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On the Model of Generation of Vortex Structures in an Isotropic Turbulent Flow

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Abstract—It is known that turbulence is characterized by intermittence which is closely related to the development of unsteady nonisotropic intense small-scale vortex structures. In this study, small fluid particles from the inertial range of isotropic turbulence are considered. It is shown that the phenomenon of rotation intensification and stretching of the particles can be analyzed theoretically. In recent experimental and numerical studies, where this phenomenon was called "the pirouette effect", its significance in the mechanism of the intense small-scale structures generation was discussed. In this study, a linear stochastic Lagrangian model for the effect is developed. In this model, the kinetic equation for the distribution function of the squared cosine of the angle between the vorticity and the eigenvector of the strain rate tensor of a fluid particle is derived and time history asymptotics of this quantity are analytically calculated at large and small times. The results are in good agreement with the recent experiments and numerical calculations. An analysis made in this study shows that the linear processes probably play the crucial role in certain processes in the isotropic turbulence, which is known to be a principally nonlinear phenomenon. The model developed makes it possible to analyze the statistics of the Lagrangian dynamics of small fluid particles in the inertial range which can be useful in some computational approaches to turbulence.

Keywords: intermittence, homogeneous and isotropic turbulence, inertial range, vorticity, strain rate tensor, Gaussian process, Furutsu–Novikov formula.

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The notion of homogeneous and isotropic turbulence introduced by Taylor [1] and then developed and generalized by Kolmogorov [2–4] is not only of an obvious theoretical interest, as an example of turbulence simplest for a theoretical analysis, but also of a practical importance, due to certain reasons. Firstly, as predicted by Kolmogorov's theory and confirmed by experiments, in an arbitrary developed turbulent flow small-scale disturbances have an universal locally-isotropic structure [4, 5]. This makes it possible to use the results of theory of isotropic turbulence for any arbitrary small-scale turbulence. Secondly, a near-isotropic turbulence is attained in the atmosphere, laboratory experiments, and numerical calculations [6], which allows one to find new nontrivial phenomena in the isotropic turbulence, which can be universal. For example, one of these universal properties is intermittence [4, 5] which, is closely related to the presence of rare, very intense, wormlike vortex structures. Precisely these structures determine the statistical characteristics of a turbulent flow [4, 5, 7], thus making it possible to theoretically substantiate the statistic properties of developed turbulence [8, 9] and to develop the quadrupole source model in the correlation theory of subsonic jet noise [10].

Till recently, under laboratory conditions the isotropic turbulence has been realized and measured only in grid flows [1, 11, 12]. The presence of high mean velocity in the turbulent flow behind the grid (relative to the laboratory) makes difficult the measurement of the Lagrangian characteristics of the isotropic turbulence. At the same time, the Lagrangian models of isotropic turbulence are used for closing the equations for the probability density [5, 13] which, for example, finds application in considering the turbulent combustion problems [14, 15]. A device developed in [16] produced "motion-less" stationary isotropic turbulence in a small region between rotating disks, which made it possible

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to perform Lagrangian measurements in isotropic turbulence invoking high-precision measurement methods. The particle image velocimetry (PIV) made it possible to measure in an analogous setup a new fine Lagrangian effect of rotation intensification and stretching of fluid particles from the inertial range [17] called the pirouette effect.

We will consider this effect in more detail. We will introduce the designation for the vorticity, or the curl of the flow velocity $\mathbf{v}(\mathbf{r}, t)$

$$\omega(\mathbf{r}, t) = [\nabla, \mathbf{v}(\mathbf{r}, t)].$$

In the experiments and numerical calculations [17, 18] a fluid particle studied was confined by a tetrahedron at whose vertices there were four test particles. Measuring the particle velocities at a tetrahedron vertex one can estimate the flow vorticity according the so-called minimal model [18, 19]. For the fluid particles from the viscous (Kolmogorov) interval this model gives the true vorticity value at the point. For fluid particles from the inertial range the vorticity considerably varies in the fluid particle from one point to another and the model gives the effective, "large-scale" vorticity $\omega(t)$, whose direction will be measured with respect to the eigenvectors of the large-scale strain rate tensor $b_{ij}(t)$ obtained using the same minimal model from the equation

$$b_{ij}(\mathbf{r}, t) = \frac{1}{2}(\partial_i v_j(\mathbf{r}, t) + \partial_j v_i(\mathbf{r}, t)),$$

where $\partial_i = \partial/\partial r_i$ is the partial derivative with respect to coordinate i. The motion of a fluid particle of the Kolmogorov scale can be decomposed into rotation about the vorticity vector and deformation along the eigenvectors of the strain rate tensor [20]. For a particle from the inertial range the large-scale vorticity and strain rate tensor determined from the minimal model also qualitatively respond this decomposition. By virtue of incompressibility, the three eigenvectors of the strain rate tensor will necessarily include the one, along which the particle stretches (with the maximum eigenvalue), the one, along which it compresses (with the minimum eigenvalue), and the intermediate vector, along which either compression or stretching can occur. Following the designations of studies [17, 18], we will designate the vector $\mathbf{e}_1(0)$ as the stretching vector, the vector $\mathbf{e}_2(0)$ as the intermediate vector, and the vector $\mathbf{e}_3(0)$ as the compression vector.

The experiments and calculations [17] showed that at small times a decrease in the angle between the $\boldsymbol{\omega}(t)$ and $\mathbf{e}_1(0)$ vectors is observable in the typical experimental realization. This means the temporal growth of the quantity α_t^2, that is, the squared cosine of the angle between the $\boldsymbol{\omega}(t)$ and $\mathbf{e}_1(0)$ vectors averaged over the ensemble of experimental realizations

$$\alpha_t^2 = \left\langle \left(\frac{\boldsymbol{\omega}(t)}{|\boldsymbol{\omega}(t)|}, \mathbf{e}_1(0) \right)^2 \right\rangle.$$

Here, the angular brackets mean the averaging over the ensemble of the realizations, while the conventional parentheses mean the scalar product of vectors. In this case, we mean the product of the unit vectors aligned with the vorticity $\boldsymbol{\omega}(t)$ and the stretching extension vector $\mathbf{e}_1(0)$. The phenomenon of increase in α_t^2 at small times was named the pirouette effect by analogy with the effect of an increase in the angular velocity of a figure skater making rotation about his own axis and thus diminishing the moment of inertia about this axis. It should be noted that qualitatively this effect is a consequence of the Helmholtz theorem [20] but the quantitative assessment of α_t^2 made in this study requires the consideration of the system dynamics in more detail. In the other study [18] the authors determined the time dependences of the squared cosines of the angles between $\boldsymbol{\omega}(t)$ and $\mathbf{e}_2(0)$ and $\boldsymbol{\omega}(t)$ and $\mathbf{e}_3(0)$

$$\beta_t^2 = \left\langle \left(\frac{\boldsymbol{\omega}(t)}{|\boldsymbol{\omega}(t)|}, \mathbf{e}_2(0) \right)^2 \right\rangle, \quad \gamma_t^2 = \left\langle \left(\frac{\boldsymbol{\omega}(t)}{|\boldsymbol{\omega}(t)|}, \mathbf{e}_3(0) \right)^2 \right\rangle.$$

In [17, 18] various qualitative considerations explaining the pirouette effect are presented, while in [21] the effect is explained by means of modeling the dynamics of the velocity gradient tensor. In those studies the fluid dynamic nonlinearity plays a significant role. The results of this study show that this effect can be explained within the framework of the linear model.

In this study, we formulate a simple theoretical model which reduces the problem of the fluid particle dynamics in the inertial range to the linear stochastic problem. The main idea of the approach developed in the study is that the large-scale vorticity and strain rate tensor are assumed to be statistically independent. The foundation for this supposition is given by the absence of vortex self-action demonstrated in [8], that is, that small-scale fluctuations have almost no effect on the vorticity dynamics concentrated in narrow tubes. Another supposition of the model concerning the Gaussian nature of the processes and random quantities determining the fluid particle dynamics is a simplification of the real process, since it is known that turbulent flow is non-Gaussian in nature [4, 5]. However, the suppositions formulated above make it possible to evaluate the pirouette effect characteristics in the linear stochastic model applying the mathematical apparatus of the theory of random processes [22]. Since the asymptotic time dependences of the correlators α_t^2, β_t^2, and γ_t^2 derived in this study are in agreement with the experimental and numerical results [17, 18], this suggests that the non-Gaussian nature of these processes does not play a crucial role in the effect under consideration.

Thus, the study is devoted to the quantitative explanation of the effect of fluid particle stretching along the vorticity line in the inertial range [17, 18] using the linear stochastic model in which the large-scale strain rate tensor may be assumed to be an external source of the vorticity dynamics. Using this approach the effect characteristics can be evaluated to validate the Lagrangian model by means of comparing the estimates obtained with the experimental and numerical results [17, 18].

The study is organized as follows. In Section 1 the basic equation of the linear stochastic model is obtained which plays the role of the Langevin equation. In Section 2 the behavior of the correlators α_t^2, β_t^2, and γ_t^2 is studied in the isotropic delta-correlated Gaussian model at large times and their exponential decay is obtained, in agreement with [17, 18]. In Sections 3 and 4 the natural Gaussian delta-correlated stochastic model of the strain rate tensor is developed to consider the effect not only in the large time limit. In Sections 5 and 6 it is shown that in the large time limit the model constructed in Sections 3 and 4 coincides with the isotropic model considered in Section 2; moreover, in Section 6 the relationship between the correlation constants of the two models is established. In Section 7 the behavior of the correlators α_t^2, β_t^2, and γ_t^2 in the stochastic model of the strain rate tensor is analyzed at small times and expressions for the linear asymptotics of the temporal behavior of the correlators are derived. However, these expressions require also the averaging over the $\boldsymbol{\omega}(0)$ direction in the $(\mathbf{e}_1(0), \mathbf{e}_2(0), \mathbf{e}_3(0))$ basis, which is made in Section 8. In Section 9 the results of the present theoretical analysis are compared with the data [17, 18]. It is shown that in the theoretical model the correlator α_t^2 growth (pirouette effect) is in general determined by only the mean value of the square of the flow vorticity. The corresponding estimates show the satisfactory agreement between the theory and the experiment. Finally, the main results of the study are formulated in Section 10.

1. DYNAMICS OF LARGE-SCALE VORTICITY

In a developed turbulent flow (in considering the structures, whose scales are much greater than the Kolmogorov scale) viscosity can be neglected. Then the vorticity dynamics are determined by the following equation

$$\partial_t \boldsymbol{\omega} + (\mathbf{v}\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\nabla)\mathbf{v}.$$

In the reference frame fitted with a fluid particle the vorticity behavior can be brought into the form [3]:

$$\frac{d\boldsymbol{\omega}(\boldsymbol{\rho}, t)}{dt} = \mathbf{B}(\boldsymbol{\rho}, t)\boldsymbol{\omega}(\boldsymbol{\rho}, t). \tag{1.1}$$

Here, $\boldsymbol{\rho}$ is the coordinate in the reference frame fitted with a certain point in the fluid particle, while $\boldsymbol{\omega}(\boldsymbol{\rho}, t)$ and $\mathbf{B}(\boldsymbol{\rho}, t)$ are the vorticity and the strain rate tensor in this coordinate system. Here, in contrast to the designation of the tensor $\mathbf{B}(\boldsymbol{\rho}, t)$, its components will be denoted by small letters; the same rule will be applied for most of tensors and matrices considered below.

We will consider an intense locally-nonisotropic structure (vortex worm) within the fluid particle. As shown in [23], at the center of the fluid particle the strain rate tensor can be decomposed into the small-scale and large-scale components determined by the velocity fields within and outside the particle, respectively. Since within the vortex worm and in its proximity the strain rate tensor is orthogonal to the vorticity (see [8, Appendix 1]), in considering its dynamics in Eq. (1.1) the tensor $\mathbf{B}(\boldsymbol{\rho}, t)$ may be replaced by the large-scale tensor $\mathbf{B}(t)$

$$\frac{d\boldsymbol{\omega}(\boldsymbol{\rho}, t)}{dt} = \mathbf{B}(t)\boldsymbol{\omega}(\boldsymbol{\rho}, t).$$

Since for fairly small fluid particles the large-scale vorticity $\boldsymbol{\omega}(t)$ is determined by the most intense structure within the fluid particle [4, 9, 24], the latter equation can also be considered as an equation of the dynamics of a small fluid particle from the inertial range[1)] replacing $\boldsymbol{\omega}(\boldsymbol{\rho}, t)$ by the large-scale vorticity $\boldsymbol{\omega}(t)$ of the entire particle

$$\frac{d\boldsymbol{\omega}(t)}{dt} = \mathbf{B}(t)\boldsymbol{\omega}(t). \tag{1.2}$$

The energy of a three-dimensional turbulent flow is transferred from the large to the small scales; for this reason, the large scale dynamics determine the dynamics of the smaller scales and, contrariwise, the small scale dynamics have only a small effect on the dynamics of the larger scales. Thus, it may be supposed that the tensor $\mathbf{B}(t)$ enters into Eq. (1.2), as an external random source independent of $\boldsymbol{\omega}(t)$. The applicability of this approach was numerically demonstrated in [24]. For the sake of simplicity we will consider the source $\mathbf{B}(t)$ as Gaussian. The considerations presented after Eq. (1.2), together with Eq. (1.2) itself, provide the foundation of the theoretical consideration of the problem. This equation can be considered as the linear model of the effect.

2. DECAY OF THE CORRELATORS α_t^2, β_t^2, AND γ_t^2 AT LARGE TIMES

Basing on Eq. (1.2) we will consider the decay of initial disturbances in a nondistinguished basis (at large times $(\mathbf{e}_1(0), \mathbf{e}_2(0), \mathbf{e}_3(0))$ becomes such a basis) which can be compared with the decay found in [17, 18].

In the problem under consideration all the five independent components $b_{ij}(t)$ can be regarded as random processes determining the dynamics of $\boldsymbol{\omega}(t)$. If these random processes are assumed to be Gaussian, then, as is known [25], it is sufficient to know pairwise correlations of the processes to describe their statistical properties. For the sake of simplicity we will assume that the $b_{ij}(t)$ process is delta-correlated [22]

$$\langle b_{ij}(t) b_{kl}(t')\rangle_{b_{\alpha\beta}(\tau)} = D_{ijkl}\delta(t - t'). \tag{2.1}$$

Here, $t \geq 0$ and $t' \geq 0$ are certain moments of time, $\delta(t - t')$ is the delta function [26, 27], and D_{ijkl} is the correlation tensor. The index $b_{\alpha\beta}(\tau)$ of the averaging brackets means that the averaging is made over the ensemble of realizations of all the five independent processes. These processes are defined at $\tau \in [0, +\infty]$. Under the assumptions made above and using Eq. (1.2) and the Furutsu–Novikov formula [22] the following equation for the vorticity distribution function $f(t, \boldsymbol{\omega})$ can be obtained

$$f(t, \boldsymbol{\omega}) = \langle \delta(\boldsymbol{\omega} - \boldsymbol{\omega}(t))\rangle_{\boldsymbol{\omega}(\tau)},$$
$$\frac{\partial f}{\partial t} = D_{ijkl}\frac{\partial}{\partial \omega_i}\left\{\omega_j \frac{\partial}{\partial \omega_k}(\omega_l f)\right\}. \tag{2.2}$$

By virtue of the statistical homogeneity and isotropicity and the flow incompressibility, the correlation tensor D_{ijkl} depends only on one constant D

$$D_{ijkl} = D\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right). \tag{2.3}$$

This fact considerably simplifies Eq. (2.2), in which it is convenient in this case to pass to spherical coordinates

$$\omega_1 = \omega\cos\theta = \omega\mu,$$

[1)] Such particles are considered in [17, 18].

$$\omega_2 = \omega \sin\theta \cos\phi = \omega \cos\phi \sqrt{1-\mu^2},$$
$$\omega_3 = \omega \sin\theta \sin\phi = \omega \sin\phi \sqrt{1-\mu^2}.$$

Thus, substituting expression (2.3) into Eq. (2.2) and making in it the change of variables, after some transformations we can obtain

$$\frac{\partial f}{\partial t} = \frac{D}{2}\left\{\frac{4}{3\omega^2}\frac{\partial}{\partial\omega}\left(\omega^4\frac{\partial f}{\partial\omega}\right) + \frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial f}{\partial\mu}\right) + \frac{1}{1-\mu^2}\frac{\partial^2 f}{\partial\phi}\right\}.$$

Thus, multiplying this equation by μ^2 and integrating it over the $\boldsymbol{\omega}$ space we can obtain an ordinary differential equation for an unknown quantity, whose role can be played by both α_t^2, and β_t^2, and γ_t^2

$$\frac{d\langle\mu^2(t)\rangle_{\boldsymbol{\omega}(\tau)}}{dt} = 2D(1 - 3\langle\mu^2(t)\rangle_{\boldsymbol{\omega}(t)}).$$

Let $\langle\mu^2(0)\rangle_{\boldsymbol{\omega}(t)}$ be an initial disturbance. Then the solution

$$\langle\mu^2(t)\rangle_{\boldsymbol{\omega}(\tau)} = \frac{1}{3} + \left(\langle\mu^2(0)\rangle_{\boldsymbol{\omega}(t)} - \frac{1}{3}\right)\exp(-2Dt). \tag{2.4}$$

An important property of this solution is the presence of the stationary point $\langle\mu^2(t)\rangle_{\boldsymbol{\omega}(t)} = 1/3$. The existence of this solution could be supposed beforehand, since the nondistinguished basis must conserve the direction isotropy. Expression (2.4) represents the asymptotics of α_t^2, β_t^2, and γ_t^2 at large times. In fact, the exponential decay at large times is observable in [17, 18].

3. STRAIN RATE TENSOR IN THE BASIS OF ITS EIGENVECTORS

We will consider the large-scale strain rate tensor $b_{ij}(t)$ in the basis of its eigenvectors $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$, and $\mathbf{e}_3(t)$, where the form it takes is simplest

$$B^{eig}(t) = b_{ij}^{eig}(t) = \begin{pmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{pmatrix}.$$

Here, the eigenvalues $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ of the strain rate tensor represent random processes. From the flow incompressibility condition we have

$$\lambda_1(t) + \lambda_2(t) + \lambda_3(t) = 0.$$

We will now so put in order the eigenvectors $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$, and $\mathbf{e}_3(t)$ that the vector $\mathbf{e}_1(t)$ shall be always associated with the greatest eigenvalue $\lambda_1(t)$, the vector $\mathbf{e}_2(t)$ with the intermediate value $\lambda_2(t)$, and the vector $\mathbf{e}_3(t)$ with the least value $\lambda_3(t)$

$$\lambda_1(t) > \lambda_2(t) > \lambda_3(t). \tag{3.1}$$

In the case of the Gaussian-distributed velocity field the strain rate tensor is also normally distributed (as the derivative of the Gaussian field)[2)]. Moreover, assuming that the velocity field has the same spectral characteristics as in an isotropic turbulent flow, we can determine the strain rate tensor distributions in an explicit form [29, 30]. From it the mathematical expectations of the eigenvalues can in turn be found

$$\langle\lambda_1(t)\rangle_{\lambda(\tau)} = \lambda, \quad \langle\lambda_2(t)\rangle_{\lambda(\tau)} = 0, \quad \langle\lambda_3(t)\rangle_{\lambda(\tau)} = -\lambda, \quad \lambda = 3\sqrt{\frac{3\langle\omega^2\rangle}{10\pi}}. \tag{3.2}$$

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2) An important "non-Gaussian" feature of a turbulent velocity field is the on-average positivity of the intermediate eigenvalue $\langle\lambda_2(t)\rangle_{\lambda(\tau)} > 0$ [6, 12, 18, 28, 29], which characterizes the irreversibility of the cascade process of energy transfer from the large to the small scales. Since in this study we use the Gaussian approximation, this effect is not taken into account. We are planning to take account for it in the future.

Here, $\langle\omega^2\rangle$ is the mean value of the squared vorticity of a fluid particle. The last averagings are made over the ensemble of realizations of the random process $\lambda(\tau) = (\lambda_1(\tau); \lambda_2(\tau); \lambda_3(\tau))$. We will analyze the dependence of $\langle\omega^2\rangle$ on the fluid particle size r_0. Obviously that $\langle\omega^2\rangle$ tends to zero with fluid particle enlargement. On the other hand, when the particle scale is of the order of the Kolmogorov scale, $\langle\omega^2\rangle$ is independent on its size and is determined according to the well-known formula $\langle\omega^2\rangle = \varepsilon/\nu = (\varepsilon/\lambda^2_{\mathrm{Kolm}})^{2/3}$, where ε is the energy deviation rate, ν is kinematic viscosity, and λ_{Kolm} is the Kolmogorov scale [4]. In the inertial range the viscosity effect is unimportant; for this reason, from theory of dimensions we obtain $\langle\omega^2\rangle \sim (\varepsilon/r_0^2)^{2/3}$. Sewing these two dependences together we obtain

$$\langle\omega^2\rangle = \begin{cases} \frac{\varepsilon}{\nu}, & r_0 \le \lambda_{\mathrm{Kolm}}, \\ \left(\frac{\varepsilon}{r_0^2}\right)^{2/3}, & r_0 > \lambda_{\mathrm{Kolm}}. \end{cases} \tag{3.3}$$

The process $b_{ij}^{eig}(t)$ can be subdivided into the time-constant random part $b_{ij}^{eig}(0)$ (initial conditions) and the process $b_{o\ ij}^{eig}(t)$ with zero mathematical expectation

$$B^{eig}(t) = B^{eig}(0) + B_0^{eig}(t) = \begin{pmatrix} \lambda_1(0) + \lambda_{1o}(t) & 0 & 0 \\ 0 & \lambda_2(0) + \lambda_{2o}(t) & 0 \\ 0 & 0 & \lambda_3(0) + \lambda_{3o}(t) \end{pmatrix},$$

$$\langle B^{eig}(t)\rangle_{\lambda(t)} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad \langle B_o^{eig}(t)\rangle_{\lambda(t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

We will assume the processes $\lambda_{1o}(t)$, $\lambda_{2o}(t)$, and $\lambda_{3o}(t)$ (process $\lambda_o(t)$) to be Gaussian noises against the background of the mean values[3)]. For the simplicity of the analysis it can be assumed that they are independent delta-correlated Gaussian noises with the same statistical characteristics. Then the pairwise correlations of the random processes $\lambda_{1o}(t)$, $\lambda_{2o}(t)$, and $\lambda_{3o}(t)$ can be defined as follows:

$$\langle\lambda_{1o}(t)\lambda_{1o}(t')\rangle_{\lambda(\tau)} = \langle\lambda_{2o}(t)\lambda_{2o}(t')\rangle_{\lambda(\tau)} = \langle\lambda_{3o}(t)\lambda_{3o}(t')\rangle_{\lambda(\tau)} = \Lambda\delta(t-t').$$

From the incompressibility condition we can determine cross-correlations in the form:

$$\langle\lambda_{io}(t)\lambda_{jo}(t')\rangle_{\lambda(\tau)} = \begin{pmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{pmatrix} \Lambda\delta(t-t') = \Lambda_{ij}\delta(t-t'), \tag{3.5}$$

where Λ_{ij} is the correlation matrix[4)] of the process $\lambda_{ij}(t)$.

4. STRAIN RATE TENSOR IN THE BASIS OF ITS ORIGINAL EIGENVECTORS

As shown in Sect. 1, Eq. (1.2) holds true in the reference frame executing translational motion together with a fluid particle (TRF). The reference frame rigidly fitted with the own basis of the strain rate tensor (ORF) is not such a system. We will choose as a TRF a coordinate system, whose axes $\mathbf{e}_1(0)$, $\mathbf{e}_2(0)$, and $\mathbf{e}_3(0)$ originally coincide with the ORF axes $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$, and $\mathbf{e}_3(t)$. In the following

[3)] Actually, this procedure is not completely correct. Even in the case of a Gaussian distributed matrix its eigenvalues are distributed in essentially non-Gaussian way [29, 31]. However, for estimations this process can be regarded as Gaussian. Moreover, we shall see that the correlator $\alpha_t^2(t)$ is actually independent of the statistical properties $\lambda_o(t)$ but is determined by the initial mean values.

[4)] From this matrix it can be readily shown that the eigenvalues are linearly dependent, in agreement with the flow incompressibility condition.

considerations the ORF can conveniently be regarded as fixed. As a result of rotation of the TRF about the ORF the strain rate tensor is transformed as follows

$$\mathbf{B}(t) = \mathbf{R}^T \mathbf{B}^{eig}(t)\mathbf{R}(t). \tag{4.1}$$

Here, $\mathbf{R}(t)$ is the time-dependent matrix of the TRF rotation about the ORF and $\mathbf{B}(t)$ is the same vector that enters into Eq. (1.2). Generally, it is known that the matrix $\mathbf{R}(t)$ can be determined from the following differential matrix equation [32]

$$\frac{d\mathbf{R}}{dt} = \mathbf{R}(t) \cdot \mathbf{O}(t). \tag{4.2}$$

Here, $\mathbf{O}(t)$ is the antisymmetric matrix of the angular velocities of the TRF rotation about the ORF

$$\mathbf{O}(t) = \begin{pmatrix} 0 & -o_3(t) & o_2(t) \\ o_3(t) & 0 & -o_1(t) \\ -o_2(t) & o_1(t) & 0 \end{pmatrix}.$$

Since at zero moment the TRF and ORF coincide, the passage matrix $\mathbf{R}(0)$ is the unit matrix, we have

$$\mathbf{R}(0) = \mathbf{E}. \tag{4.3}$$

We will assume that the components of $\mathbf{O}(t)$ are delta-correlated, Gaussian, and independent

$$\langle o_i(t) o_j(t') \rangle_{o(\tau)} = \Omega \delta_{ij} \delta(t - t'). \tag{4.4}$$

In Eq. (4.4) the averaging is made over realizations of the three-dimensional random-process vector $\mathbf{o}(\tau) = (o_1(\tau), o_2(\tau), o_3(\tau))$. Substituting Eq. (3.4) into Eq. (4.1) we can obtain the following decomposition of the strain rate tensor into two terms

$$b_{ij}(t) = b^0_{ij}(t) + b_{ij_o}(t).$$

Here, the following designations are introduced

$$b^0_{ij}(t) = \lambda_{(n)}(0) r_{in}(t) r_{jn}(t), \quad b_{ij_o}(t) = \lambda_{(n)_o}(0) r_{in}(t) r_{jn}(t). \tag{4.5}$$

The tensor index in the parentheses means that the summation is over it, together with the indices which are not contained in parentheses, for example

$$b^0_{ij}(t) = \lambda_{(n)}(0) r_{in}(t) r_{jn}(t) = \lambda_1(0) r_{i1}(t) r_{j1}(t) + \lambda_2(0) r_{i2}(t) r_{j2}(t) + \lambda_3(0) r_{i3}(t) r_{j3}(t).$$

The averaging must be made both over the initial conditions $\lambda(0)$ and over the random processes $\lambda_o(t)$ and $\mathbf{o}(t)$, which represent the same process. This means that conditions (3.5) and (4.4) must be supplemented with the pairwise cross-correlations. For the sake of simplicity we may assume that $\lambda_o(t)$ and $\mathbf{o}(t)$ are independent

$$\langle \lambda_{io}(t) o_j(t') \rangle_{\lambda(\tau), o(\tau)} = 0. \tag{4.6}$$

In what follows the set of conditions (3.5), (4.4), and (4.6) will be named the Gaussian, delta-correlated model of the strain rate tensor.

5. DECAY OF THE AVERAGED TENSOR IN THE BASIS OF THE ORIGINAL EIGENVECTORS

In Sections 3 and 4 we formulated the Gaussian delta-correlated model of the strain rate tensor. To apply it to the stochastic equation (1.2) it is necessary to find the mathematical expectation and the pairwise correlation of the strain rate tensor. In this section we will find the mathematical expectation $\langle b_{ij}(t)\rangle_{\lambda(\tau),o(\tau)} = \langle b_{ij}(t)\rangle_{proc}$ (for the sake of brevity the averaging over a process is denoted by the subscript *proc*).

Since

$$\langle b_{ij_o}(t)\rangle_{proc} = \langle \lambda_{(n)_o}(t) r_{in}(t) r_{jn}(t)\rangle_{proc} = \langle \lambda_{(n)_o}(t)\rangle_{\lambda(\tau)}\langle r_{in}(t) r_{jn}(t)\rangle_{o(t)} = 0,$$

the unknown parameters can be simplified

$$\langle b_{ij}(t)\rangle_{proc} = \lambda_{(n)}(0)\langle r_{in}(t) r_{jn}(t)\rangle_{\lambda(\tau)}.$$

To determine them the distribution function of the rotation matrix can be introduced

$$f(\mathbf{R}, t) = \langle \delta(\mathbf{R} - \mathbf{R}(t))\rangle_{o(\tau)}.$$

Using Eqs. (4.2) with the initial conditions (4.3) we obtain the following equation with the initial conditions for the function $f(\mathbf{R}, t)$

$$\frac{\partial f}{\partial t} = \Omega\left(r_{pm}\frac{\partial}{\partial r_{pl}} r_{km}\frac{\partial}{\partial r_{kl}} f - r_{pl}\frac{\partial}{\partial r_{pm}} r_{km}\frac{\partial}{\partial r_{kl}} f\right), \tag{5.1}$$

$$f(\mathbf{R}, 0) = \delta(\mathbf{R} - \mathbf{E}). \tag{5.2}$$

Given the distribution function, the unknown correlator can be calculated as an integral over the nine-dimensional space

$$\lambda_{(n)}(0)\langle r_{in}(t) r_{jn}(t)\rangle_{\omega(\tau)} = \lambda_{(n)}(0)\int r_{in} r_{jn} f(\mathbf{R}, t)\, d\mathbf{R}.$$

Here, $d\mathbf{R} = dr_{11}\, dr_{12}\, \ldots\, dr_{33}$ is the nine-dimensional differential. We will multiply the right and left sides of Eq. (5.1) and the initial condition (5.2) by $\lambda_{(n)} r_{in} r_{jn}$ and integrate the expressions thus obtained with respect to $d\mathbf{R}$. As a result of this integration, we arrive at the system of ordinary differential equations

$$\lambda_{(n)}(0)\frac{d}{dt}\langle r_{in}(t) r_{jn}(t)\rangle_{o(\tau)} = 2\Omega(\lambda_{(t)}(0) - \lambda_{(n)}(0))\delta_{ll}\langle r_{in}(t) r_{jn}(t)\rangle_{o(\tau)},$$

$$\lambda_{(n)}(0)\langle r_{in}(t) r_{jn}(t)\rangle_{o(\tau)} = \lambda_{(n)}(0)\delta_{in}\delta_{jn}.$$

The flow incompressibility condition, together with Eq. (4.5), make it possible to bring this equation into the form:

$$\frac{d}{dt}\langle b^0_{ij}(t)\rangle_{proc} = -6\Omega\langle b^0_{ij}(t)\rangle_{proc},$$

$$\langle b^0_{ij}(t)\rangle_{proc} = \begin{pmatrix} \lambda_1(0) & 0 & 0 \\ 0 & \lambda_2(0) & 0 \\ 0 & 0 & \lambda_3(0) \end{pmatrix}.$$

This is a linear equation, whose solution is given by the formula

$$\langle b_{ij}(t)\rangle_{proc} = \begin{pmatrix} \lambda_1(0)\exp(-6\Omega t) & 0 & 0 \\ 0 & \lambda_2(0)\exp(-6\Omega t) & 0 \\ 0 & 0 & \lambda_3(0)\exp(-6\Omega t) \end{pmatrix}. \tag{5.3}$$

Thus, in the Gaussian delta-correlated model the diagonal components associated with the distinguishness of the originally chosen basis decay with time. We note that, firstly, despite the delta-correlation of the control processes, the inertial exponential decay of the diagonal components of the tensor is observable. Secondly, the role of incompressibility in the universality of this process is very important. In fact, in the case of the unit tensor (obviously, with a nonzero trace), which remains the same in all reference frames, none decay of diagonal components can be observable.

6. ISOTROPIZATION OF THE CORRELATION TENSOR IN THE BASIS OF THE ORIGINAL EIGENVECTORS

We will consider the pairwise correlator of the strain rate tensor

$$\langle (b_{ij}(t) - \langle b_{ij}(t)\rangle_{proc})(b_{ij}(t') - \langle b_{ij}(t')\rangle_{proc})\rangle_{proc}.$$

To determine it requires the distribution function dependent on two moments of time

$$f(\mathbf{R}, \mathbf{R}', t, t') = \langle \delta(\mathbf{R} - \mathbf{R}(t))\delta(\mathbf{R}' - \mathbf{R}(t'))\rangle_{o(\tau)}.$$

However, in the Gaussian delta-correlated model the distribution function $f(R, t)$ is also sufficient to determine this correlator. Omitting some simple transformations we obtain

$$\langle (b_{ij}(t) - \langle b_{ij}(t)\rangle_{proc})(b_{kl}(t') - \langle b_{kl}(t')\rangle_{proc})\rangle_{proc}$$
$$= \langle \lambda_{(n)_o}(t)\lambda_{(m)_o}(t')\rangle_{\lambda(t)}\langle r_{in}(t)r_{jn}(t)r_{km}(t')r_{lm}(t')\rangle_{o(\tau)}.$$

Then for the transformed pairwise correlator we obtain

$$\langle \lambda_{(n)_o}(t)\lambda_{(m)_o}(t')\rangle_{\lambda(\tau)}\langle r_{in}(t)r_{jn}(t)r_{km}(t')r_{lm}(t')\rangle_{o(\tau)}$$
$$= \delta(t-t')\Lambda_{(m)(n)}\langle r_{in}(t)r_{jn}(t)r_{km}(t')r_{lm}(t')\rangle_{o(\tau)} = \delta(t-t')\Lambda_{(m)(n)}\langle r_{in}(t)r_{jn}(t)r_{km}(t)r_{lm}(t)\rangle_{o(\tau)}$$
$$= \delta(t-t')\Lambda_{(m)(n)}\int r_{in}r_{jn}r_{km}r_{lm}f(\mathbf{R}, t)\, d\mathbf{R}.$$

This means that the pairwise correlator depends only on the one-moment distribution function $f(\mathbf{R}, t)$. The last expression can be transformed using the particular form of the correlation tensor Λ_{mn} from Eq. (3.5)

$$\langle b_{ij_o}(t)b_{kl_o}(t')\rangle_{proc} = \Lambda\delta(t-t')\left(\frac{3}{2}\langle nn(n)(n)\rangle(t) - \frac{1}{2}\delta_{ij}\delta_{kl}\right). \quad (6.1)$$

Here, for the sake of convenience, the following designation is introduced

$$\langle r_{ip}(t)r_{jq}(t)r_{kr}(t)r_{ls}(t)\rangle_{o(\tau)} = \langle pqrs\rangle(t).$$

Thus, now it is necessary to find the correlator $\langle nn(n)(n)\rangle$. Multiplying the right and left sides of Eqs. (5.1) and (5.2) by $r_{in}r_{jn}r_{k(n)}r_{l(n)}$ and integrating with respect to $d\mathbf{R}$ we obtain the linear equation and the boundary condition

$$\frac{d}{dt}\langle nn(n)(n)\rangle = -20\langle nn(n)(n)\rangle(t) + 4(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}),$$
$$\langle nn(n)(n)\rangle(0) = \delta_{ij}\delta_{kl}\delta_{(i)(k)}.$$

The solution of this equation is given by the formula

$$\langle nn(n)(n)\rangle(t) = \delta_{ij}\delta_{kl}\delta_{(i)(k)}\exp(-20\Omega t) + \frac{1}{5}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl})(1 - \exp(-20\Omega t)).$$

Substituting this expression in Eq. (6.1) we can obtain the expression for the pairwise correlation

$$\langle b_{ij_o}(t)b_{kl_o}(t')\rangle_{proc} = \Lambda\Bigg(\frac{3}{2}\delta_{ij}\delta_{kl}\delta_{(i)(k)}\exp(-20\Omega t)$$
$$+ \frac{3}{10}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl})(1 - \exp(-20\Omega t)) - \frac{1}{2}\delta_{ij}\delta_{kl}\Bigg)\delta(t-t'). \quad (6.2)$$

After a long time this expression goes over into expression (2.3) for the isotropic tensor

$$\langle b_{ij_o}(t)b_{kl_o}(t)\rangle_{proc} \xrightarrow{t\to\infty} \langle b_{ij}(t)b_{kl}(t)\rangle_{proc} \xrightarrow{t\to\infty} \frac{3}{10}\Lambda\left((\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl}\right)\delta(t-t').$$

For constant D from Eq. (2.3) we obtain

$$D = \frac{3}{10}\Lambda. \tag{6.3}$$

The results obtained, together with expression (5.3) for the decay of the mathematical expectations means the consistency of the Gaussian model of the strain rate tensor devised in Sections 3 and 4 and the isotropic Gaussian model of Section 2 applicable for considering the effect only at large times.

For small times the inertial exponential decay of the pairwise correlation at the delta-correlation of control processes means that the inertia forces associated with rotation of the reference frame of the tensor eigenvalues do not introduce corrections in the first-order time expansion of the correlators α_t^2, β_t^2, and γ_t^2, which will be used in the next section in deriving the expression for the pirouette effect in the model constructed.

7. LINEAR ASYMPTOTICS OF THE CORRELATORS α_t^2, β_t^2, AND γ_t^2 AT SMALL TIMES

Now, with expressions (5.3) and (6.2), we can make an analysis of the pirouette effect observable at small times. In the $t \to 0$ limit expressions (5.3) and (6.2) give the mathematical expectation and pairwise correlations of the strain rate tensor in the basis of its eigenvectors

$$\langle b_{ij}(t)\rangle_{proc} = \begin{pmatrix} \lambda_1(0) & 0 & 0 \\ 0 & \lambda_2(0) & 0 \\ 0 & 0 & \lambda_3(0) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$$\langle b_{ij_o}(t)b_{kl_o}(t)\rangle_{proc} = \Lambda\left(\frac{3}{2}\delta_{ij}\delta_{kl}\delta_{(i)(k)} - \frac{1}{2}\delta_{ij}\delta_{kl}\right)\delta(t-t') = D_{ijkl}\delta(t-t'). \tag{7.1}$$

Applying the Furutsu–Novikov formula to Eq. (1.2) we can obtain the following second-order equation for the vorticity distribution function $f(t,\omega)$

$$\frac{\partial}{\partial t}f = -\lambda_{(i)}\frac{\partial}{\partial\omega_i}\omega_i f + D_{ijkl}\frac{\partial}{\partial\omega_i}\omega_j\frac{\partial}{\partial\omega_k}\omega_l f. \tag{7.2}$$

We will determine the Green function for Eq. (7.2)

$$\frac{\partial}{\partial t}G = -\lambda_{(i)}\frac{\partial}{\partial\omega_i}\omega_i G + D_{ijkl}\frac{\partial}{\partial\omega_i}\omega_j\frac{\partial}{\partial\omega_k}\omega_l G, \tag{7.3}$$

$$G(0,\boldsymbol{\omega}-\boldsymbol{\omega}_0) = \delta(\boldsymbol{\omega}-\boldsymbol{\omega}_0). \tag{7.4}$$

Given the Green function $G(t,\boldsymbol{\omega}-\boldsymbol{\omega}_0)$ and the initial distributions of the eigenvalues and vorticity, we can find the unknown correlators α_t^2, β_t^2, and γ_t^2. For example, for the correlator α_t^2 we have

$$\alpha_t^2 = \int\limits_{R^7}\frac{\omega_1^2}{\omega_1^2+\omega_2^2+\omega_3^2}G(t,\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot f_0(\boldsymbol{\omega}_0)\cdot f_{12}(\lambda_1,\lambda_2)\,d\eta_2\,d\eta_3\,d\omega_0\,d\lambda_1\,d\lambda_2, \tag{7.5}$$

where $f_0(\boldsymbol{\omega}) = f(0,\boldsymbol{\omega})$ is the initial distribution function, whose expression will be obtained in Sect. 8, and $G(t,\boldsymbol{\omega}-\boldsymbol{\omega}_0)$ is the Green function. It also depends on the parameters λ_1 and λ_2. In Eq. (7.3) we can make the change of variables, which will make it linear and lead it to the so-called canonical form [26]:

$$\eta_1 = \frac{1}{\sqrt{3}}(\exp(\omega_1)+\exp(\omega_2)+\exp(\omega_3)),$$

$$\eta_2 = \frac{1}{\sqrt{2}}(\exp(\omega_1)-\exp(\omega_3)),$$

$$\eta_3 = \frac{1}{\sqrt{6}}(-\exp(\omega_1) + 2\exp(\omega_2) - \exp(\omega_3)). \quad (7.6)$$

As a result of this change[5)] we obtain the following equation

$$\frac{\partial G}{\partial t} = \frac{\sqrt{2}}{2}(\lambda_3 - \lambda_1)\frac{\partial G}{\partial \eta_2} - \frac{3}{\sqrt{6}}\lambda_2\frac{\partial G}{\partial \eta_3} + \frac{3}{2}\Lambda\left(\frac{\partial^2 G}{\partial \eta_2^2} + \frac{\partial^2 G}{\partial \eta_3^2}\right). \quad (7.7)$$

We note that Eq. (7.7) does not include η_1. After change (7.6) the initial condition (7.4) can be rewritten, in view of the properties of the delta function [27], as follows:

$$G(0, \boldsymbol{\eta} - \boldsymbol{\eta}^0) = \exp(-\sqrt{3}\eta_1^0)\delta(\boldsymbol{\eta} - \boldsymbol{\eta}^0). \quad (7.8)$$

Here, $\boldsymbol{\eta}_0$ represents the initial conditions for $\boldsymbol{\eta}$. They are obtained by substituting $\boldsymbol{\omega}_0$ into Eq. (7.6). Equation (7.7) with the initial condition (7.8) is solved using the Fourier transformation

$$G(t, \boldsymbol{\eta} - \boldsymbol{\eta}^0) = \frac{\exp(-\sqrt{3}\eta_1^0)}{6\pi\Lambda t}\delta(\eta_1 - \eta_1^0)\exp\left(-\frac{(2(\eta_2 - \eta_2^0) - \sqrt{2}(\lambda_1 - \lambda_3)t)^2}{24\lambda t}\right)$$
$$\times \exp\left(-\frac{\sqrt{6}(\eta_1 - \eta_3^0) - 3\lambda_3 t)^2}{36\Lambda t}\right).$$

Making the change of variables (7.6) in integral (7.5) and substituting in it the expression obtained for the Green function, after some simple transformations we obtain

$$\alpha_t^2 = \int\limits_{R^7} \frac{(\omega_1^0)^2\exp(\sqrt{2}\eta_2)}{(\omega_1^0)^2\exp(\sqrt{2}\eta_2) + (\omega_2^0)^2\exp(\sqrt{6}\eta_3) + (\omega_3^0)^2\exp(-\sqrt{2}\eta_2)} f_0(\boldsymbol{\omega}_0) f_{12}(\lambda_1, \lambda_2)$$
$$\times \exp\left(-\frac{(2\eta_2 - \sqrt{2}(2\lambda_1 + \lambda_2)t)^2}{24\Lambda t}\right)\exp\left(-\frac{(2\eta_3 - \sqrt{6}\lambda_2 t)^2}{24\Lambda t}\right) d\eta_2\, d\eta_3\, d\omega_0\, d\lambda_1\, d\lambda_2.$$

In order to obtain the asymptotics of the last integral near $t = 0$ we will make in it one more substitution

$$\zeta_2 = \frac{2\eta_2 - \sqrt{2}(\lambda_1 - \lambda_3)t}{2\sqrt{\Lambda t}}, \quad \zeta_3 = \frac{2\eta_3 - \sqrt{6}\lambda_2 t}{2\sqrt{\Lambda t}}.$$

As a result, we obtain

$$\alpha_t^2 = \int\limits_{R^2}\int\limits_{R^3} \Omega_1^{-1}\Omega_2 \exp\left(-\frac{\zeta_2^2 + \zeta_3^2}{6}\right) f_0(\boldsymbol{\omega}_0) f_{12}(\lambda_1, \lambda_2)\, d\zeta_2\, d\zeta_3\, d\boldsymbol{\omega}_0\, d\lambda_1\, d\lambda_2,$$
$$\Omega_1 \equiv (\omega_1^0)^2\exp(\sqrt{2\Lambda t}\zeta_2 + 2\lambda_1 t) + (\omega_2^0)^2\exp(\sqrt{6\Lambda t}\zeta_3 + 2\lambda_2 t)$$
$$+ (\omega_3^0)^2\exp(-\sqrt{2\Lambda t}\zeta_2 + 2\lambda_3 t),$$
$$\Omega_2 \equiv (\omega_1^0)^2\exp(\sqrt{2\Lambda t}\zeta_2 + 2\lambda_1 t).$$

In this expression the fraction Ω_2/Ω_1 can be expanded into the Taylor series in the vicinity of $t = 0$[6)]. In view of the fact that this expression is rather cumbersome, we will not present its particular form and restrict ourselves to its general form:

$$\Omega_2/\Omega_1 \approx A_0(\boldsymbol{\omega}_0/|\boldsymbol{\omega}_0|) + a_1(\omega_0/|\omega_0|, \lambda, \Lambda, \zeta)\sqrt{t} + A_1(\boldsymbol{\omega}_0/|\boldsymbol{\omega}_0|, \lambda, \Lambda, \zeta)t + O(t^{3/2}). \quad (7.9)$$

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5)Generally, this change is valid only in the spatial sector $\{\omega_1 \geq 0; \omega_2 \geq 0; \omega_3 \geq 0\}$. However, in view of the oddness (invariance with respect to the change ω_1 for $-\omega_1$) and self-similarity (invariance with respect to the change ω_1 for $\alpha\omega_1$) of expression (7.5) for the correlator α_t^2 and the similar expressions for the correlators β_t^2 and γ_t^2, the integration can be performed only within this sector.

6)Rigorously speaking, it makes sense to expand the integrands only up to the first order, since to take account for the quadratic corrections requires the solution of an equation with the full correlation tensor from Eq. (6.2).

Here, the functions $A_0(\boldsymbol{\omega}_0/|\boldsymbol{\omega}_0|)$, $a_1(\boldsymbol{\omega}_0/|\boldsymbol{\omega}_0|, \lambda, \Lambda, \zeta)$, and $A_1(\boldsymbol{\omega}_0/|\boldsymbol{\omega}_0|, \lambda, \Lambda, \zeta)$ represent polynomials with respect to all their arguments. It can be readily shown that in the polynomial a_1 either the power of ζ_1 or the power of ζ_2 is odd. This means that after the integration with respect to ζ the result will contain only integer powers of t. For the sake of convenience we introduce the following designation for the averaging over the initial directions $\boldsymbol{\omega}(t)$

$$\langle H(\boldsymbol{\omega})\rangle_0 = \int\limits_{R^3} H(\boldsymbol{\omega}_0) f_0(\boldsymbol{\omega}_0)\, d\boldsymbol{\omega}_0.$$

Making in Eq. (7.9) the final averaging over the initial conditions $\lambda(t)$ and using Eq. (3.2) we obtain

$$\alpha_t^2 = \left\langle \frac{\omega_1^2}{\omega^2} \right\rangle_0 + 2\lambda t\left(\left\langle \frac{\omega_1^2\omega_2^2}{\omega^4} \right\rangle_0 + 2\left\langle \frac{\omega_1^2\omega_3^2}{\omega^4} \right\rangle_0\right)$$
$$+ 12\Lambda t\left(\left\langle \frac{\omega_1^2\omega_2^4}{\omega^6} \right\rangle_0 - \left\langle \frac{\omega_1^4\omega_2^2}{\omega^6} \right\rangle_0 - \left\langle \frac{\omega_1^4\omega_3^2}{\omega^6} \right\rangle_0 + \left\langle \frac{\omega_1^2\omega_3^4}{\omega^6} \right\rangle_0\right) + O(t^2). \tag{7.10}$$

Similar expressions can be obtained for the correlators β_t^2 and γ_t^2. The meaning of the coefficients of λt and Λt can be conveniently illustrated assuming that the initial vorticity function is isotropic. In this case, all coefficients of quantity Λt are zero. For the expressions α_t^2, β_t^2, and γ_t^2 the coefficients of the product λt are $8/15$, 0, and $-8/15$, respectively. Thus, clearly that the term with Λt tries to return the system to the isotropic regime (and for this reason it does not work in the case of the isotopic distribution), while the term with λt reflects the property (3.1) of the special choice of the eigenvalues.

8. AVERAGING OVER INITIAL DIRECTIONS OF VORTICITY

In order to end averaging (7.10) it is necessary to know the initial vorticity distribution function $f_0(\boldsymbol{\omega}) = f(0,\ \boldsymbol{\omega})$. Without going into the nonlinear reasons of the anisotropy of the vorticity direction distribution in the basis of eigenvectors [31], we will assume it to be a certain arbitrary ellipsoidal distribution

$$f_0(\boldsymbol{\omega}) = f_{\omega_0}\left(\frac{\omega_1^2}{\alpha^2} + \frac{\omega_2^2}{\beta^2} + \frac{\omega_3^2}{\gamma^2}\right), \tag{8.1}$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

The initial correlator values α_0^2, β_0^2, and γ_0^2 determined from experiments and numerical calculations make it possible to determine the parameters of the distributions α^2, β^2, and γ^2 without defining concretely the function f_{ω_0}. In fact, let us write the system of equations relating the initial correlator values with the unknown parameters

$$\alpha_0^2 = \left\langle \frac{\omega_1^2}{\omega^2} \right\rangle_0 = \int\limits_{R^3} \frac{\omega_1^2}{\omega^2} f_{\omega_0}\left(\frac{\omega_1^2}{\alpha^2} + \frac{\omega_2^2}{\beta^2} + \frac{\omega_3^2}{\gamma^2}\right) d\boldsymbol{\omega},$$
$$\beta_0^2 = \left\langle \frac{\omega_2^2}{\omega^2} \right\rangle_0 = \int\limits_{R^3} \frac{\omega_2^2}{\omega^2} f_{\omega_0}\left(\frac{\omega_1^2}{\alpha^2} + \frac{\omega_2^2}{\beta^2} + \frac{\omega_3^2}{\gamma^2}\right) d\boldsymbol{\omega},$$
$$\gamma_0^2 = \left\langle \frac{\omega_3^2}{\omega^2} \right\rangle_0 = \int\limits_{R^3} \frac{\omega_3^2}{\omega^2} f_{\omega_0}\left(\frac{\omega_1^2}{\alpha^2} + \frac{\omega_2^2}{\beta^2} + \frac{\omega_3^2}{\gamma^2}\right) d\boldsymbol{\omega}.$$

To take these integrals we can make the change transforming the ellipsoidal distribution into the isotropic one and to pass to the spherical coordinate system

$$\omega_1 = \alpha x, \quad \omega_2 = \beta y, \quad \omega_3 = \gamma z,$$
$$x = r\cos\phi\sqrt{1-\mu^2}, \quad y = r\sin\phi\sqrt{1-\mu^2}, \quad z = r\mu.$$

Then the distribution normalization condition can be written as follows:

$$4\pi \cdot \alpha\beta\gamma \int_0^\infty f_{\omega_0}(r) r^2\, dr = 1.$$

For this reason, the independence of the averaged quantity of the variable r makes it possible to express α_0^2, β_0^2, and γ_0^2 for any distribution f_{ω_0} in terms of the elliptic functions

$$F = F\left(\arccos\frac{\gamma}{\beta}, \sqrt{\frac{\beta^2(\alpha^2-\gamma^2)}{\alpha^2(\beta^2-\gamma^2)}}, \right), \quad E = E\left(\arccos\frac{\gamma}{\beta}, \sqrt{\frac{\beta^2(\alpha^2-\gamma^2)}{\alpha^2(\beta^2-\gamma^2)}}, \right),$$

$$\alpha_0^2 = \frac{\alpha^2}{\alpha^2-\beta^2}\left(1 + \frac{\gamma^2(\beta^2-\alpha^2)F - \alpha^2(\beta^2-\gamma^2)E}{\alpha\sqrt{\beta^2-\gamma^2}(\alpha^2-\gamma^2)}\right), \tag{8.2}$$

$$\beta_0^2 = \frac{\beta^2}{\beta^2-\gamma^2}\left(\frac{\beta^2-\gamma^2}{\beta^2-\alpha^2} - \frac{\alpha\sqrt{\beta^2-\gamma^2}}{\beta^2-\alpha^2}E\right),$$

$$\gamma_0^2 = \frac{\gamma^2}{\gamma^2-\alpha^2}\frac{\alpha}{\sqrt{\beta^2-\gamma^2}}(E-F).$$

The solution of system (8.2) cannot be written in terms of elementary functions but an approximate solution of the system can be readily calculated for any particular initial conditions. For example, let us take the initial conditions from [2, 4]

$$\alpha_0^2 = 0.350, \quad \beta_0^2 = 0.385, \quad \gamma_0^2 = 0.265. \tag{8.3}$$

Then we obtain the following values of the distribution parameters

$$\alpha^2 = 0.355, \quad \beta^2 = 0.42, \quad \gamma^2 = 0.225. \tag{8.4}$$

Thus, determining in any particular case the values of the distribution parameters (8.1) we can obtain the numerical values of the coefficients of λt and Λt in expansion (7.10).

9. COMPARISON WITH THE EXPERIMENT

Thus, making all necessary averagings, instead of Eq. (7.10) we obtain the following expression for α_t^2 and the analogous expressions for β_t^2, and γ_t^2

$$\alpha_t^2 \approx 0.350 + 0.385\lambda t - 0.025\Lambda t, \tag{9.1}$$

$$\beta_t^2 \approx 0.385 - 0.030\lambda t - 0.085\Lambda t,$$

$$\gamma_t^2 \approx 0.265 - 0.355\lambda t + 0.110\Lambda t.$$

Here, $\lambda = 3\sqrt{3/10\pi}(\varepsilon/r_0^2)^{1/3}$ (cf. Eqs. (3.2) and (3.3)) and Λ is a parameter-to fit of the same order. Clearly that the parameter values obtained ensure both the growth of α_t^2 (pirouette effect [17]), and the permanent slow decay of the correlator β_t^2 observable also in the experiment and the numerical calculations [4], and the linear decrease in γ_t^2 (see [18]). The correlation coefficient Λ has almost no effect on the growth of α_t^2. Thus, Eq. (9.1) for α_t^2 can be quantitatively verified from experimental data. Using the time scale $t_0 = (r_0^2/\varepsilon)^{1/3}$ we obtain the universal growth of the correlator α_t^2 observable in [17]

$$\alpha_t^2 = 0.350 + 0.36\frac{t}{t_0}.$$

Within the framework of the model constructed one more phenomenon numerically revealed in [18] can be explained. It turns out that if a tetrahedron accompanies a fluid particle retaining its shape, then neither characteristic behavior of the vorticity can be observable. In fact, in this case the fluid particle gradually “flows out” of the tetrahedron and is replaced by another fluid. The extensible intensifying vortex worm also partially comes out from the fluid particle and, for this reason, the worm length l in the tetrahedron remains constant. Thus, from the Helmholtz theorem we obtain

$$\omega \cdot S_\omega \cdot l = \text{const},$$

where ω is the vorticity within the vortex worm and S_ω is its cross-sectional area. As noted in Sect. 1, in the model constructed precisely this quantity is associated with the large-scale vorticity of the tetrahedron.

SUMMARY

The linear stochastic model of the dynamics of fluid particles from the inertial range is formulated. In this model, the large-scale strain rate tensor is the external source of the large-scale vorticity dynamics. Using this model the behavior of the vorticity direction in the Lagrangian reference scale is analytically investigated in the delta-correlated Gaussian approximation. The results of the analysis performed are in quantitative agreement with experimental and numerical results: at large times the vorticity direction exponentially rapidly becomes isotropic (2.4) with respect to the originally chosen direction, while at small times the vorticity is equalized (9.1) with the stretching vector (pirouette effect). The initial non-isotropicity of the vorticity direction with respect to the eigenvectors of the strain rate tensor is taken into account in the form of the initial conditions in the linear problem.

The importance of the results is in the fact itself of the applicability of the linear stochastic model in the fundamentally nonlinear turbulence process. It is shown that the linear processes play, possibly, the main role at the initial stage of the formation of intense large-scale structures in small-scale turbulence.

The results obtained in the study require in the future taking account for non-Gaussian effects of statistical irreversibility and intermittence in turbulent flows.

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