspension flow. Mobilization and washing-out of retained particles take place in reverse water flow. Such problem appear in the oil-producing industry and the industrial filter maintenance [4, 6, 27, 28].

In what follows, we will assume that incomplete particle mobilization takes place in the displacement of suspension by pure water which moves in the opposite direction and the deposit remains partially in the porous medium. The initial nonuniformly-distributed deposit affects the filtration process of forward suspension flow. In [29] the numerical solution of the problem with initial deposit was obtained. However, so far there are no analytical solutions of this problem.

In the present study filtration of a suspension which displaces pure water from the porous medium with initial deposit is considered. The moving boundary is the concentration front of suspended particles in the suspension. The dependence of the initial deposit on the coordinate in the model which takes into account the dependence of the porosity and the fractional flux on the accumulated deposit leads to the curvilinear boundary. The exact solution of the problem on the concentration front is obtained for the initial deposit nonuniformly distributed in the porous medium. The asymptotic solution is constructed in the neighborhood of the concentration front; its principal term coincides with the exact solution on the front. A similar solution of the problem without initial deposit in the presence of the rectilinear boundary was obtained in [18] and the asymptotics were constructed for a small deposit in [30].

1. MATHEMATICAL MODEL. GENERAL PROPERTIES OF THE SOLUTION

The model supposes insignificant diffusion/dispersion. Small concentrations of suspended and confined particles have no effect on the volume balance of general flow [16, 17, 28]. The high particle concentrations obey the Amagata law of additivity of specific volumes in mixing (see [17, 28]):

$$\rho(c) = c\rho_r + (1 - c)\rho_w,$$

where ρ, ρ_w, and ρ_r are the brine, water, and rock densities, respectively. The rock and solvent components are incompressible and, consequently, ρ_w and ρ_r are constant.

The filtration equation can be applied to capture of particles in small pore throats and confinement of fine particles [3–5, 7–11, 13–16]. The model uses only the size-exclusion capture mechanism for the individual particles and rules out formation of arched bridges by several particles at the entry in large pores. The deposit formation rate is proportional to the advective particle flux. It is assumed that the retained particles do not separate from the porous medium frame. Other assumptions include incompressibility of any suspension component, namely, both the suspended and retained particles and water. The fluid that transports the particles is Newtonian.

It is assumed that flow is single-phase and the suspension injected in the porous medium contains the same water as the reservoir water initially saturating the rock. The linear one-dimensional suspension flow which corresponds to the laboratory investigations or brine injections in broken wells is considered.

From the above-mentioned assumptions there follows the system of equations of long-term deep-bed filtration which consists of the balance equation for the suspended- and retained-particle concentrations c and σ, the kinetic deposit growth equation, and Darcy's law:

$$\frac{\partial(\varphi_a(\sigma)c)}{\partial T} + U\frac{\partial(f_a(\sigma)c)}{\partial X} + \frac{\partial\sigma}{\partial T} = 0, \tag{1.1}$$

$$\frac{\partial\sigma}{\partial T} = \frac{1}{l}f_a(\sigma)f_n(\sigma)Uc, \tag{1.2}$$

$$U = -\frac{k(\sigma)}{\mu}\frac{\partial P}{\partial X}. \tag{1.3}$$

Here, φ_a is the admissible porosity, f_a is the fractional flow through the accessible pores, $f_n = 1 - f_a$ is the fractional fluid flow through the unaccessible pores, l is the characteristic microlength of the porous medium, U is the flow velocity, k is the permeability coefficient, μ is the dynamic viscosity, P is the pressure. The system is considered in the domain $\{(X,T):\ 0 < X < L,\ T > 0\}$, where L is the length of the porous medium.

The detailed derivation of the macroscopic system (1.1)–(1.3) from the microlevel equations is given in [13–15, 17]. In [1, 3–5, 7–9, 11, 16] this system is derived phenomenologically.

In what follows, we will solve the problem using a given and constant injection rate U. Due to incompressibility of general flow $U(X,T) = \text{const}$. In this case equations (1.1) and (1.2) form the closed system.

We introduce the dimensionless variables

$$x = \frac{X}{L},\quad t = \frac{UT}{\varphi L},\quad C = \frac{c}{c^0},\quad S = \frac{\sigma}{\varphi c^0},\quad g(S) = \frac{\varphi_a(\varphi c^0 S)}{\varphi},$$

$$f(S) = f_a(\varphi c^0 S),\quad \Lambda(S) = \frac{L}{l}f_a f_n(\varphi c^0 S),$$

where φ is the porosity and c^0 is the concentration of the suspended particles of the injected suspension.

Equations (1.1) and (1.2) take the form:

$$\frac{\partial(g(S)C)}{\partial t} + \frac{\partial(f(S)C)}{\partial x} + \frac{\partial S}{\partial t} = 0, \tag{1.4}$$

$$\frac{\partial S}{\partial t} = \Lambda(S)C. \tag{1.5}$$

Here, $C(x,t)$ and $S(x,t)$ are the dimensionless concentrations of the suspended and retained particles; the accessibility factor $g(S)$, the accessible fractional flux factor $f(S)$, and the filtration coefficient $\Lambda(S)$ are given positive continuous differentiable functions.

It is assumed that the suspension with a constant suspended-particle concentration is injected to the filter inlet $x = 0$; at the initial time $t = 0$ there are no suspended particles in the porous medium and there is a deposit $s_0(x)$ unevenly distributed over the filter. The corresponding initial and boundary conditions take the form:

$$C\big|_{x=0} = 1, \tag{1.6}$$

$$C\big|_{t=0} = 0, \tag{1.7}$$

$$S\big|_{t=0} = s_0(x), \tag{1.8}$$

where $s_0(x)$ is a nonnegative continuous differentiable function.

The conditions (1.6)–(1.8) determine the unique solution of the problem in the domain $\Omega = \{(x,t):\ 0 < x < 1,\ t > 0\}$. The solution has a discontinuity since the conditions (1.6) and (1.7) are not adjusted in the origin. In what follows, it will be shown that the curve of discontinuity Γ is specified by the equation

$$t_\Gamma(x) = \int\limits_0^x \frac{g(s_0(x))}{f(s_0(x))}\, dx. \tag{1.9}$$

The curve Γ divides the domain Ω into two subdomains

$$\Omega_0 = \{(x,t):\ 0 < x < 1,\ 0 < t < t_\Gamma(x)\}, \quad \Omega_S = \{(x,t):\ 0 < x < 1,\ t > t_\Gamma(x)\}.$$

Suspension filtration takes place in the domain Ω_S; there are no suspended particles in the domain Ω_0 and the retained particle concentration is independent of time. The suspended-particle concentration front propagates through the porous medium along the curve Γ with the velocity

$$v = \frac{f(s_0(x))}{g(s_0(x))}. \tag{1.10}$$

Theorem 1. The solution of the problem (1.4)–(1.8) is

1) constant in the domain Ω_0: $C(x,t)\big|_{(x,t)\in\Omega_0} = 0$ and $S(x,t)\big|_{(x,t)\in\Omega_0} = s_0(x)$;

2) positive in the domain Ω_S: $C(x,t)\big|_{(x,t)\in\Omega_0} > 0$ and $S(x,t)\big|_{(x,t)\in\Omega_0} > s_0(x)$.

Proof. Using (1.5), we can write Eq. (1.4) in the form:

$$g(S)\frac{\partial C}{\partial t} + f(S)\frac{\partial C}{\partial x} + g'(S)\frac{\partial S}{\partial t}C + f'(S)\frac{\partial S}{\partial x}C + \Lambda(S)C = 0. \tag{1.11}$$

The characteristic system corresponding to Eq. (1.11) is written in the form:

$$\dot t = g(S), \quad \dot x = f(S), \tag{1.12}$$

$$\dot C + \left(g'(S)\frac{\partial S}{\partial t} + f'(S)\frac{\partial S}{\partial x} + \Lambda(S)\right) C = 0, \tag{1.13}$$

where the dot denotes differentiation with respect to the parameter τ along the characteristic.

We will consider two families of the characteristics $t(\tau)$, $x(\tau)$, $C(\tau)$. In the domain Ω_0 (ahead of the concentration front) the characteristics go out from the points $(x_0, 0)$ on the coordinate axis OX. For the system (1.12), (1.13) the initial conditions can be specified as follows:

$$t(0) = 0, \quad x(0) = x_0, \quad C(0) = 0. \tag{1.14}$$

In the domain Ω_S (behind the concentration front) the characteristics go out from the points $(0, t_0)$ on the time axis. We specify the conditions

$$t(0) = t_0, \quad x(0) = 0, \quad C(0) = 1. \tag{1.15}$$

a) In the domain Ω_0 equation (1.13) with condition (1.14) has the solution $C(\tau) = 0$. Then from Eq. (1.5) there follows $\partial S/\partial t = 0$ and $S(x,t) = s_0(x)$. In the plane (x,t) equations (1.12) with conditions (1.14) determine the characteristics

$$t = \int\limits_0^\tau g(s_0(x))d\tau = \int\limits_{x_0}^x \frac{g(s_0(x))}{f(s_0(x))}dx. \tag{1.16}$$

The characteristic (1.16) taken at $x_0 = 0$, which coincides with the curve Γ specified by the relation (1.9), is the boundary of the domain Ω_0.

b) In the domain Ω_S the solution of the problem (1.13), (1.15) can be written in the form:

$$C = e^{-\int\limits_0^\tau (g'(S)\partial S/\partial t + f'(S)\partial S/\partial x + \Lambda(S))d\tau}. \tag{1.17}$$

From (1.17) there follows the inequality $C(\tau) > 0$. Then from Eq. (1.15) we obtain

$$\frac{\partial S}{\partial t} > 0 \tag{1.18}$$

and $S(x,t) > s_0(x)$. Theorem 1 is proved.

As will be shown below, the system of partial differential equations (1.4), (1.5) can be reduced to a first-order ordinary differential equation for the deposit concentration $S(x,t)$.

In fact, the expression for $C(x,t)$ obtained from Eq. (1.5)

$$C = \frac{\partial S/\partial t}{\Lambda(S)} \tag{1.19}$$

after its substitution in Eq. (1.4) yields

$$\frac{\partial\left(g(S)\dfrac{\partial S/\partial t}{\Lambda(S)}\right)}{\partial t} + \frac{\partial\left(f(S)\dfrac{\partial S/\partial t}{\Lambda(S)}\right)}{\partial x} + \frac{\partial S}{\partial t} = 0. \tag{1.20}$$

We introduce the notation

$$a(S) = \frac{g(S)}{\Lambda(S)}, \quad b(S) = \frac{f(S)}{\Lambda(S)}.$$

Then

$$\frac{\partial\big(b(S)\partial S/\partial t\big)}{\partial x} = b'(S)\frac{\partial S}{\partial x}\frac{\partial S}{\partial t} + b(S)\frac{\partial^2 S}{\partial t\partial x} = \frac{\partial\big(b(S)\partial S/\partial x\big)}{\partial t}. \tag{1.21}$$

Substitution of (1.21) in Eq. (1.20) leads to the equation

$$\frac{\partial\left(a(S)\partial S/\partial t\right)}{\partial t} + \frac{\partial\left(b(S)\partial S/\partial x\right)}{\partial t} + \frac{\partial S}{\partial t} = 0. \tag{1.22}$$

Integration of both sides of Eq. (1.22) with respect to the variable t yields

$$a(S)\frac{\partial S}{\partial t} + b(S)\frac{\partial S}{\partial x} + S = K(x). \tag{1.23}$$

The integration function $K(x)$ can be determined from the conditions (1.7) and (1.8) at $t = 0$. We obtain

$$K(x) = b(s_0)s_0' + s_0. \tag{1.24}$$

Using (1.24), we can write Eq. (1.23) in the form:

$$a(S)\frac{\partial S}{\partial t} + b(S)\frac{\partial S}{\partial x} + (S - s_0 - b(s_0)s_0') = 0. \tag{1.25}$$

For uniqueness of the solution of Eq. (1.25) it is necessary to impose a condition at $x = 0$. From the condition (1.6) it follows that at $x = 0$ equation (1.5) takes the form:

$$\frac{\partial S}{\partial t} = \Lambda(S). \tag{1.26}$$

Dividing both sides of Eq. (1.26) by $\Lambda(S)$ and integrating it with respect to the variable t over the interval $[0,t]$ with the initial condition (1.8), we obtain

$$\int\limits_{s_0(0)}^{S^{(0)}(t)} \frac{dS}{\Lambda(S)} = t; \quad S^{(0)}(t) = S(x,t)\big|_{x=0}. \tag{1.27}$$

Thus, the deposit concentration $S(x,t)$ must satisfy Eq. (1.25) with the condition (1.27).

Consequence 1 from Theorem 1. Let $s_0'(x) \leq 0$. Then in the domain Ω_S

$$\frac{\partial S}{\partial x} < 0. \tag{1.28}$$

Proof. We will consider the terms on the left-hand side of Eq. (1.25). In accordance with Theorem 1 and inequality (1.18), the expressions $a(S)\dfrac{\partial S}{\partial t}$ and $S - s_0$ are positive in the domain Ω_S. Since $s_0'(x) \leq 0$, then the inequality (1.28) follows from Eq. (1.25). Consequence 1 is proved.

In accordance with Consequence 1, the deposit concentration decreases with increase in the x coordinate, if the initial deposit $s_0(x)$ does not increase with x.

In what follows, the problem (1.4)–(1.8) will be considered on the boundary Γ and in its neighborhood in the domain Ω_S.

2. EXACT SOLUTION ON THE CONCENTRATION FRONT

We will determine the suspended-particle concentration on the front Γ. For this purpose it is necessary to express the value of the partial derivative $\partial S/\partial x$ on Γ. It should be noted that when $(x,t) \in \Omega_S$ the limiting value of $\partial S/\partial x|_\Gamma$ is not equal to $s_0'(x)$ since the derivatives of the function $S(x,t)$ are discontinuous on the boundary.

Statement 1. In the domain Ω_S the solution $S(x,t)$ of the problem (1.4)–(1.8) must satisfy the relation

$$\left.\frac{\partial S}{\partial x}\right|_\Gamma = s_0'(x) - \frac{g\big(s_0(x)\big)}{f\big(s_0(x)\big)}\frac{\partial S}{\partial t}. \tag{2.1}$$

Proof. Let **l** be the tangential vector to curve Γ at a certain point $M(x,t)$. The partial derivative of the function $S(x,t)$ in the direction l is equal to

$$\frac{\partial S}{\partial l} = \frac{\partial S}{\partial t}\sin\varphi + \frac{\partial S}{\partial x}\cos\varphi, \tag{2.2}$$

where φ is the angle between the vector **l** and the *OX* axis (Fig. 2).

On the other hand, by definition

$$\frac{\partial S}{\partial l} = \lim_{\substack{M_1\to M\\ MM_1\in l}} \frac{S(M_1) - S(M)}{|M_1M|}. \tag{2.3}$$

Denote $a = |MM_x| = |MM_1|\cos\varphi$. If the curve of the boundary Γ is convex downwards on the segment $(x, x+a)$, then the point $M_1(x+a, t+\Delta t) \in \Omega_0$ (Fig. 2a). Since the concentration $S(x,t) = s_0(x)$ in the domain Ω_0, then

$$S(M_1) = s_0(x+a). \tag{2.4}$$

If the curve of the boundary Γ is convex upwards on the segment $(x, x+a)$, then the point $M_1 \in \Omega_S$ (Fig. 2b) and $|M_1M_2| = o(a)$ by definition of the tangent. (The function $\beta = o(a)$ if $\lim\limits_{a\to 0}\beta/a = 0$.)

From Eq. (1.5) it follows that the derivative $\partial S/\partial t$ is continuous and the estimate holds

$$S(M_1) = S(M_2) + o(a) = s_0(x+a) + o(a). \tag{2.5}$$

Substituting (2.4) and (2.5) in the limit (2.3), in both cases we obtain

$$\frac{\partial S}{\partial l} = s_0'(x)\cos\varphi. \tag{2.6}$$

From equating (2.2) and (2.6) there follows the expression for the derivative $\partial S/\partial x$

$$\left.\frac{\partial S}{\partial x}\right|_\Gamma = s_0'(x) - \frac{\partial S}{\partial t}\tan\varphi. \tag{2.7}$$

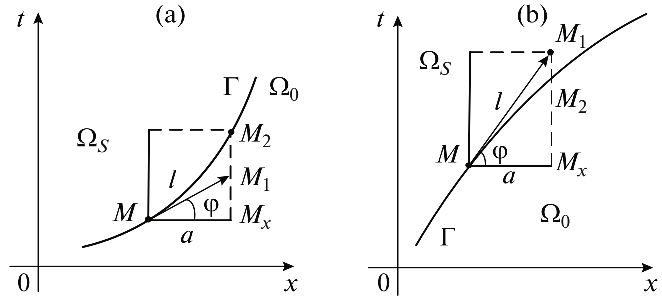

Fig. 2. Convexity of the boundary Γ downwards (a) and upwards (b).

From (1.9) we have

$$\tan\varphi = \left.\frac{dt}{dx}\right|_{\Gamma} = \frac{g\big(s_0(x)\big)}{f\big(s_0(x)\big)}. \tag{2.8}$$

Statement 1 follows from formulas (2.7) and (2.8).

Theorem 2. On the concentration front Γ the suspended-particle concentration takes the form:

$$C(x,t)\big|_{t=t_\Gamma(x)} = \frac{u(x)}{1+v(x)}, \tag{2.9}$$

where

$$u(x) = \frac{f(s_0(0))}{f(s_0(x))} e^{-\int\limits_0^x \frac{\Lambda(s_0(z))}{f(s_0(z))}\,dz}, \quad v(x) = \int\limits_0^x \left(\frac{g\big(s_0(y)\big)}{f\big(s_0(y)\big)}\right)'_S \Lambda(s_0(y))u(y)\,dy. \tag{2.10}$$

Proof. Using (2.1) and (1.5), equation (1.11) on the boundary Γ can be represented in the form:

$$g(s_0)\frac{\partial C}{\partial t} + f(s_0)\frac{\partial C}{\partial x} + \big(f'(s_0)s_0'(x) + \Lambda(s_0)\big)C + \left(g'(s_0) - f'(s_0)\frac{g(s_0)}{f(s_0)}\right)\Lambda(s_0)C^2 = 0. \tag{2.11}$$

The change of variables $\tau = t - t_\Gamma(x)$, $x = x$ reduces (2.11) to the ordinary differential equation

$$f(s_0)\frac{\partial C}{\partial x} + \big(f'(s_0)s_0'(x) + \Lambda(s_0)\big)C + \left(g'(s_0) - f'(s_0)\frac{g(s_0)}{f(s_0)}\right)\Lambda(s_0)C^2 = 0. \tag{2.12}$$

The formulas (2.9) and (2.10) specify the solution of the Bernoulli equation (2.12) with the condition (1.6). Theorem 2 is proved.

Consequence 2. The sufficient condition of existence of the solution on the concentration front is

$$\frac{\partial}{\partial S}\left(\frac{g(S)}{f(S)}\right)\bigg|_{S=s_0(x)} \geq 0 \tag{2.13}$$

or

$$\left.\frac{\partial v}{\partial S}\right|_{S=s_0(x)} \leq 0. \tag{2.14}$$

Proof. The denominator of the solution (2.9) may not vanish. Since the functions $\Lambda(s_0(y))$ and $u(y)$ are positive, the condition (2.13) ensures positiveness of the denominator in (2.9). In accordance with (1.10), the inequality (2.14) follows from (2.13).

3. ASYMPTOTICS IN THE NEIGHBORHOOD OF THE CONCENTRATION FRONT

Let the coefficients of the system (1.4), (1.5) admit the expansion in the Taylor formula with the coefficients dependent on x

$$g(S) = g_0 + g_1(S - s_0) + \frac{g_2}{2}(S - s_0)^2 + \frac{g_3}{6}(S - s_0)^3 + O(S - s_0)^4,$$
$$f(S) = f_0 + f_1(S - s_0) + \frac{f_2}{2}(S - s_0)^2 + \frac{f_3}{6}(S - s_0)^3 + O(S - s_0)^4,$$
$$\Lambda(S) = \Lambda_0 + \Lambda_1(S - s_0) + \frac{\Lambda_2}{2}(S - s_0)^2 + O(S - s_0)^3. \tag{3.1}$$

Here, the subscript 0 denotes the value of the function for $S = s_0(x)$ and the nonzero subscript $j = 1,\ 2$, and 3 of the functions $g,\ f$, and Λ denotes the corresponding derivative of the jth order taken at $S = s_0(x)$.

In the neighborhood of the concentration front Γ the asymptotic solution can be sought in the domain Ω_S in the form:

$$S(x,t) = s_0(x) + \sum_{i=1}^{n} \frac{\big(t - t_\Gamma(x)\big)^i}{i!} s_i(x) + O\big(t - t_\Gamma(x)\big)^{n+1}, \tag{3.2}$$

$$C(x,t) = c_0(x) + \sum_{i=1}^{n} \frac{\big(t - t_\Gamma(x)\big)^i}{i!} c_i(x) + O\big(t - t_\Gamma(x)\big)^{n+1}. \tag{3.3}$$

The expansions (3.2) and (3.3) make it possible to represent the solution of the system of two partial differential equations which depends on two variables in the form of Taylor series in powers of the small parameter $t - t_\Gamma(x)$. The series coefficients $c_i(x)$ depend on a single variable x and must satisfy a system of first-order ordinary differential equations. The asymptotics make it possible to obtain the local solution in the explicit form with the necessary accuracy.

We substitute the expansions (3.1)–(3.3) in Eqs. (1.4) and (1.5) and equate the terms of the same power of $(t - t_\Gamma(x))$. For the first terms of asymptotics there follow the algebraic relations

$$s_1 = \Lambda_0 c_0, \tag{3.4}$$
$$s_2 = \Lambda_0 c_1 + \Lambda_0 \Lambda_1 c_0^2, \tag{3.5}$$
$$s_3 = \Lambda_0 c_2 + 3\Lambda_0 \Lambda_1 c_0 c_1 + (\Lambda_2 \Lambda_0 + \Lambda_1^2)\Lambda_0 c_0^3$$

and the recurrent system of differential equations

$$(f_0 c_0)' + \Lambda_0 c_0 + \beta \Lambda_0 c_0^2 = 0, \tag{3.6}$$
$$(f_0 c_1)' + \Lambda_0(3\beta c_0 + 1)c_1 + (\beta\Lambda_1 + \gamma\Lambda_0)\Lambda_0 c_0^3 + (f_1 \Lambda_0 c_0^2)' + \Lambda_0 \Lambda_1 c_0^2 = 0, \tag{3.7}$$
$$(f_0 c_2)' + \Lambda_0(4\beta c_0 + 1)c_2 + (f_2 \Lambda_0^2 c_0^3)' + (f_1 \Lambda_0 \Lambda_1 c_0^3)' + 3(f_1 \Lambda_0 c_0 c_1)' + (\Lambda_0 \Lambda_1^2 + \Lambda_0^2 \Lambda_2)c_0^3$$
$$+ (\beta \Lambda_0^2 \Lambda_2 + \beta \Lambda_0 \Lambda_1^2 + 3\gamma \Lambda_0^2 \Lambda_1 + \delta \Lambda_0^3)c_0^4 + (3\Lambda_0 \Lambda_1 + 6\gamma \Lambda_0^2 c_0 + 6\beta \Lambda_0 \Lambda_1 c_0)c_0 c_1 + 3\beta \Lambda_0 c_1^2 = 0. \tag{3.8}$$

Here, $\alpha = g_0/f_0,\ \beta = g_1 - \alpha f_1,\ \gamma = g_2 - \alpha f_2$, and $\delta = g_3 - \alpha f_3$.

From (1.6) we obtain the conditions for finding the unique solutions of ordinary differential equations (3.6)–(3.8)

$$c_0\big|_{x=0} = 1,$$
$$c_1\big|_{x=0} = 0, \quad c_2\big|_{x=0} = 0. \tag{3.9}$$

For the next terms of the asymptotics the equations and the initial conditions can be similarly constructed.

Equation (3.6) for the principal term of the asymptotics coincides with Eq. (2.12) on the concentration front; the solution is given by the formulas (2.9) and (2.10).

Example 1. Let $g(S) = g_0 = 1$ and $f(S) = f_0 = 1$. Equation (3.6) takes the form:

$$f_0 c_0' + \Lambda_0 c_0 = 0. \qquad (3.10)$$

The solution of Eq. (3.10) under the condition (3.9) is

$$c_0 = \exp\left(-\int_0^x \Lambda\big(s_0(z)\big)\,dz\right).$$

For the constant filtration coefficient $\Lambda(S) = \Lambda_0 = \text{const}$ the exact solution of the problem (1.4)–(1.8) in the domain Ω_S coincides with the principal terms of the asymptotics (3.2), (3.3)

$$C(x,t) = c_0 = e^{-\Lambda_0 x}; \quad S(x,t) = s_1(t-x) = \Lambda_0 e^{-\Lambda_0 x}(t-x).$$

Example 2. $s_0 = k = \text{const}$. In this case the coefficients of expansions (3.1) are independent of x. The solution of Eqs. (3.6) and (3.7) takes the form:

$$c_0 = \frac{1}{(1+\beta)e^{\Lambda_0/f_0 x} - \beta}, \qquad (3.11)$$

$$c_1 = \frac{\big(0.5\gamma\Lambda_0(1+e^{\Lambda_0/f_0 x}) + (1+\beta)\sigma e^{\Lambda_0/f_0 x}\big)(1-e^{\Lambda_0/f_0 x})}{\big((1+\beta)e^{\Lambda_0/f_0 x} - \beta\big)^3}. \qquad (3.12)$$

Here, $\sigma = \Lambda_1 - 2f_1/f_0$.

The terms of asymptotics of the deposit concentration s_1 and s_2 can be determined from the algebraic equations (3.4) and (3.5)

$$s_1 = \frac{\Lambda_0}{(1+\beta)e^{\Lambda_0/f_0 x} - \beta}, \qquad (3.13)$$

$$s_2 = \frac{\Lambda_0\left(\frac{1}{2}\gamma\Lambda_0(1+e^{\Lambda_0/f_0 x}) + (1+\beta)\sigma e^{\Lambda_0/f_0 x}\right)(1-e^{\Lambda_0/f_0 x})}{\big((1+\beta)e^{\Lambda_0/f_0 x} - \beta\big)^3} + \frac{\Lambda_0\Lambda_1}{\big((1+\beta)e^{\Lambda_0/f_0 x} - \beta\big)^2}. \qquad (3.14)$$

Substituting (3.11)–(3.14) in the expansions (3.2) and (3.3), we obtain the asymptotic solution in the neighborhood of the concentration front $t = t_\Gamma(x)$

$$C(x,t) = \frac{1}{(1+\beta)e^{\Lambda_0/f_0 x} - \beta} + \frac{\left(\frac{1}{2}\gamma\Lambda_0(1+e^{\Lambda_0/f_0 x}) + (1+\beta)\sigma e^{\Lambda_0/f_0 x}\right)(1-e^{\Lambda_0/f_0 x})}{\big((1+\beta)e^{\Lambda_0/f_0 x} - \beta\big)^3}(t - t_\Gamma(x)) + O(t - t_\Gamma(x))^2,$$

$$S(x,t) = s_0(x) + \frac{\Lambda_0}{(1+\beta)e^{\Lambda_0/f_0 x} - \beta}(t - t_\Gamma(x))$$

$$+ \Lambda_0 \frac{\left(\frac{1}{2}\gamma\Lambda_0(1+e^{\Lambda_0/f_0 x}) + (1+\beta)\sigma e^{\Lambda_0/f_0 x}\right)(1-e^{\Lambda_0/f_0 x}) + \Lambda_1\big((1+\beta)e^{\Lambda_0/f_0 x} - \beta\big)}{\big((1+\beta)e^{\Lambda_0/f_0 x} - \beta\big)^3}(t - t_\Gamma(x))^2 + O(t - t_\Gamma(x))^3.$$

4. NUMERICAL SIMULATION

For calculations we used the values of the coefficients of Eqs. (1.4) and (1.5) for particles of three dimensions calculated by Z. You on the basis of the experimental data [18] (see Table 1).

The numerical calculation of the problem was carried out using the finite-difference method by means of the explicit TVD-scheme with the SUPERBEE limiter function [31] for the linear distribution of the initial deposit

$$s_0(x) = (1 - 0.1x). \qquad (4.1)$$

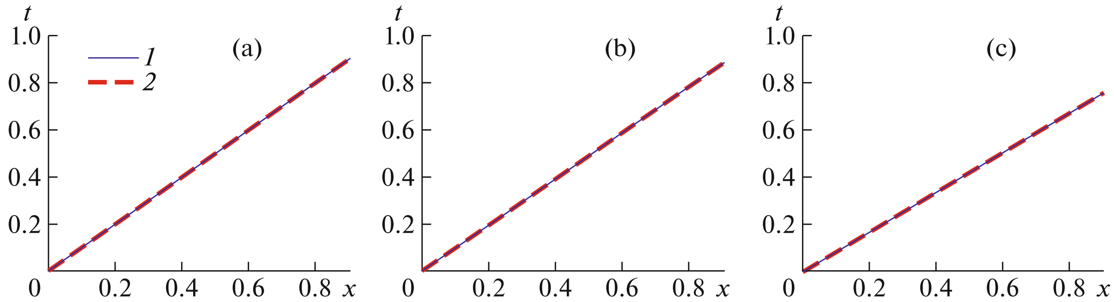

Fig. 3. Concentration fronts: $r_1 = 1.5675$ (a); $r_2 = 2.179$ (b); and $r_3 = 3.168$ (c); curves *1* and *2* correspond to the numerical solution and asymptotics, respectively.

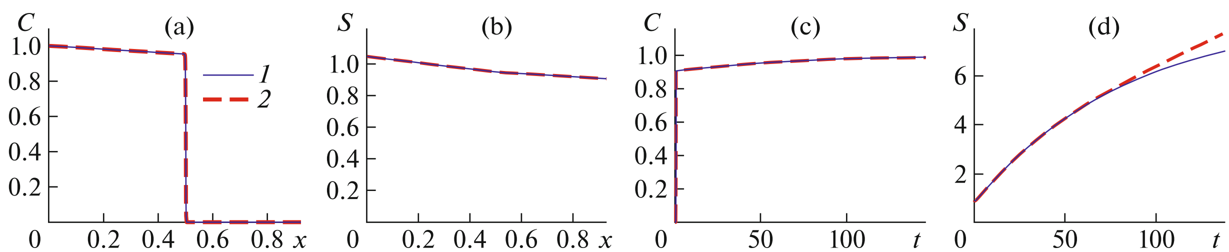

Fig. 4. Particles with $r_1 = 1.5675$: $C(x,t)\big|_{t=0.5}$ (a); $S(x,t)\big|_{t=0.5}$ (b); $C(x,t)\big|_{x=1}$ (c); and $S(x,t)\big|_{x=1}$ (d); curves *1* and *2* correspond to the numerical solution and asymptotics, respectively.

For the problem with deposit the relation between the time and coordinate steps was taken from the Courant convergence condition [29].

In Fig. 3 we have plotted the graphs of the concentration fronts for particles of three types.

As a result of smallness of the coefficients of powers of S (see Table 1), all three calculated curves *1* have almost no deviation from straight lines *2* given by the formula $t = g(0)x/f(0)$.

In Figs. 4–6 we have plotted the graphs of the concentrations of suspended and retained particles of three types as functions of the spatial coordinate x at $t = 0.5$ and time t at the outlet of the porous medium at $x = 1$.

The graphs of the numerical solution and asymptotics almost coincide at $t = 0.5$.

At $t = 0.5$ the graphs of the suspended particles concentrations are discontinuous on the concentration front Γ (Figs. 4a–6a) and the graphs of the retained particles concentrations have a break

Table 1

Particle radius, μm	Coefficients of equations
$r_1 = 1.5675$	$g(S) = 0.9987 + 9.1 \times 10^{-13}S - 3.73 \times 10^{-8}S^2 + 6.1 \times 10^{-5}S^3$
	$f(S) = 0.9999 + 1.8 \times 10^{-5}S - 2.05 \times 10^{-7}S^2 + 2.848 \times 10^{-4}S^3$
	$\Lambda(S) = 0.11 - 0.01351S + 4.49 \times 10^{-5}S^2 + 1.163 \times 10^{-3}S^3$
$r_2 = 2.179$	$g(S) = 0.9743 - 8.88 \times 10^{-14}S + 1.27 \times 10^{-11}S^2 - 1.24 \times 10^{-9}S^3$
	$f(S) = 0.9947 + 6.27 \times 10^{-5}S - 2.9 \times 10^{-8}S^2 + 6.21 \times 10^{-10}S^3$
	$\Lambda(S) = 0.51 - 5.956 \times 10^{-3}S + 2.29 \times 10^{-6}S^2 + 1.35 \times 10^{-8}S^3$
$r_1 = 3.168$	$g(S) = 0.7635 + 2.44 \times 10^{-15}S + 3.2 \times 10^{-14}S^2 + 3.6 \times 10^{-13}S^3$
	$f(S) = 0.9075 + 2.315 \times 10^{-4}S + 2.27 \times 10^{-8}S^2 - 3.42 \times 10^{-8}S^3$
	$\Lambda(S) = 1.551 - 3.467 \times 10^{-3}S - 1.16 \times 10^{-6}S^2 - 1.16 \times 10^{-7}S^3$

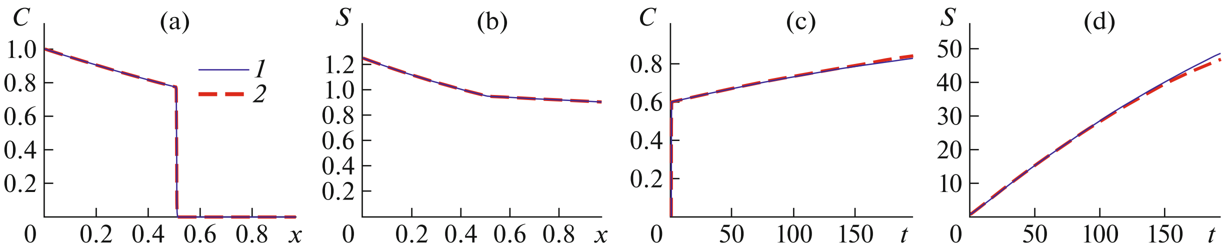

Fig. 5. Particles with $r_2 = 2.179$: $C(x,t)\big|_{t=0.5}$ (a); $S(x,t)\big|_{t=0.5}$ (b); $C(x,t)\big|_{x=1}$ (c); and $S(x,t)\big|_{x=1}$ (d); curves *1* and *2* correspond to the numerical solution and asymptotics, respectively.

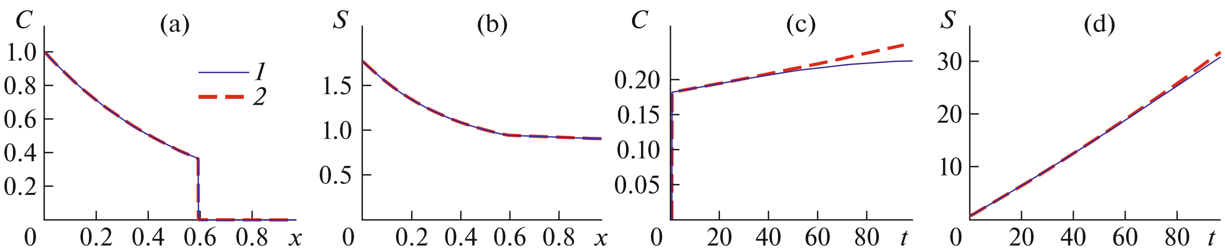

Fig. 6. Particles with $r_3 = 3.186$: $C(x,t)\big|_{t=0.5}$ (a); $S(x,t)\big|_{t=0.5}$ (b); $C(x,t)\big|_{x=1}$ (c); and $S(x,t)\big|_{x=1}$ (d); curves *1* and *2* correspond to the numerical solution and asymptotics, respectively.

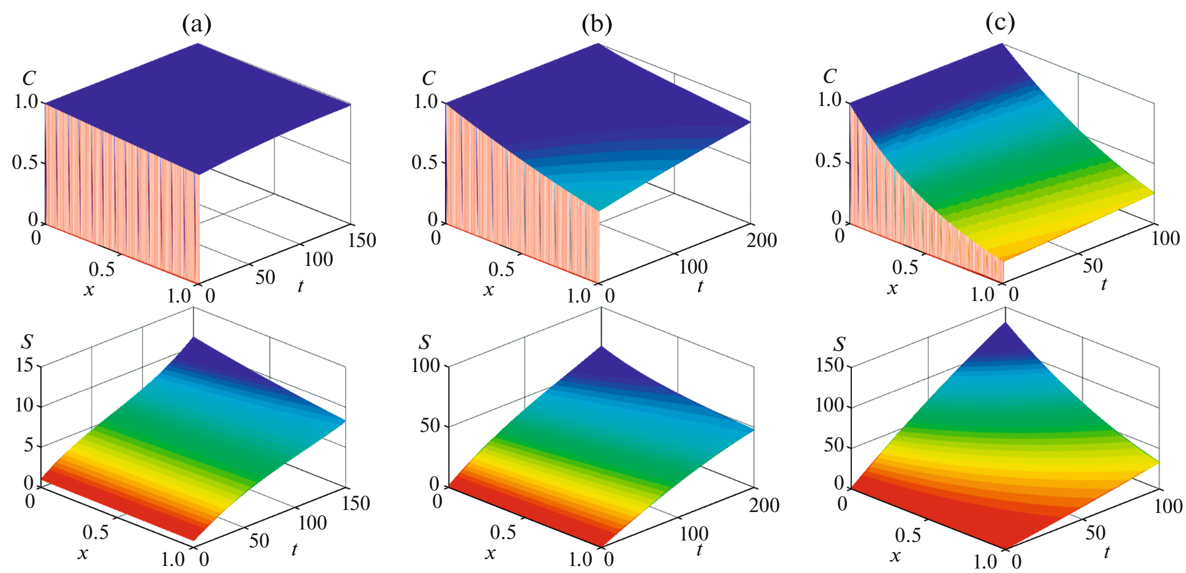

Fig. 7. Three-dimensional graphs of $C(x,t)$ and $S(x,t)$; $r_1 = 1.5675$ (a), $r_2 = 2.179$ (b), and $r_3 = 3.168$ (c).

(Figs. 4b–6b). The points of discontinuity and break can be determined from the relation (1.6) at $t_\Gamma(x) = 0.5$ for the initial deposit (4.1): $x_1 = 0.50$, $x_2 = 0.51$, and $x_3 = 0.59$ (see also Fig. 3).

The graphs of the concentrations at $x = 1$ (Figs. 4b–4d) show the time interval of applicability of asymptotics at the outlet of the porous medium. Depending on the type of particles, the asymptotics are similar to the numerical solution on the time intervals from 0.5 to 200.

In Fig. 7 we have reproduced the three-dimensional graphs of the concentrations of suspended and retained particles (numerical solution).

SUMMARY

The one-dimensional problem of monodisperse suspension deep-bed filtration in a porous medium is considered. Incompressible Newtonian single-phase flow, namely, water with solid particles which are not subjected to molecular diffusion and physicochemical forces of interaction with the porous medium frame, displaces pure water from the porous medium with nonuniformly distributed deposit. The size-exclusion particle-capture mechanism is considered in the absence of deposited-particle mobilization. As distinct from the standard model, the pore accessibility and the fractional solid-particle flow are assumed to vary with accumulation of deposit. In this case the boundary of the concentration front Γ of suspended particles is curvilinear.

It is proved that the solution of the filtration problem must satisfy the natural physical conditions, namely, the retained-particle concentration is constant in time ahead of the concentration front and increases with time behind the front; the suspended-particle concentration is equal to zero ahead of the front and is positive behind the concentration front.

The exact solution on the curvilinear concentration front specifies the suspended-particle concentration on the mobile boundary of water and suspension. The sufficient condition of existence of the solution on the concentration front is obtained.

It is shown that the process of suspension filtration in the porous medium without initial deposit proceeds more intensively near the inlet; the deposit is nonuniformly distributed and its concentration decreases with increase in the spatial coordinate. If the incomplete retained-particle mobilization is assumed to be proportional to the deposit concentration during reverse pure water pumping, then the retained deposit also decreases with increase in x. The problem of the periodic change of suspension injection and reverse water flow with regard to particle mobilization is more complex and should be studied later.

The asymptotic solution whose principal term coincides with the exact solution on the boundary Γ is constructed behind the concentration front. The numerical calculation of the problem showed that the asymptotics and the numerical solution are similar. Depending on the type of the suspended particles in suspension, the asymptotic solution has acceptable accuracy at the porous medium outlet up to time of 50–200.

The exact and asymptotic solutions of the filtration problem give the concentrations of suspended and retained particles as functions of the external parameters in the explicit form. This makes it possible to predict the experimental results and reduce the amount of laboratory studies intended to optimize the filtration process [32].

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