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Heat Exchange in a Cylindrical Channel with Stabilized Laminar Fluid Flow

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Abstract—Based on the determination of the temperature perturbation front and additional boundary conditions, an approximate analytical solution is obtained for the stationary heat exchange problem when fluid flows in a cylindrical channel with a constant parabolic velocity profile (the Gretz–Nusselt problem), which allows us to investigate the temperature distribution in the fluid in a wide range of distances from the pipe inlet, including small and very small distances. Based on the data of numerical calculations of temperature change at a certain value of the spatial variable using the solution obtained by solving the inverse heat conduction problem, the Peclet number was found (in the case where it is unknown in the solution obtained), from which we can determine the velocity profile and the flow rate of the liquid. Graphs of the distribution of isotherms and the their velocities in space over time are plotted.

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In the process of solving the boundary problem of unsteady heat conduction in the case of laminar flow of a fluid in a cylindrical channel, it is physically and reasonably divided into two tasks, unsteady and steady-state. The boundaries between them are determined by the known relations between the time and longitudinal spatial variables [1, 2]. Each of the tasks defined in the fixed boundaries of the specified variables is solved separately. The problem of heat conduction for a solid cylinder is unsteady (heat exchange proceeds as in a stationary fluid), its exact analytical solutions are known. The greatest difficulty is the solution of the stationary problem with allowance for fluid motion (the Gretz–Nusselt problem). It was first solved by Gretz in 1885 [3] and, independently, by Nusselt in 1910. The resulting improved solution of the Gretz–Nusselt problem [1] is represented by an infinite functional series that converges poorly at small values of the longitudinal coordinate, i.e., in areas where temperature changes are of great interest. In addition, the dependence on the transverse coordinate contains Bessel functions of different (including fractional) order. Such a solution is unsuitable for applications, and therefore the development of approximate analytical methods for solving such problems may be of practical importance.

Below, we describe the method for the approximate solution of the Gretz–Nusselt problem based on the introduction of a temperature perturbation front and additional boundary conditions [5–8]. Its distinguishing feature is the simplicity of the expressions obtained with the possibility of finding a solution with almost a given accuracy, including for very small values of the longitudinal spatial variable.

1. Problem statement and its solution. We consider the steps of constructing an approximate analytical solution of the stationary heat transfer problem when an incompressible fluid flows in a cylindrical channel with the following assumptions and notation: the physical properties of the fluid are constant, the velocity profile does not change along the length of pipe, the temperature of fluid at the pipe inlet is constant over the cross section and is equal to t_0, the internal surface temperatures of the pipe walls are constant and equal to t_c, internal heat sources and energy dissipation are not taken into account, heat transfer by heat conduction along the pipe axis is neglected. A diagram of the stabilized laminar flow of fluid in a circular pipe is shown in the left part of Fig. 1.

The mathematical formulation of the problem has the following form [1, 2]

$$\omega_x t_\eta = a\xi^{-1} t_\xi + a t_{\xi\xi}; \quad t = t(\xi,\eta), \quad 0 < \xi < r, \quad \eta > 0, \tag{1.1}$$

$$\eta = 0: \quad t = t_0; \quad \xi = r: \quad t = t_c, \quad \xi = 0: \quad t_\xi = 0, \tag{1.2}$$

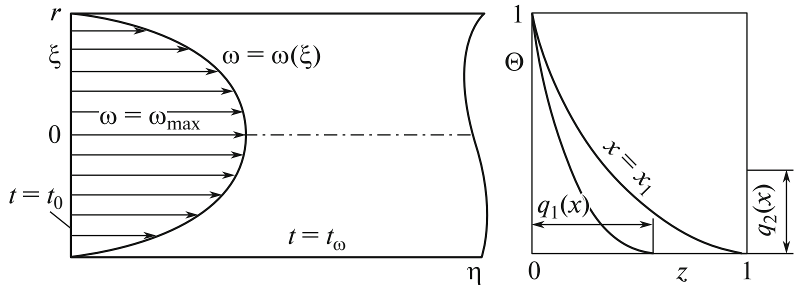

Fig. 1.

where t is the temperature, η and ξ are the transverse and longitudinal coordinates, r is the radius of pipe, t_0 is the temperature of liquid at the inlet of tube (at $\eta = 0$), t_w is the temperature of the wall (at $\xi = r$), a is the thermal diffusivity, $\omega_x = 2\overline{\omega}(1 - \xi^2 r^{-2})$ is the distribution of velocity by the coordinate ξ $(0 \le \xi \le r)$, $\overline{\omega} = \omega_{\max}/2$ is the average velocity, and $\omega_{\max}$ is the maximum velocity of flow.

We introduce the following dimensionless variables and parameters:

$$\Theta = \frac{t - t_0}{t_w - t_0}, \quad z = 1 - \frac{\xi}{r}, \quad x = \frac{\eta}{2\,\mathrm{Pe}\,r}, \quad \mathrm{Pe} = \frac{\overline{\omega} r}{a},$$

where $\Theta = \Theta(z, x)$ is the relative excess temperature, z is the dimensionless transverse coordinate, taken from the pipe wall, x is the dimensionless longitudinal coordinate, and Pe is the Peclet number.

Let us consider the construction of the solution of the Gretz–Nusselt problem based on the introduction of a temperature perturbation front and additional boundary conditions [5–8], dividing the process of heating the medium into two stages along the x coordinate: $0 \le x \le x_1$ and $x_1 \le x < \infty$. To do this, we introduce a boundary moving along the coordinate z dividing the initial region into two sub-regions: heated $0 \le z \le q_1$ and unheated $q_1 \le z \le 1$. Here, $q_1 = q_1(x)$ is the function that determines the boundary advance along the coordinate z depending on the longitudinal x coordinate (see the right side of Fig. 1). At the same time, in the area not affected by heating, the temperature t_0 $(\Theta|_{x=0} = 0)$ at the inlet to the channel is maintained. The first stage of the heat exchange process ends when the moving boundary reaches the middle of the channel, i.e., at $x = x_1$. At the second stage, the temperature varies throughout the volume of the medium $0 < z < 1$, and a new function $q_2 = q_2(x)$ is introduced into consideration, which characterizes the temperature change over time at the center of the channel, i.e., $q_2 = \Theta|_{z=1}$.

2. Construction of the solution at the first stage. The problem of the heating of liquid in the first stage of the process can be formulated in the form (see the right side of Fig. 1)

$$(2z - z^2)\Theta_x = \Theta_{zz} - (1 - z)^{-1}\Theta_z; \quad 0 \le z \le q_1, \quad 0 \le x \le x_1, \tag{2.1}$$

$$z = 0: \quad \Theta = 1; \quad z = q_1: \quad \Theta = 0, \quad \Theta_z = 0. \tag{2.2}$$

The last two equalities (2.2) are the conditions of thermal insulation of the moving boundary (conditions of conjugation of the heated and unheated zones). The first of these means the equality of the temperature of fluid at the moving boundary to its temperature at the inlet to the channel, the second, the absence of heat flow outside the temperature disturbance front.

In problem (2.1), (2.2), the first of the boundary conditions (1.2) is missing due to the fact that the task outside the temperature perturbation front is not defined at all, i.e., there is no need to fulfill this condition across the entire width of the channel. Here, the second of conditions (2.2) is quite sufficient, according to which the values of temperature at the front the temperature perturbation and at the channel entrance are equal. In problem (2.1), (2.2), the third of the boundary conditions (1.2) is also missing, since it does not affect the heat exchange process at its first stage.

Let us note that problem (2.1), (2.2) does not belong to the class of problems that take into account the final velocity of the thermal wave. The construction of their solutions reduces to the integration of the hyperbolic (wave) heat equation. The physical temperature perturbation front introduced in problem (2.1), (2.2) is an analog of a moving isotherm (but not a thermal wave). Since the temperature on the tem-

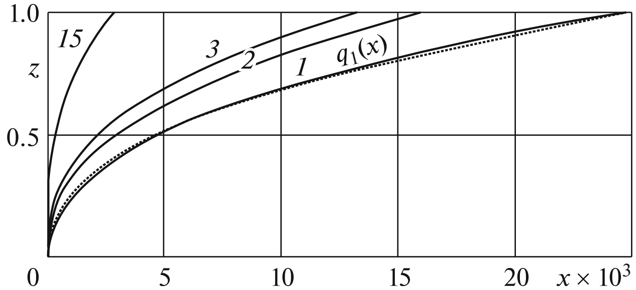

Fig. 2

perature perturbation front during its movement along the coordinate z is maintained, determined at the pipe inlet (the last condition (2.2)), it is an analog of the zero isotherm. Below, it will be shown that the rate of movement of the zero isotherm at $n \to \infty$ when approaching an infinite value and, therefore, $x_1 \to 0$. This result is due to the fact that in this case, the parabolic equation (2.1) is solved, in which the infinite velocity of heat propagation is already laid in the derivation process, and the accuracy of equation (2.1) increases with an increase in the number of approximations.

We require that the desired solution of problem (2.1), (2.2) satisfy not the original equation (2.1), but the average over the thickness of the thermal layer $0 \le z \le q_1$. Defining the integral of equation (2.1) over a variable in the range from $z = 0$ to $z = q_1$, we obtain the integral equation

$$\int_0^{q_1} (2z - z^2)\Theta_x \, dz = \int_0^{q_1} [\Theta_{zz} - (1 - z)^{-1}\Theta_z] dz \tag{2.3}$$

called the integral of heat balance [5, 6, 8].

Solution of problem (2.1), (2.2) is taken in the form

$$\Theta = \Theta(z, x) = \sum_{k=0}^{n} a_k z^k, \tag{2.4}$$

where $a_k = a_k(q_1)$ are unknown coefficients, for determination of which the boundary conditions (2.2) are used. Substituting the row (2.4) (limited to its first three members) in these conditions, relative to a_k $(k = 0, 1, 2)$ we have the system of three algebraic linear equations; solving it we will obtain

$$\Theta = (1 - z/q_1)^2. \tag{2.5}$$

To determine the unknown function $q_1 = q_1(x)$, we substitute expression (2.5) into integral equation (2.3) and obtain

$$\frac{\partial}{\partial x}\int_0^{q_1} (2z - z^2)(1 - zq_1^{-1}) dz = 2q_1^{-1} - \int_0^{q_1} 2(z - 1)^{-1}(zq_1^{-2} - q_1^{-1}) dz.$$

From that, the nonlinear ordinary differential equation (ODE) relative to $q_1(x)$ follows

$$q_1^2(2q_1^2 - 9q_1 + 10)q_{1x} = 60. \tag{2.6}$$

Using the initial condition $q_1|_{x=0} = 0$, we obtain the nonlinear algebraic equation

$$24q_1^5 - 135q_1^4 + 200q_1^3 - 3600x = 0 \tag{2.7}$$

numerical solution of which is shown in Fig. 2 (dashed curve).

To obtain an approximate analytical solution of Eq. (2.7) we will use the method described previously in [7, 8], according to which we will find

$$q_1 = 5.017x^{0.436}. \tag{2.8}$$

Assume here $q_1 = 1$, we will find the dimensionless distance from the inlet into the tube $x = x_1 = 0.0247$, at which the front of temperature perturbation q_1 reaches the center of the pipe $z = 1$.

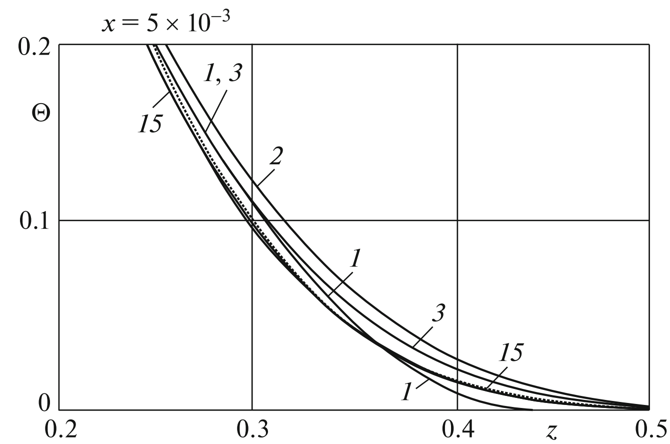

Fig. 3.

Curves *1*, *2*, *3*, and *15* presented in Fig. 2 correspond to the first, second, third and fifteenth approximations. Results of calculations by the formula (2.8) (curve *1)* disagree with numerical solution of Eq. (2.6) by less than 1%. Substituting solution of (2.8) into Eq. (2.5), we obtain the solution of problem (2.1), (2.2) in the first approximation

$$\Theta = (1 - 0.1993zx^{-0.436})^2. \tag{2.9}$$

It exactly satisfies the boundary conditions (2.2) and the heat balance integral (2.3). Equation (2.1) in this case is performed only on average.

The results of calculations by the formula (2.9) in comparison with the exact solution [1] and with the calculation by the finite difference method at $x = 0.005$ are shown in Fig. 3. In the considered range of variation of z coordinates, the exact and numerical solutions coincide.

It is necessary to clarify the use of classic exact solutions. Two such solutions were given in [1]. For one of them, the eigenvalues of the Sturm–Liouville boundary problem are found from a power algebraic equation, the degree of which is determined by the number of approximations. Solving this equation with high degrees of unknowns (with a large number of approximations) is difficult. For example, using modern computer technology, it is possible to obtain only the eighth degree equation (eight eigenvalues, which are completely insufficient for building solutions for small values of the x coordinate). These difficulties were noted also by the authors of the proposed methods [1]. In this connection, an asymptotic solution of the Sturm–Liouville equation as applied to the boundary value problem (1.1), (1.2) was constructed [9]. Three solutions were obtained, each of which is valid only for a specific region of variation of the transverse coordinate y — for the near-wall region (small y), the middle part, and the center of the pipe. There were no specific indications on the clearer division of these areas. Note that the solutions obtained are, in fact, approximate, rather complex, and they include infinite series containing Bessel functions, including those of fractional order. These features of this method significantly complicate its use and, moreover, do not allow obtaining a solution that is uniform for the entire range of variation of the transverse coordinate y.

For comparison with the solutions obtained in the present work, the results obtained by the method developed by the authors [1, 3, 4] were used, but only for the values of the longitudinal coordinate $x \geq 0.05$. For smaller quantities, this method was not applied for the reasons stated above.

In order to compare the results of solutions for values using the finite difference method, a numerical solution of problem (1.1) was obtained. The following steps were taken along the x and z coordinates:

$$\Delta x, \Delta z = \begin{cases} 10^{-5}, & 0.02 \quad \text{at} \quad 10^{-2} \leq x \leq 10^{-3}, \\ 10^{-9}, & 10^{-3} \quad \text{at} \quad 10^{-3} \leq x \leq 10^{-5}. \end{cases}$$

Analysis of the results of first approximation, found by formula (2.9), allows us to conclude that at $x = 0.005$ they differ from the results obtained by the sweep method by about 8% (see curve *1* in Fig. 3).

To increase the accuracy, we find the solution to problem (2.1), (2.2) in the second approximation. In this case, we will use the six members of series (2.4), for finding the unknown coefficients of which the basic boundary conditions (2.2) will not be enough, and it is necessary to introduce three additional

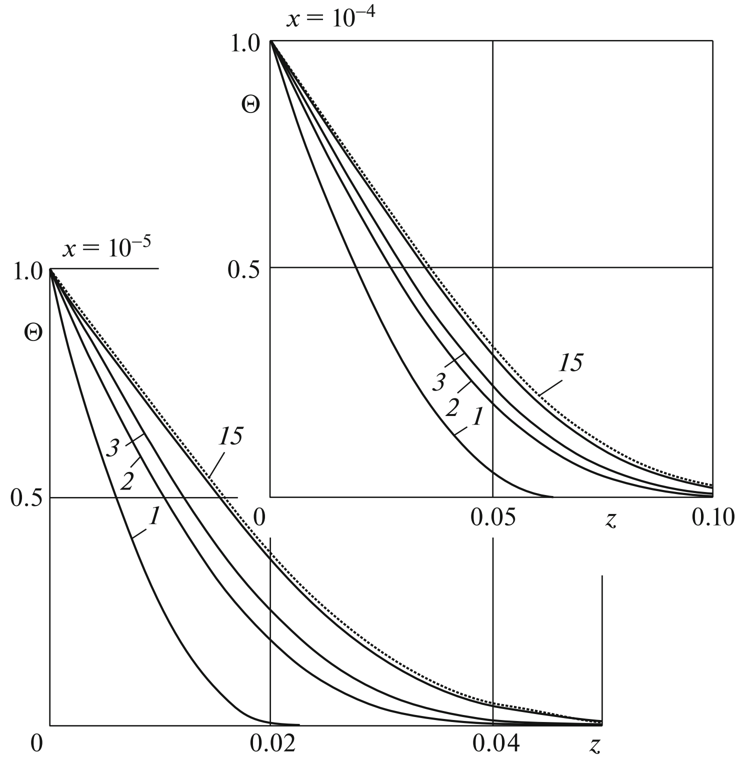

Fig. 4.

boundary conditions (ABC). To find them, we use conditions (2.2) and equation (2.1). Writing equation (2.1) as applied to the point $z = 0$, we obtain the first ABC

$$z = 0: \quad \Theta_{zz} - \Theta_z = 0. \tag{2.10}$$

To find the second ABC we differentiate the relation (2.2) by the variable x

$$z = q_1: \quad \Theta_x = \Theta_z q_{1x} + \Theta_x = 0. \tag{2.11}$$

Relation (2.11), when taking into account the relation (2.2), will be written in the form

$$z = q_1: \quad \Theta_x = 0. \tag{2.12}$$

Comparing relation (2.12) with Eq. (2.1), applied to point $z = q_1$ we obtain the second ABC

$$z = q_1: \quad \Theta_{zz} = 0. \tag{2.13}$$

To find the third ABC, we differentiate the third relation (2.2) by the variable x, taking into account that variable z is a function of x:

$$z = z_{q_1}: \quad \Theta_{zx} = \Theta_{zz} q_{1_x} + \Theta_{zx} = 0. \tag{2.14}$$

When taking into account the conditions (2.13), the relation (2.14) will appear in the form

$$z = q_1: \quad \Theta_{zx} = 0. \tag{2.15}$$

Differentiating Eq. (2.1) by the variable z and comparing the results with relation (2.15), when taking into account the equalities (2.12) and (2.13) we obtain the third ABC

$$z = q_1: \quad \Theta_{zzz} = 0. \tag{2.16}$$

Using equality (2.6) in all main boundary conditions (2.2) and ABC (2.10), (2.13), and (2.16), with respect to the coefficients a_k, we obtain a system of six algebraic linear equations of triangular form. As a result, from the expression (2.4) follows

$$\Theta = 1 - \frac{1}{q_1^2 + 8q_1}\left[20z - 10z^2 + \frac{20(q_1+2)}{q_1^2}z^3 - \frac{5(3q_1+8)}{q_1^3}z^4 + \frac{4(q_1+3)}{q_1^4}z^5\right] \quad (2.17)$$

and from the heat balance integral (2.3) with respect to q_1 we obtain a nonlinear ODE, whose approximate analytical solution [7, 8] with the initial condition $q_1|_{x=0} = 0$ has the form

$$q_1 = 5.5116x^{0.4129} \Rightarrow x = x_1 = 0.01602. \quad (2.18)$$

Relations (2.17) and (2.18) determine the solution of problem (2.1), (2.2) in the second approximation (see curve *2* in Fig. 3). The results of calculations according to the formula (2.17) in a comparison with the calculations by the sweep method are given in Figs. 3 and 4. Their analysis allows us to conclude that the second approximation significantly clarifies the solution. In particular, at $x = 5\times10^{-3}$ (Fig. 3), the discrepancy between the results decreases from 8% (in the first approximation) to 4%.

To further improve the accuracy, we will find a solution in the third approximation. In relation (2.6) in this case it is necessary to use nine members of the series, therefore, to find the nine unknown coefficients, in addition to conditions (2.2), (2.10), (2.13), and (2.16), it is necessary to use three more ABC

$$z = 0: \quad \Theta_{zzz} - \Theta_{zz} - \Theta_z = 0; \quad z = q_1: \quad \Theta_{zzzz} = 0, \quad \Theta_{zzzzz} = 0. \quad (2.19)$$

Similarly, you can obtain any desired ABC, and in each subsequent approximation three ABC are added: one at $z = 0$ and two others at $z = q_1$. The use of a smaller number of them does not give a noticeable increase in the accuracy of the solution in this approximation.

In all subsequent approximations, starting with the fourth, ABC at the points $z = 0$ and $z = q_1$ are determined by the following general formulas:

$$z = 0: \quad \frac{\partial^{2(i-2)}\Theta}{\partial z^{2(i-2)}} = 0, \quad z = q_1: \quad \frac{\partial^i\Theta}{\partial z^i} = 0, \quad \frac{\partial^{i+1}\Theta}{\partial z^{i+1}} = 0, \quad i = 4,5,6,\ldots. \quad (2.20)$$

To find the solution of problem (2.1), (2.2) in the third approximation, the main boundary conditions (2.2) and ABC (2.10), (2.13), (2.16), and (2.19) are used. Similarly to the previous, we find

$$q_1 = 5.341x^{0.3874} \Rightarrow x = x_1 = 0.01323. \quad (2.21)$$

Using the method described above for constructing ABC, analytical solutions in the first, second, third, and fifteenth approximations were obtained.

The results of calculations of the displacement of the temperature perturbation front q_1 along the transverse coordinate z, depending on the value of longitudinal coordinate x in different approximations, are given in Fig. 2 (the approximation number is indicated on the curves). Their analysis allows us to conclude that with an increase in the number of approximations n, the value of the longitudinal coordinate x, at which the temperature perturbation front reaches the value of the coordinate $z = 1$, decreases, and at $n \to \infty$, $x \to 0$. This result is in full accordance with the hypothesis of an infinite rate of heat propagation underlying the derivation of the parabolic Eq. (2.1). An increase in the rate of movement of the temperature perturbation front with an increase in the number of approximations indicates an increase in the accuracy of the solutions obtained [5].

The results of calculations by the formula (2.4) in comparison with the calculation by the sweep method (dotted line) at $x = 10^{-4}$ and $x = 10^{-5}$ are given in Fig. 4. Their analysis allows us to conclude that with an increase in the number of approximations, the accuracy of the solution increases significantly.

3. Construction of solution at the second stage. The ABC method based on the heat balance integral can also be applied to the second stage of the heat exchange process. The second stage, which corresponds to the values of the longitudinal coordinate $x \geq x_1$, is characterized by a change in temperature over the entire cross section of channel, up to the state when all the liquid warms to the wall temperature. For this stage, the concept of a thermal layer loses its meaning, and the temperature in the center of channel is taken as an additional desired function $\Theta|_{z=1} = q_2$ (see the right part of Fig. 1).

The mathematical formulation of the problem for second stage of the process has the following form

$$(2z - z^2)\Theta_x = \Theta_{zz} - (1-z)^{-1}\Theta_z \quad 0 \leq z \leq 1, \quad x \geq x_1, \quad (3.1)$$

$$z = 0: \quad \Theta = 1; \quad z = 1: \quad \Theta = q_2, \quad \Theta_z = 0. \quad (3.2)$$

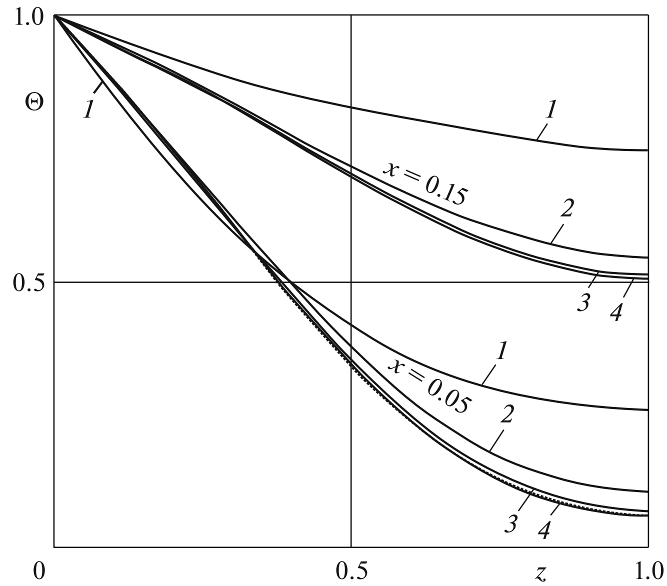

Fig. 5.

The initial condition of problem (3.1), (3.2) will be the equality (2.5) at $q_1|_{x=x_1} = 1$, however, its special satisfaction is not necessary, since it will be achieved in the process of solving the problem (3.1), (3.2). It is linked with the fact that at $x = x_1$, i.e., when $q_1|_{x=x_1} = 1$ and $q_2|_{x=x_1} = 0$, the problem statement (2.1), (2.2), and (3.1), (3.2) fully coincide.

Averaging differential equation (3.1) over the entire medium $(0 \le z \le 1)$, we obtain the integral of heat balance

$$\int_0^1 (1-z)(2z-z^2)\Theta_x dz = \int_0^1 [(1-z)\Theta_{xx} - \Theta_z]dz. \quad (3.3)$$

Problem solution (3.1), (3.2) will be taken as polynomial

$$\Theta = \sum_{k=0}^{n} b_k z^k, \quad (3.4)$$

where $b_k = b_k(q_2)$ are unknown coefficients. After determining the coefficients b_k $(k = 0,1,2)$ from boundary conditions (3.2) we obtain the equation

$$\Theta = 1 + (q_2 - 1)(2 - z)z. \quad (3.5)$$

Substituting this into the integral of heat balance (3.3), relative to unknown function q_2 we obtain an ODE. Its solution at the initial condition $q_2|_{x=x_1} = 0$ will be

$$q_2 = 1 - \exp[12(x_1 - x)], \quad (3.6)$$

where x_1 is the dimensionless distance from the inlet into the pipe at which the temperature perturbation front q_1 reaches the center of the pipe $z = 1$. From solving the problem in the first approximation at the first stage of process $x_1 = 0.0247$.

Substituting expression (3.6) into equality (3.5), we find

$$\Theta = z(z-2)\exp[12(x_1 - x)] + 1. \quad (3.7)$$

By direct substitution, we can verify that relation (3.7) exactly satisfies the boundary conditions (3.2), the initial condition, and the heat balance integral (3.3). Equation (3.1), as follows from equality (3.3), in this case is satisfied only on average.

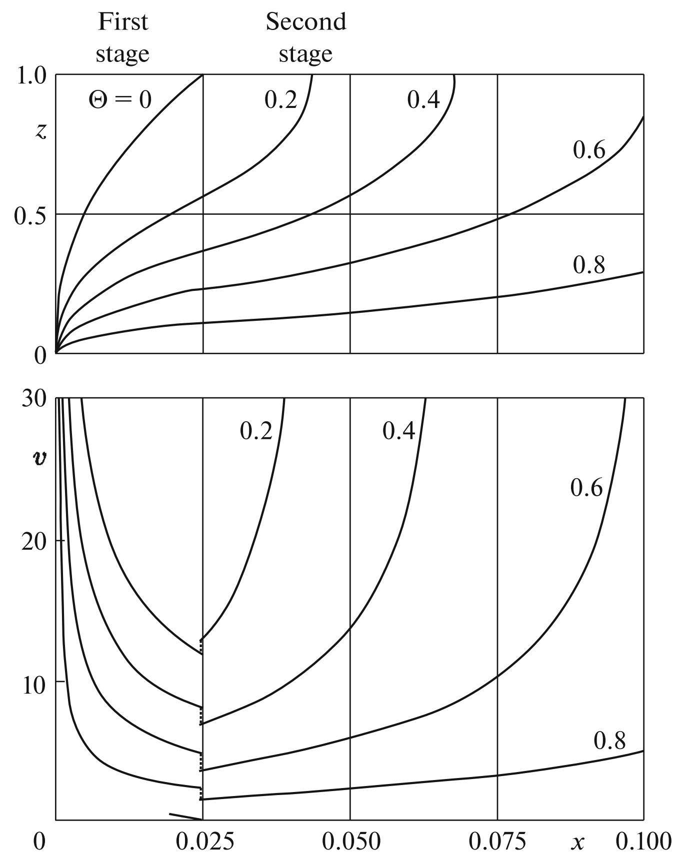

Fig. 6.

The results of calculations according to the formula (3.7) (curve *1*) in comparison with the exact solution [1] (curve *4*), as well as with the calculation by the sweep method (dashed line) at $x = 0.05$ and $x = 0.15$ are given in Fig. 5. Their analysis allows us to conclude that in the range of x values in the second stage of the process ($0.0247 \le x < \infty$), the exact solution nearly coincides with the numerical one. The difference in temperature obtained by formula (3.7) and from the exact solution (we will consider it in this range of values of the x coordinate to be the most reliable) exceeds 10% (for $0.05 \le x < \infty$).

To improve the accuracy, we will find a solution to problem (3.1), (3.2) in the second approximation with the involvement of ABC. To find them, we differentiate the boundary conditions (3.2) by the variable x. Then we have

$$z = 0: \quad \Theta_x = 0, \quad z = 1: \quad \Theta_x = q_{2x}, \quad \Theta_{zx} = 0. \tag{3.8}$$

Comparing Eq. (3.1) with the first condition (3.8), we obtain the first ABC coinciding with ABC (2.10).

To obtain the second ABC, we write Eq. (3.1) for the point $z = 1$. Opening the uncertainty arising in the second term of the right-hand side of equation (3.1) by the L'Hôpital's rule, we obtain the condition $z = 1$: $\Theta_x = 2\Theta_{zz}$. Comparing it with second condition (3.8), we find the second ABC

$$z = 1: \quad \Theta_{zz} = \frac{1}{2} q_{2x}. \tag{3.9}$$

Similarly we get the third ABC

$$z = 1: \quad \Theta_{zzz} = 0. \tag{3.10}$$

Substituting the series (3.4), limited to its first six members, to the basic boundary conditions (3.2) and ABC (2.10), (3.9), and (3.10), we obtain a system of six triangular algebraic linear equations for unknown coefficients b_k $(k = 0, 1, \ldots, 5)$, after which we find the expression

$$\Theta = 1 + 0.111(q_2 - 1)z(16z^4 + 55z^3 + 60z^2 + 10z + 20) + 0.333zq_{2x}(-z^4 + 3.25z^3 - z^2 + 0.25z + 0.5). \quad (3.11)$$

Substituting it into the integral of heat balance (3.3), relative to unknown function q_2 we obtain ABC

$$129q_{2xx} + 6484q_{2x} + 40320q_2 = 40320.$$

Integrating this with boundary conditions

$$x = x_1: \quad q_2 = 0, \quad x = x_1: \quad q_{2x} = 0,$$

we obtain

$$q_2 = 1 + \exp[-7.27(x_1 - x)] + \exp[-43(x_1 - x)]. \quad (3.12)$$

The value $x_1 = 0.01602$ was obtained at the second approximation of first stage of the process.

Relations (3.11) and (3.12) represent the solution of problem (3.1), (3.2) in the second approximation.

Note that the constant factors at $(x_1 - x)$ are slightly different from the first two eigenvalues $\lambda_1 = 7.31$ and $\lambda_2 = 44.6$, which are obtained by solving analytical problem (1.1) using exact analytical methods [1].

The results of calculations according to the formula (3.11) are given in Fig. 5 (curve *2*). Their analysis allows us to conclude, that compared with the first approximation, the accuracy of solving the problem has increased significantly. The maximum discrepancy of the results of the second approximation from the exact solution does not exceed 5% (in the range $0.05 \le x < \infty$).

Let us find the solution of problem (3.1), (3.2) in the third approximation. Similar to the previous relatively unknown function q_2, we obtain a linear third-order ODE. Its solution after determining the integration constants from the boundary conditions

$$x = x_1: \quad q_2 = 0, \quad q_{2x} = 0, \quad q_{2xx} = 0$$

has the form

$$q_2 = A_1 \exp[\nu_1(x - x_1)] + A_2 \exp[\nu_2(x - x_1)] + A_3 \exp[\nu_3(x - x_1)] + 1,$$
$$A_1 = -1.370; \quad A_2 = 0.4412; \quad A_3 = -0.1654; \quad \nu_1 = -7.321; \quad \nu_2 = -42.79; \quad \nu_3 = -243.2. \quad (3.13)$$

Relations (3.4) and (3.13), taking into account the found values of the coefficients b_k $(k = 0,1,\ldots,8)$, represent the solution of problem (3.1), (3.2) in the third approximation.

The results of calculations using formula (3.4) in the third approximation ($n = 8$) are given in Fig. 5 (curve *3*). Their analysis indicates a significant increase in the accuracy of the solution. In the range $x_1 = 0.01323 \le x < \infty$, the discrepancy with the exact solution does not exceed 1.5%.

A characteristic feature of the approximate analytical solutions is the simplicity of their construction with the polynomial dependence of temperature on the coordinate z, in contrast to the classical exact analytical solutions, where such dependence is represented by trigonometric and Bessel functions. Polynomial dependence allows us to obtain a solution in the form of a field of isotherms, as well as to determine the speed of movement of the isotherms.

The principle of construction of isotherms is considered in the example of the first approximation for the first and second stages of the process. Expressing the coordinate z as a function of temperature Θ and coordinate x, solutions (2.5) and (3.5) can be reduced to the form

$$z(\Theta, x) = q_1(1 + \sqrt{\Theta}) \quad \text{and} \quad z(\Theta, x) = (q_2 + \sqrt{(q_2 - \Theta)(q_2 - 1)} - 1)/(q_2 - 1). \quad (3.14)$$

The graphs of the distribution of isotherms over the coordinate z, depending on the value of the longitudinal coordinate, found by formulas (3.14) are given in the upper part of Fig. 6.

Defining the first derivatives of functions (3.14), we obtain the formulas for determining the velocities of the isotherm movement along the coordinate z depending on the coordinate for the first and second stages of the process, respectively.

$$v(\Theta, x) = (1 - \sqrt{\Theta})q_{1x} \quad \text{and} \quad v(\Theta, x) = \frac{1}{2}(q_2 - 1)^{-\frac{3}{2}}(q_2 - \Theta)(\Theta - 1)q_{2x}. \quad (3.15)$$

The velocity curves of the isotherms found by formulas (3.15) are given in the lower part of Fig. 6.

Note that the zero isotherm $\Theta = 0$ coincides with the graph of the motion of the temperature perturbation front along the coordinate z as a function of the coordinate x in the first approximation (see Fig. 2). In fact, at $\Theta(z, x) = 0$ the first expression (3.14), it completely coincides with formula (2.8), which characterizes the displacement of the temperature perturbation front. From this it follows that, in the physical sense, the temperature perturbation front is an analog of the isotherm moving along the coordinate z depending on the x coordinate, i.e., this is the zero isotherm. Due to the infinite propagation speed of the thermal perturbation underlying the parabolic equation (1.1), with a larger number of approximations of the integral method x_1 the value for the first stage of the process will approach zero, and, therefore, the zero isotherm will increase infinitely, which fully agrees with the hypothesis of infinite velocity propagation of the thermal perturbation inherent in Eq. (1.1).

Analysis of distribution of the isotherm movement speeds allows us to conclude that at $z \to 0$ the isotherm velocity tends to infinity, which is connected with the infinite value of the heat flux due to the assignment of the first kind of boundary condition at the point $z = 0$ (thermal shock). The isotherm velocities grow without limit when approaching the point $z = 1$, which is caused by setting the boundary condition of the adiabatic wall at this point (i.e., lack of heat exchange), and therefore all isotherms are perpendicular to the surface $z = 1$ (see the upper part of Fig. 6).

Due to the low accuracy of first approximation at both stages of the isotherm process, defined by formulas (3.14), there is a small kink at point $x = x_1 = 0.0247$ i.e., at the junction of the graphs of the first and second stages of the process (see the upper part of Fig. 6). In this regard, in the lower part of Fig. 6, at the intersection point of the velocities is a jump, which in the second approximation is barely observed, as well as a kink in the isotherms.

4. Solution of the inverse problem of heat transfer. The advantages of the approximate analytical solutions obtained above are that by using experimental data on the change in the coordinate x of the temperature of liquid in any one point of the pipe, by solving the inverse heat conduction problem, the Peclet number $\mathrm{Pe} = \overline{\omega} r/a$ and, therefore, the average flow rate of liquid $\overline{\omega}$ can be found. According to the average velocity, the volume $G_v = \overline{\omega} S$ or mass flow rate $G_m = \rho\overline{\omega} S$ of the fluid, can be obtained, where ρ is its density, and $S = \pi r^2$ is the cross-sectional area of the pipe.

Suppose that during the experiment the temperature of the liquid on the axis of the pipe of radius $r = 0.01$ m ($z = 1$) is measured. Measurements are taken at the arbitrarily selected points along the longitudinal coordinate (for example, at $\eta_1 = 10$, $\eta_2 = 11$, $\eta_3 = 12$, $\eta_4 = 13$ m).

As experimental data, we will use the results of the numerical solution of problem (1.1), according to which, at the indicated points, the temperature has values $\Theta = 0.50808$, 0.55881, 0.60445, 0.64545, respectively. By approximating the experimental data with a cubic parabola, we obtain a function describing the temperature variation in the liquid along the coordinate η in the range $10 \le \eta \le 13$

$$\Theta(1, \eta) = 0.1304\eta - 0.004943\eta^2 - 7.278 \times 10^{-5}\eta^3 - 0.3748. \tag{4.1}$$

From the solution of the inverse problem using the relation (3.7), one can reconstruct the Pe number. Substituting expression (4.1) into the left side of solution (3.7) at $z = 1$, taking into account the introduced dimensionless parameters, and determining the integral of the obtained relation in the range $10 \le \eta \le 13$, we find

$$\int_{10}^{13} \Theta(1, \eta) d\eta = \int_{10}^{13} \left\{1 + \left(1 - \exp\left[12(x_1 - 0.5\eta(\mathrm{Pe}\, r)^{-1})\right] - 1\right)\right\} d\eta. \tag{4.2}$$

Defining integrals, we will have a transcendental equation relative to Pe, solving which we find Pe = 2952, which is less than the exact value 3333.3 (at which the numerical calculation was performed) by 12%. The deviation of calculation results from the exact value of the Peclet number in the second approximation is 7% (Pe = 3564), and in the third, is less than 1% (Pe = 3384).

According to known values of Pe, r, and a, the average velocity of flow can be found, which in the third approximation at $a = 15 \times 10^{-8}$ m^2/s is $\overline{\omega} = 0.0507$ m/s (the exact value is 0.05 m/s).

5. Discussion of the results. From the determination of the temperature perturbation front (assuming a finite rate of heat propagation) and the introduction of additional boundary conditions, an approximate analytical solution of the heat exchange problem was obtained with stabilized laminar fluid flow in the channel. Separation of the heat transfer process into two stages along the longitudinal coordinate made it possible to simplify the process of constructing the solution of a complex nonlinear problem, reducing it to solving two problems, including ordinary differential equations, whose solution is much simpler than the original partial differential equation. The introduction of the temperature perturbation front made it possible to abandon the fulfillment of the initial condition along the channel width, replacing it with an initial condition that obtains only at the channel entrance. In this way, it is possible to avoid the linear superposition of particular solutions used in classical methods, which leads to a significant complication of the analytical solution (having the form of an infinite series) and, as a consequence, to its poor convergence at small values of the coordinate x. As a result of the division of the heat exchange process into two stages in time, for each of them, fairly simple analytical solutions were obtained, allowing us to perform temperature state calculations in almost the entire range of the longitudinal variable (with a stabilized velocity profile, including at the channel entrance, see above the list of assumptions made).

The solutions obtained do not contain infinite series and special functions, which made it possible to reconstruct the Peclet number and, consequently, the average flow rate of the fluid, by which the flow rate can be determined, using the temperature change in time at the point $z = 1$ known from the numerical experiment. Investigations of the temperature state of the medium in the fields of isothermal lines with the determination of movement velocities of isotherms using a transverse variable depending on the longitudinal coordinate were also carried out.

It should be noted that when increasing the number of approximations due to the complication of additional boundary conditions, the difficulties of obtaining approximate analytical solutions grow. In particular, the differential equations are complicated with respect to the desired functions $q_1 = (x)$ and $q_2 = (x)$, and applying these to the second stage of the process, the order of the equation also increases, which creates additional difficulties in obtaining its analytical solution. However, this only leads to an increase in the amount of computational work.

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