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Plane-Parallel Advective Flow in a Horizontal Layer of an Incompressible Fluid with an Internal Linear Heat Source

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Abstract—We present a new exact solution of the Navier-Stokes equations in the Oberbeck-Boussinesq approximation describing a plane-parallel advective flow in a plane horizontal layer of an incompressible fluid with solid boundaries. At the boundaries, a linear temperature distribution is defined in the presence of an internal heat source that is linear with respect to the horizontal coordinate. Examples of such solutions are given. The possibility of an analytical determination of the velocity and temperature of such flows is demonstrated. The velocity profile has not a cubic profile, which is usual for advective flows, but a more complex form depending on the source type.

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Specific convective flows arise in the presence of an internal heat source while some of them can be analytically described. There is an overview of studies of such flows [1] in a vertical layer [2–4], an inclined liquid layer [5] with internal heat sources distributed uniformly throughout the volume, as well as with a heat source whose density decreases exponentially with distance from boundaries. Such a distribution can occur when passing across a layer of light flux, the absorption of which in the liquid occurs according to Burger's law and all absorbed energy is released in the form of heat [6]. Thermocapillary convection was studied in the presence of an internal heat source in the liquid layer under conditions of weightlessness [7]. The effect of distributed point heat sources on the dynamics of a convective flow in a vertical cylinder was analytically analyzed [8]. The problem of the response of a stably stratified liquid (gaseous) medium to the effect of heat sources and the momentum extended along the vertical harmoniously varying with time is analytically solved [9].

Advective flows arise in a flat horizontal liquid layer under the action of a longitudinal temperature gradient [1]. They lack the vertical velocity component, i.e., the velocity vector in the flow is oriented perpendicular to the buoyancy force, which is the main cause of the motion. This property does not change under various boundary conditions for the velocity [10]. In the case when the temperature at the boundaries of the layer is a linear function of the longitudinal coordinate proportional to the constant horizontal temperature gradient at the boundaries of the layer, the flow is described analytically as an exact solution of the Navier-Stokes equations [11, 12]. There is an overview of such plane-parallel advective flows [13–15] under various boundary conditions. The flows are stationary and, as a rule, closed, with zero consumption. A procedure is given for obtaining exact solutions of the Navier-Stokes equations describing a wide class of closed advective flows in a rotating plane layer of an incompressible fluid [16]. It was noted that analytical solutions can be found in the case of both a linear temperature distribution and a linear distribution of the heat flux on the horizontal boundaries of the layer. Moreover, one of these solutions was presented (in the absence of rotation) [17]. It was shown [18, 19] that analytical solutions describing advective flows can be used to develop quasi-two-dimensional models used in technological and geophysical applications [20]. A new class of exact solutions describing the distribution of temperature and concentration at the boundaries of the liquid layer according to a quadratic law is presented [21].

1. Mathematical model. Let us consider a plane infinite horizontal layer of an incompressible fluid with a width of $2h$ with solid boundaries in a homogeneous gravitational field. The fluid motion is described by the convection equations in the Boussinesq approximation [1] in the Cartesian coordinate system $Oxyz$ (z is the vertical coordinate, and x and y are the horizontal coordinates). In the layer, there is a heat source linearly varying along the x coordinate.

Selecting h, h^2/ν, $g\beta Ah^3/\nu$, Ah, and $\rho_0 g\beta Ah^3$ (where ν is the kinematic viscosity, β is the coefficient of thermal expansion, g is the acceleration of gravity, and ρ_0 is the average density) as measurement units of the length, time, velocity, temperature, and pressure, we obtain the initial equations in the following dimensionless form:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathrm{Gr}\,\mathbf{v}\nabla\cdot\mathbf{v} = -\nabla P + \nabla^2\mathbf{v} + \theta, \quad \nabla\cdot\mathbf{v} = 0,$$
$$\frac{\partial T}{\partial t} + \mathrm{Gr}\,\mathbf{v}\cdot\nabla T = \frac{1}{\mathrm{Pr}}\nabla^2 T + xf(z), \tag{1.1}$$
$$\theta = (0,0,T), \quad \mathrm{Gr} = \frac{g\beta Ah^4}{\nu^2}, \quad \mathrm{Pr} = \frac{\nu}{\chi},$$

where $\mathbf{v}$ is the velocity vector, T is the temperature, p is the pressure, depending on time and spatial coordinates x, y, z, Gr is the Grashof number, Pr is the Prandtl number, χ is the coefficient of thermal diffusivity, and the $f(z)$ function describes the behavior of the thermal source along the vertical. On solid boundaries:

$$z = \pm 1: \quad \mathbf{v} = 0, \quad T = x. \tag{1.2}$$

2. Advective flow. Taking into account the boundary conditions (1.2) and the incompressibility condition for the fluid (the second equation of system (1.1)), we seek the exact solution of the problem in the following form:

$$u = u_0(z), \quad \mathrm{v} \equiv 0, \quad w \equiv 0, \quad T = T_0 \equiv x\tau_0(z) + \tau_1(z), \quad p = p_0(x,z), \tag{2.1}$$

which leads to a system of the following equations with boundary conditions:

$$\frac{\partial p_0}{\partial z} = T_0, \quad u_0''(z) = \frac{\partial p_0}{\partial x}, \quad \mathrm{Gr}\,\mathrm{Pr}\,\tau_0(z)u_0(z) = x\tau_0''(z) + \tau_1''(z) + \mathrm{Pr}\,xf(z), \tag{2.2}$$

$$u_0(\pm 1) = 0, \quad \tau_0(\pm 1) = 1, \quad \tau_1(\pm 1) = 0. \tag{2.3}$$

Note that this $\tau_1(z)$ is the fluid temperature at $x = 0$.

Taking into account the linear dependence of the right-hand side of the last equation (2.2) on x, we conclude that this equation is divided into two equations:

$$\tau_0''(z) = -\mathrm{Pr}\,f(z), \quad \mathrm{Gr}\,\mathrm{Pr}\,\tau_0(z)u_0(z) = \tau_1''(z). \tag{2.4}$$

When boundary conditions are taken into account, we find:

$$\tau_0(z) = 1 + \mathrm{Pr}[\hat{F}^+ + \hat{F}^- z - F(z)]; \quad F(z) = \iint f(z)dzdz, \quad \hat{F}^\pm = \frac{F(1) \pm F(-1)}{2}. \tag{2.5}$$

Differentiating the first equation (2.2) with respect to x and the second equation with respect to z, we eliminate the pressure and obtain the boundary-value problem for the velocity:

$$u_0'''(z) = \tau_0(z), \quad u_0(\pm 1) = 0, \quad \int_{-1}^{1} u_0(z)dz = 0, \tag{2.6}$$

the general solution of which has the following form:

$$u_0(z) = C_1 + C_2 z + C_3\frac{z^2}{2} + \frac{z^3}{6} + \mathrm{Pr}\left[\hat{F}^+\frac{z^3}{6} + \hat{F}^-\frac{z^4}{24} - U(z)\right], \quad U(z) = \iiint F(z)dzdzdz. \tag{2.7}$$

Taking the boundary conditions of the problem (2.6) into account, we obtain an expression for the velocity:

$$u_0(z) = \frac{z^3 - z}{6}[1 + \mathrm{Pr}\,\hat{F}^+] + \mathrm{Pr}\left[\frac{5z^4 - 6z^2 + 1}{120}\hat{F}^- + \hat{U}^+ + \hat{U}^- z\right]$$
$$+ \mathrm{Pr}\left[\frac{3}{4}(z^2 - 1)\left(2\hat{U}^- - \int_{-1}^{1} U(z)dz\right) - U(z)\right]; \quad \hat{U}^\pm = \frac{U(1) \pm U(-1)}{2}. \tag{2.8}$$

Now, solving the boundary-value problem

$$\tau_1''(z) = \mathrm{Gr}\,\mathrm{Pr}\,\tau_0(z)u_0(z), \quad \tau_1(\pm 1) = 0,$$

we find

$$\tau_1(z) = \mathrm{Gr}\,\mathrm{Pr}[\vartheta(z) - \vartheta^+ - \vartheta^- z],$$
$$\vartheta(z) = \iint \tau_0(z)u_0(z)dzdz, \quad \vartheta^\pm = \frac{\vartheta(1) \pm \vartheta(-1)}{2}. \tag{2.9}$$

At Pr = 0, this solution coincides with the solution [12] describing the advective plane-parallel flow in a horizontal layer with solid boundaries.

Let us consider the simplest examples of flows. In all examples, $\tau_1(z) = \mathrm{Gr}\,\mathrm{Pr}\,\tau_{11}(z)$.

Linear local source: $f(z) = \delta(z) = \begin{cases} 1, & \text{where} \quad z = 0, \\ 0, & \text{where} \quad z \neq 0, \end{cases} \quad F(z) = |z|,$

$$\tau_0(z) = 1 + \mathrm{Pr}[1 - |z|], \quad u_0(z) = \frac{z^3 - z}{6} + \mathrm{Pr}\left[\frac{4z^3 - z^3|z| - 3z}{24}\right],$$

$$\tau_{11}(z) = \frac{3z^5 - 10z^3 + 7z}{360} + \mathrm{Pr}\frac{-5z^5|z| + 10z^3|z| + 12z^5 - 35z^3 + 18z}{720}$$
$$+ \mathrm{Pr}^2\frac{-70z^5|z| + 105z^3|z| + 10z^7 + 84z^5 - 210z^3 + 81z}{10080}.$$

Distributed linear source: $f(z) \equiv 1$, $F(z) = \dfrac{z^2}{2}$

$$\tau_0(z) = 1 + \mathrm{Pr}\frac{1 - z^2}{2}, \quad u_0(z) = \frac{z^3 - z}{6} + \mathrm{Pr}\frac{10z^3 - z^5 - 9z}{120},$$

$$\tau_{11}(z) = \frac{3z^5 - 10z^3 + 7z}{360} + \mathrm{Pr}\frac{-11z^7 + 63z^5 - 133z^3 + 81z}{5040}$$
$$+ \mathrm{Pr}^2\frac{35z^9 - 660z^7 + 2394z^5 - 3780z^3 + 2011z}{604800}.$$

The case: $f(z) = |z|$, $F(z) = \dfrac{z^2|z|}{6}$

$$\tau_0(z) = 1 + \mathrm{Pr}\frac{1 - z^2|z|}{6}, \quad u_0(z) = \frac{z^3 - z}{6} + \mathrm{Pr}\frac{20z^3 - z^5|z| - 19z}{720},$$

$$\tau_{11}(z) = \frac{3z^5 - 10z^3 + 7z}{360} + \mathrm{Pr}\frac{z^5(16 - 9z^2)|z| + 48z^5 - 156z^3 + 101z}{17280}$$
$$+ \mathrm{Pr}^2\left[\frac{z^5(76 - 45z^2)|z|}{518400} + \frac{z(12z^{10} + 1320z^4 - 4180z^2 + 2507)}{5702400}\right].$$

The case: $f(z) = z^2$, $F(z) = \dfrac{z^4}{24}$

$$\tau_0(z) = 1 + \mathrm{Pr}\frac{1 - z^4}{24}, \quad u_0(z) = \frac{z^3 - z}{6} + \mathrm{Pr}\frac{35z^3 - z^7 - 34z}{5040},$$

$$\tau_{11}(z) = \frac{3z^5 - 10z^3 + 7z}{360} + \mathrm{Pr}\frac{-z^3(3z^6 - 5z^4 - 21z^2 + 69) + 46z}{30240}$$
$$+ \mathrm{Pr}^2\frac{z^3(7z^{10} - 546z^6 + 884z^4 + 1911z^2 - 6188) + 3932z}{132088320}.$$

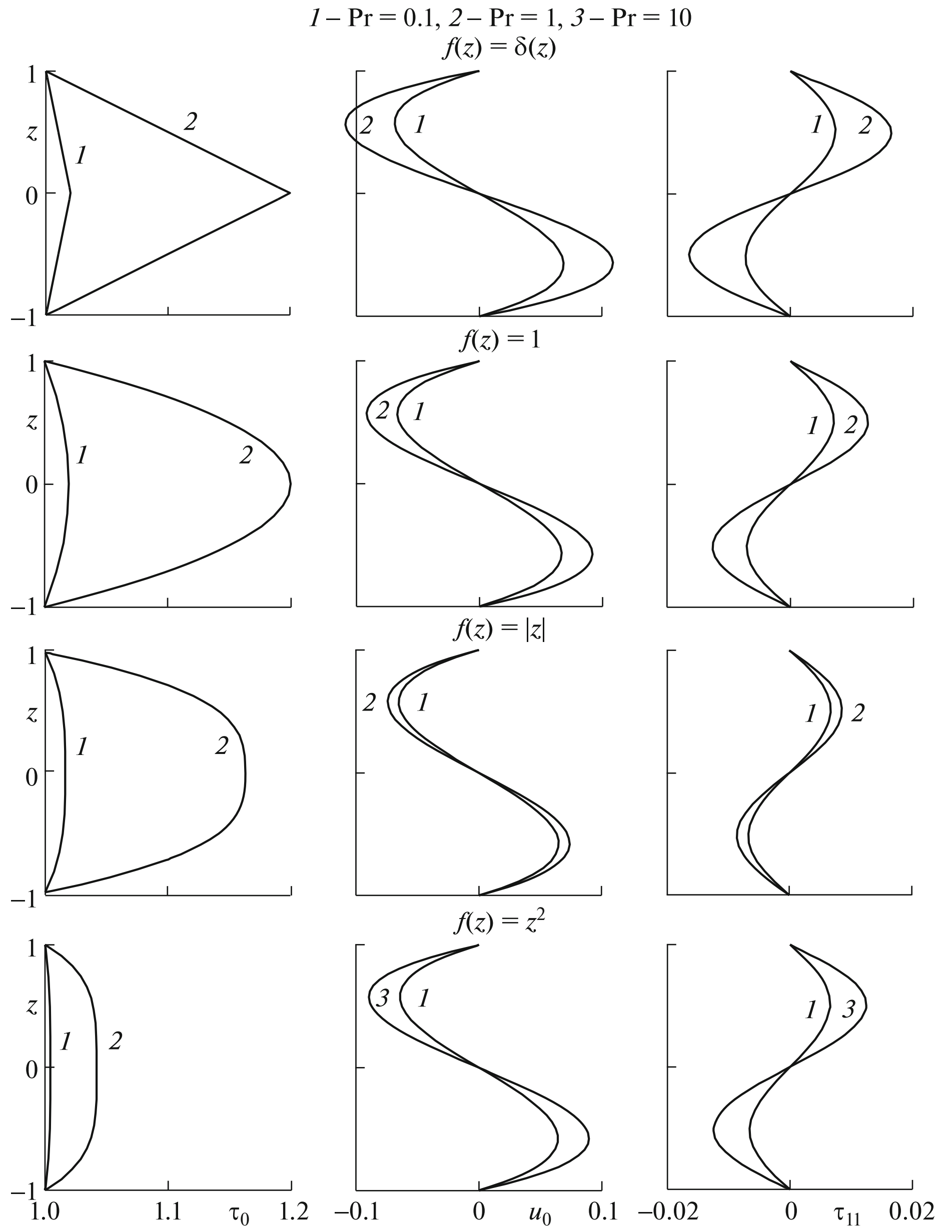

Fig. 1.

In all four examples, the liquid moves from right to left in the upper half of the layer, and the liquid moves from left to right in the lower half of the layer (a middle column of the figure fragments). The $\tau_{11}(z)$ temperature component is positive in the upper half of the layer and negative in the lower half of the layer (right fragments). At small values of the Prandtl number, the velocity reaches its extreme values at $z = \pm 0.5$ while the extremum points move insignificantly toward the layer center with increasing Prandtl number. Obviously, the maximum $u_0(z)$ increases linearly with increasing Prandtl number while $\tau_{11}(z)$ increases according to the quadratic law. In each following example, the velocity and temperature modules decrease.

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