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Helical Vortex Lines in Axisymmetric Viscous Incompressible Fluid Flows

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Abstract—This paper considers the steady and unsteady swirling axisymmetric flows of a homogeneous viscous incompressible fluid. The possibility of the existence of helical vortex lines on the surface of revolution homeomorphic to a torus is investigated. An example of unsteady flow in which there are helical vortex lines is given. It is proved that the existence of helical vortex lines lying on the surface of revolution homeomorphic to a torus is impossible in a steady axisymmetric flow of a viscous incompressible fluid.

Keywords: incompressible fluid, Navier–Stokes equations, axisymmetric flows, helical vortex lines, maximum principle, circumferential circulation

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Swirling axisymmetric flows are intermediate between 2D and 3D flows. On the one hand, they have a "3D" interaction of radial-axial (meridional) and circumferential motions. On the other hand, symmetry imposes additional constraints on motion parameters, which simplifies the study compared to the general 3D case. The study of swirling axisymmetric flows is not only theoretical, but is also of practical interest. Firstly, because some real flows can be considered axisymmetric, for example, the flows in pipes and axisymmetric channels, in tornadoes, nozzles, whirlpools when draining fluid from tanks, in flowing around bodies of revolution, etc. Secondly, because the verification of the laws of axisymmetric flows can be useful for verifying 3D numerical schemes when calculating axisymmetric flows. The regularities of such flows are associated with the shapes of flow lines and vortex lines. Interest in the regularities of the shape of flow lines was demonstrated in [1–4], in which a number of important results were obtained. This paper is devoted to the study of the shape of vortex lines.

1. BASIC NOTATION AND MOTION EQUATIONS

Let us consider a laminar axisymmetric flow of a homogeneous viscous incompressible fluid in a potential field of mass forces. The following dimensionless variables will be used further: $\mathbf{V}$ is the speed, $\mathbf{\Omega} = \operatorname{rot}\mathbf{V}$ is the vorticity, p is the pressure related to density, Π is the potential of body forces, and Re is the Reynolds number. Fluid motion is described by equations in the Gromeka–Lamb form

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{\Omega} \times \mathbf{V} = -\frac{1}{\mathrm{Re}} \operatorname{rot} \mathbf{\Omega} - \nabla \left[p + \frac{\mathbf{V}^2}{2} + \Pi \right], \quad \operatorname{div} \mathbf{V} = 0 \tag{1.1}$$

Let us introduce a cylindrical coordinate system r, φ, z with the origin at the point O so that the axis Oz coincides with the axis of symmetry of the flow. Denote by $\mathbf{e}_r$, $\mathbf{e}_\varphi$, and $\mathbf{e}_z$ the right triple of unit vectors in the radial, circumferential, and axial directions. The velocity vector is $\mathbf{V} = V_r\mathbf{e}_r + V_\varphi\mathbf{e}_\varphi + V_z\mathbf{e}_z = \mathbf{V}_r + \mathbf{V}_\varphi + \mathbf{V}_z$

The functions V_r, V_φ, V_z, p, and Π will be considered dependent only on the variables r, z and t. The smoothness of velocity and pressure sufficient for research will be assumed as a natural property of these physical parameters of the fluid.

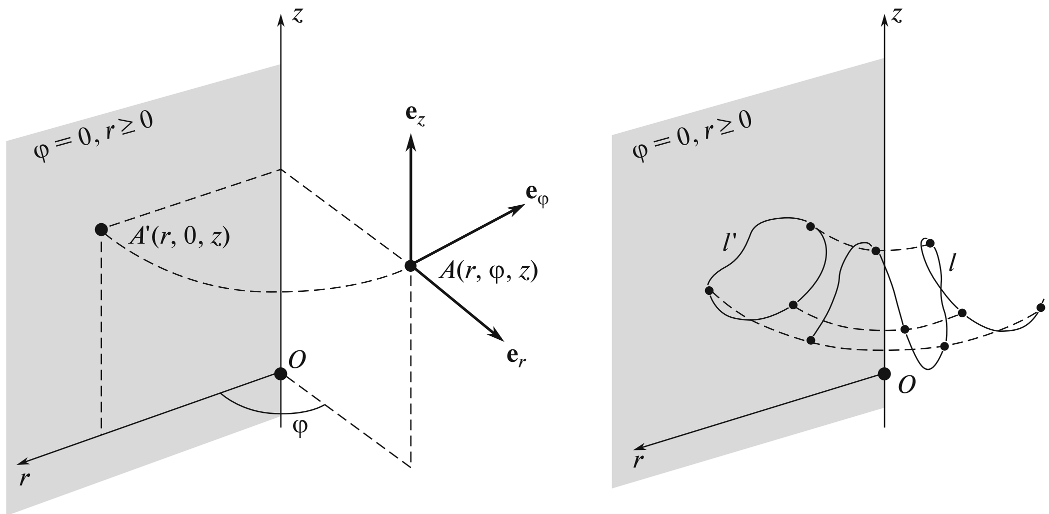

Fig. 1.

2. LINES WITH CLOSED *AC*-PROJECTION

The point A' with coordinates $(r, 0, z)$ obtained as a result of projection along a circle formed by rotating the point A around the axis of symmetry onto the main meridional half-plane $\{\varphi = 0, r \geq 0\}$ will be called a projection along a circle (or *AC*-projection) of the point A with coordinates (r, φ, z) (left panel of Fig. 1). A line whose *AC*-projection is the smooth boundary of a bounded simply-connected planar domain (loop) that lies in the half-plane $\{\varphi = 0, r \geq 0\}$ will be called a line with a closed *AC*-projection. Moreover, different points of the line can have the same *AC*-projections. The right side of Fig. 1 shows the loop l', which is a closed *AC*-projection of the open line l.

Let us consider the line whose *AC*-projection is a closed loop in more detail. Such a line lies on a surface homeomorphic to a torus formed by rotating the aforementioned loop around the axis of symmetry (left panel of Fig. 2). The line is "wound" on the surface and can either close on itself after a finite number of revolutions and have a finite length (right panel of Fig. 2) or not close on itself at any number of revolutions, then it will have infinite length. In any case, it will be helical.

Let us give an example of an unsteady flow in which there are vortex lines with a closed *AC*-projection. Consider an axisymmetric vector field.

$$\mathbf{V} = \{J_1(r)\sin z\mathbf{e}_r - \sqrt{2}J_1(r)\cos z\mathbf{e}_\varphi + J_0(r)\cos z\mathbf{e}_z\}\exp(-2t/\mathrm{Re}) \tag{2.1}$$

where J_0 and J_1 are the Bessel functions of zero and first order, respectively.

We define the pressure field for an arbitrary potential $\Pi = \Pi(r, z, t)$ in addition to the velocity field (2.1)

$$p = p_0 - \left(\frac{\mathbf{V}^2}{2} + \Pi\right), \qquad p_0 = \text{const} \tag{2.2}$$

Let us show that formulas (2.1) and (2.2) give an exact solution of Navier–Stokes equations (1.1). Testing of the incompressibility condition in the case of $\dfrac{\partial}{\partial\varphi} \equiv 0$ is not difficult, because $(rJ_1(r))' = rJ_0(r)$ (hereinafter, the prime means differentiation with respect to the argument r). Let us test first equation (1.1). We have

$$\begin{aligned}\mathbf{\Omega} &= -\frac{\partial V_\varphi}{\partial z}\mathbf{e}_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\mathbf{e}_\varphi + \frac{1}{r}\frac{\partial(rV_\varphi)}{\partial r}\mathbf{e}_z \\ &= \{-\sqrt{2}J_1(r)\sin z\mathbf{e}_r + (J_1(r)\cos z - J_0'(r)\cos z)\mathbf{e}_\varphi - \sqrt{2}J_0\cos z\mathbf{e}_z\}\exp(-2t/\mathrm{Re}) \\ &= -\sqrt{2}\{J_1(r)\sin z\mathbf{e}_r - \sqrt{2}J_1(r)\cos z\mathbf{e}_\varphi + J_0(r)\cos z\mathbf{e}_z\}\exp(-2t/\mathrm{Re})\end{aligned}$$

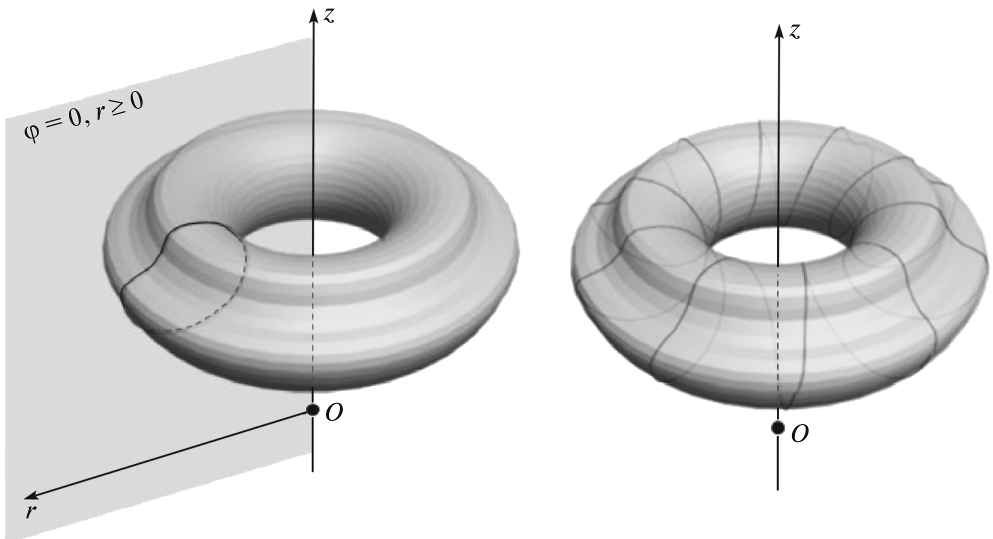

Fig. 2.

(equality $J_0'(r) = -J_1(r)$ is considered). Hence, $\mathbf{\Omega} = -\sqrt{2}\mathbf{V}$. Therefore, $\mathbf{\Omega} \times \mathbf{V} = 0$ and $\operatorname{rot}\mathbf{\Omega} = 2\mathbf{V}$. Thus, the test of first equation (1.1) considering the equality $\dfrac{\partial \mathbf{V}}{\partial t} = -\dfrac{2}{\mathrm{Re}}\mathbf{V}$ reduces to testing the following equation

$$-2\mathbf{V}/\mathrm{Re} + 0 = -2\mathbf{V}/\mathrm{Re} - \nabla(p + \mathbf{V}^2/2 + \Pi)$$

which is satisfied by virtue of Eq. (2.2).

Thus, formulas (2.1) and (2.2) give an exact solution of the Navier–Stokes equations. Since $\mathbf{\Omega} = -\sqrt{2}\mathbf{V}$, the vortex lines and the flow lines of this solution coincide. Therefore, the *AC*-projections of the vortex lines represent the vector lines of the meridional velocity component

$$\mathbf{V}_{rz} = \{J_1(r)\sin z\mathbf{e}_r + J_0(r)\cos z\mathbf{e}_z\}\exp(-2t/\mathrm{Re})$$

Let us write it in the following form

$$\mathbf{V}_{rz} = -\frac{1}{r}\left\{\frac{\partial(rJ_1(r)\cos z)}{\partial z}\mathbf{e}_r - \frac{\partial(rJ_1(r)\cos z)}{\partial r}\mathbf{e}_z\right\}\exp(-2t/\mathrm{Re}) \tag{2.3}$$

Let r_1 be the first root of equation $J_0(r) = 0$. We have the following relations for the first and second derivatives of the function $rJ_1(r)$ at the point $r = r_1$

$$r = r_1 : (rJ_1(r))' = rJ_0(r) = 0, \quad (rJ_1(r))'' = (rJ_0(r))' = r_1J_0'(r_1) = -r_1J_1(r_1) < 0$$

Therefore, the function $rJ_1(r)\cos z$ is represented by the Taylor formula

$$rJ_1(r)\cos z = r_1J_1(r_1)(1 - (r^2 + z^2)/2) + o(r^2 + z^2)$$

in the meridional half-plane in the vicinity of the point $(r = r_1, z = 0)$

Hence, for some sufficiently small $\varepsilon > 0$ following equation

$$rJ_1(r)\cos z = r_1J_1(r_1) - \varepsilon \tag{2.4}$$

sets the closed loop C_ε enclosing the point $(r = r_1, z = 0)$. The first differentials of the coordinates of r and z points located on the loop C_ε are related by

$$\frac{\partial(rJ_1(r)\cos z)}{\partial r}dr + \frac{\partial(rJ_1(r)\cos z)}{\partial z}dz = 0 \tag{2.5}$$

which is obtained from Eq. (2.4). Along with equality (2.3), this relation means that the AC-projection of one of the flow lines, and, hence, the AC-projection of one of the vortex lines, coincides with the loop C_ε.

The above example shows that the vortex lines of an unsteady axisymmetric flow of a viscous incompressible fluid can have a closed AC-projection.

Note that solution of (2.1), (2.2) should be considered as an extension to the viscous unsteady case of the solution obtained by Gromeka for the steady flow of an ideal fluid [5]. Gromeka's solution is obtained from expression (2.1) if we drop the unsteady factor $\exp(-2t/\mathrm{Re})$. Thus, the vortex lines can have a closed AC-projection both in unsteady flows of a viscous fluid and in steady flows of an ideal fluid.

It will be shown below that in the steady case, if there is viscosity, the existence of such vortex lines is impossible.

3. MAXIMUM PRINCIPLE FOR CIRCUMFERENTIAL CIRCULATION

Let us proceed to the study of steady flows of a viscous fluid. The meridional component of the vorticity $\mathbf{\Omega}_{rz}$ is the rotor of the peripheral velocity ($\mathbf{\Omega}_{rz} = \mathrm{rot}\,\mathbf{V}_\varphi$). The circumferential component of first equation (1.1) is as follows

$$\mathbf{\Omega}_{rz} \times (\mathbf{V}_r + \mathbf{V}_z) = -\frac{1}{\mathrm{Re}}\mathrm{rot}\,\mathbf{\Omega}_{rz}$$

We obtain the following equation by multiplying this equality by a scalar vector $2\pi r\mathbf{e}_\varphi$

$$(2\pi r\mathbf{e}_\varphi \cdot [\mathbf{\Omega}_{rz} \times (\mathbf{V}_r + \mathbf{V}_z)]) = -\frac{1}{\mathrm{Re}}(2\pi r\mathbf{e}_\varphi \cdot \mathrm{rot}\,\mathbf{\Omega}_{rz}) \tag{3.1}$$

We transform its left side using the cyclicity property of the mixed product of vectors and the expression for $\mathbf{\Omega}_{rz}$ in a cylindrical coordinate system:

$$(2\pi r\mathbf{e}_\varphi \cdot [\mathbf{\Omega}_{rz} \times (\mathbf{V}_r + \mathbf{V}_z)]) = ((\mathbf{V}_r + \mathbf{V}_z)\cdot \mathbf{\nabla})(2\pi r V_\varphi)$$

The right side of Eq. (3.1) is as follows

$$-\frac{1}{\mathrm{Re}}2\pi r(\mathbf{e}_\varphi \cdot \mathrm{rot}\,\mathbf{\Omega}_{rz}) = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2}{\partial z^2}\gamma + \frac{\partial^2}{\partial r^2}\gamma - \frac{1}{r}\frac{\partial}{\partial r}\gamma\right); \quad \gamma = 2\pi r V_\varphi$$

and Eq. (3.1) can be written as

$$\left(\left(\mathbf{V}_r + \frac{1}{\mathrm{Re}}\frac{1}{r}\mathbf{e}_r + \mathbf{V}_z\right)\cdot\mathbf{\nabla}\right)\gamma = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2}{\partial r^2}\gamma + \frac{\partial^2}{\partial z^2}\gamma\right) \tag{3.2}$$

The value of $\gamma = 2\pi r V_\varphi$ was called [6] the circumferential circulation. The parameter γ is closely related to the shape of the vortex lines. This follows from the vector equality that can be easily verified in the cylindrical coordinate system

$$\mathbf{\Omega}_{rz} = -\frac{1}{2\pi r}[\mathbf{e}_\varphi \times \mathbf{\nabla}\gamma] \tag{3.3}$$

For axisymmetric flows, equality (3.3) holds for viscous and nonviscous fluids and for compressible and incompressible fluids. It is true for steady and unsteady axisymmetric flows. From equality (3.3) it follows that the gradient γ is orthogonal to the vector lines $\mathbf{\Omega}_{rz}$, i.e., γ retains its value on these lines. However, the vector lines $\mathbf{\Omega}_{rz}$ on the half-plane $\{\varphi = 0, r \geq 0\}$ are AC-projections of the vortex lines. Therefore, the circumferential circulation in axisymmetric flows of any type is preserved on the AC-projections of the vortex lines. This fact will be used in the next section.

The last proposition can be illustrated by the example of the exact solution of (2.1), (2.2). In this solution, the circumferential circulation is $\gamma = -2\pi r\sqrt{2}J_1(r)\cos z$. Equation (2.5) is satisfied on a closed loop C_ε, which is a AC-projection of the vortex line, which means the constancy of γ on the loop C_ε.

Equation (3.2) is elliptic. The usual means of investigating extremal properties of elliptic equations is the E. Hopf theorem [7, 8]. This theorem offers different versions of the maximum principle for solutions of quasilinear elliptic equations depending on the properties of the coefficients of these equations. However, there is a requirement for the boundedness of coefficients in the conditions of the theorem, and one of the coefficients in Eq. (3.2) has a feature on the axis of symmetry ($r = 0$), and the Hopf theorem cannot be applied. This gap has recently been eliminated [9], and the maximum principle for circumferential circulation, in which the region under consideration can have boundary points on the Oz axis, was proved.

Let the axisymmetric laminar flow of an incompressible fluid with nonzero viscosity be steady in the absence of external mass forces and let $\bar{G}$ *be an arbitrary bounded closed flow region lying in the meridional half-plane* $\{r \geq 0, \varphi = 0\}$, *then either the circumferential circulation is constant*, *or its minimum and maximum are reached at the boundary and only at the boundary of the region* $\bar{G}$.

4. VORTEX LINES WITH A CLOSED *AC*-PROJECTION IN A STEADY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID

The maximum principle formulated above allows us to prove the following proposition.

There are no vortex lines with a closed AC-projection, *which is the boundary of the simply connected closed region* $\bar{G}$ *which lies entirely inside the flow*, *in a steady axisymmetric flow of a homogeneous viscous incompressible fluid*.

Proof. Assume that the *AC*-projection of the vortex line is the boundary $\bar{G}$. Then, the circumferential circulation $\bar{G}$ has the same value on the entire boundary of γ according to equality (3.3) (see the text after formula (3.3)). Therefore, it follows from the maximum principle for circumferential circulation (true only if the viscosity coefficient is not zero) that the value of γ is constant throughout the region of $\bar{G}$, and inside $\bar{G}$ expression (3.3) gives $\mathbf{\Omega}_{rz} = 0$. Hence, the equality $\mathbf{\Omega}_{rz} = 0$ is also satisfied on the boundary of $\bar{G}$ by virtue of smoothness. However, in this case, the vector lines of the field $\mathbf{\Omega}$ in the form of circles, which are figures of revolution around the axis of symmetry, pass through the points of the boundary $\bar{G}$, and this boundary is not the *AC*-projection of any one vortex line. The proposition is proved.

If the viscosity coefficient is zero, then the maximum principle and this proof cannot be applied. Therefore, helical lines are possible in an ideal fluid, which is confirmed by Gromeka's solution (2.1), (2.2).

The obtained result means that the existence of helical vortex lines "wound" on such a torus in the steady case in the axisymmetric flows of a viscous incompressible fluid is impossible if the entire interior of the "torus" is filled with fluid.

5. CONCLUSIONS

We have shown that there cannot be helical vortex lines that lie on such a homeomorphic to a torus surface of revolution around the axis of symmetry which covers the region that lies entirely inside the flow in a steady axisymmetric flow of a homogeneous viscous incompressible fluid. This effect is significantly associated with stationarity, which is confirmed by the above example of unsteady flow in which the said vortex lines exist.

REFERENCES

1. Arnold, V.I., Sur la topologie des ecoulements stationnaires des fluides parfaits, *C. R. Acad. Sci.,* Paris, 1965, vol. 261, no. 1, pp. 17–20.
2. Arnol'd, V.I., On the topology of three-dimensional steady flows of an ideal fluid, *J. Appl. Math. Mech.,* 1966, vol. 30, no. 1, pp. 223–226.
3. Kozlov, V.V., Notes on steady vortex motions of continuous medium, *J. Appl. Math. Mech.,* 1983, vol. 47, no. 2, pp. 288–289.
4. Kozlov, V.V., *Obshchaya teoriya vikhrei* (General Vortex Theory), Izhevsk: Udmurdskii universitet, 1998.
5. Gromeka, I.S., Some cases of incompressible liquid motion, in *Sobranie sochinenii* (Collection of Scientific Works), Moscow: USSR Acad. Sci., 1952, pp. 76–148.
6. Sizykh, G.B., Evolution of vorticity in swirling axially symmetric flows of viscous incompressible liquid, *Uch. Zap. TsAGI,* 2015, vol. 46, no. 3, pp. 14–20.
7. Hopf, E., Elementare Bemerkungen über die Losungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, *Sitzungsberichte der Preussischen Akademie der Wissenschaften,* 1927, vol. 19, pp. 147–152.
8. Miranda, C., *Partial Differential Equations of Elliptic Type,* Berlin, Heidelberg: Springer-Verlag, 1970.
9. Besportochnyi, A.I., Burmistrov, A.N., and Sizykh, G.B., Variant of Hopf theorem, *Tr. MFTI,* 2016, vol. 8, no. 1, pp. 115–122.

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