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A Hybrid Technique For the Solution of Unsteady Maxwell Fluid with Fractional Derivatives Due to Tangential Shear Stress*

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Abstract—The article describes the unsteady motion of viscoelastic fluid for a Maxwell model with fractional derivatives. The flow is produced by cylinder, considering time dependent quadratic shear stress ft^2 on Maxwell fluid with fractional derivatives. The fractional calculus approach is used in the constitutive relationship of Maxwell model. By applying Laplace transform with respect to time t and modified Bessel functions, semianalytical solutions for velocity function and tangential shear stress are obtained. The obtained semianalytical results are presented in transform domain, satisfy both initial and boundary conditions. Our solutions particularized to Newtonian and Maxwell fluids having typical derivatives. The inverse Laplace transform has been calculated numerically. The numerical results for velocity function are shown in Table by using MATLAB program and compared them with two other algorithms in order to provide validation of obtained results. The influence of fractional parameters and material constants on the velocity field and tangential stress is analyzed by graphs.

Keywords: Maxwell fluid; Velocity function; Shear stress; Laplace transformation; Modified Bessel function.

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1. INTRODUCTION

The analysis of non-Newtonian fluids has been an essential subject in the field of fluid mechanics because of advancements in technological applications. The study of the motion of non-Newtonian fluids is very tortuous considering non-linear dependence as compared to Newtonian fluids. Because of the complicated constitutive relationship, several researchers have not analyzed the flow response of non-Newtonian fluids [1]. Many applications of non-Newtonian fluids involve extrusion of polymer fluids, food stuff, suspension and colloidal solutions, exotic lubricants, slurry fuels, synthetic propellants and many others. These types of fluids are treated as viscoelastic fluids. Many models and constitutive equations are presented which exhibit all properties of viscoelastic fluids. Rivlin and Ericksen [2], Truesdell and Noll [3] have been classified viscoelastic fluids by presenting constitutive relations for the stress tensor which is a function of velocity gradient.

During the past few years, the researchers have been studied the flows of non-Newtonian fluids [4–10], because of their technological applications as well as their interesting mathematical features. Ting [11] has been found definite solutions related to motions of second grade fluid in cylindrical geometry, Waters and Kings [12] for Oldroyd-B fluids and Srivastava [13] for Maxwell fluids. The Maxwell model is simplest type of model, which describes rheological effects of viscoelastic fluid. But the typical relation between shear stress and shear rate is not properly described by Maxwell model [14, 15]. Some types of elementary unsteady pipe flow of rate type fluid along with sine oscillation flow is analyzed by Rahaman and Ramkissoon [16]. Andrienko *et al.* [17] also found the instantaneous velocities drastically increase at certain frequencies of the oscillating pressure gradient for the motion of Maxwell fluid in a tube. Moreover, Rio *et al.* [18] analyzed the effects of elasticity on the dynamics of Maxwell fluid model in porous tube with the vibrating pressure gradient.

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During the modeling of tangled fluid dynamics like relaxation, oscillation and properties of viscoelastic reaction, the fractional calculus has experienced appreciably success. From the qualitative point of view, several researchers indicated that the classical models is inadequate for the description of rate type fluids. For the modeling of viscoelastic behavior of real materials, fractional-order laws of deformation are proposed. Accordingly, the fractional calculus approach is meaningful for polymer solids and polymer solutions. Rheological constitutive equations with fractional derivatives are obtained by interchanging the ordinary derivatives of strain and stress of known models by fractional order derivatives. Some references are mentioned here to study motion of non-Newtonian fluids with fractional derivatives [19–27].

In fluid mechanics, the system of cylindrical coordinates is adequate to solve many physical problems, including geophysical and meteorological problems, flows in rotating cavities or flows in pipes. The physiological fluids having a complex rheology, are described by the non-Newtonian models including the Maxwell fluid model. The presented paper describes the motion of Maxwell fluid governed by fractional differential equations. The semianalytical solutions of velocity function and time dependent tangential shear stress of Maxwell fluid through a circular cylinder of radius R are obtained. The flow of Maxwell fluid governed by fractional differential equations. The flow is produced by time dependent shear stress ft^2 of circular cylinder. We established semianalytical solutions for velocity function and time dependent shear stress by using Laplace transforms and modified Bessel functions. For the validation of our obtained numerical solutions for inverse Laplace transform, a comparison with two existing numerical packages is presented in Table 1. Furthermore, our obtained results can be characterized to attain precise solutions for Newtonian along with ordinary Maxwell fluids. In the last section, graphical illustrations represent the impact of fractional parameters and material constants on the velocity and shear stress of Maxwell fluid.

2. BASIC GOVERNING EQUATIONS

The constitutive equations related to motion of Maxwell fluid [19] are presented by

$$\mathbf{T} = -\rho\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda\frac{D\mathbf{S}}{Dt} = \mu\mathbf{A}. \tag{1.1}$$

Here, the indeterminate spherical stress and the Cauchy stress are represented by $-\rho\mathbf{I}$ and $\mathbf{T}$, respectively. Whereas $\mathbf{S}$ is the extra-stress tensor, λ is the material constant, μ represents dynamic viscosity of the fluid, $\mathbf{A} = \mathbf{L} + \mathbf{LT}$ with $\mathbf{L}$ the velocity gradient and $D\mathbf{S}/Dt$ is defined as

$$\frac{D\mathbf{S}}{Dt} = D_t^\beta\mathbf{S} + \mathbf{v}\cdot\mathbf{S} - \mathbf{LS} - \mathbf{SL}^T. \tag{1.2}$$

Here ∇ is gradient operator and $\mathbf{v}$ is velocity vector. Here transpose operation is represented by the superscript T and the Riemann-Liouville fractional differential operator is written as

$$D_t^\beta f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int\limits_0^t \frac{f(\eta)}{(t-\eta)^\beta}d\eta, & 0 \le \beta < 1; \\ \frac{df(t)}{dt}, & \beta = 1. \end{cases} \tag{1.3}$$

Here $\Gamma(.)$ denotes the gamma function. When $\beta \to 1$ the presented model can be specialized to ordinary Maxwell model because $D_t^\beta = df/dt$. The classical Newtonian model can also be obtained by proposed model for $\lambda \to 0$ and $\beta \to 1$.

Consider the velocity function $\mathbf{G}$ and the extra-stress $\mathbf{S}$ for the movement of fluid [19] as

$$\mathbf{V} = g(r,t)\mathbf{e}_z, \quad \mathbf{S} = \mathbf{S}(r,t). \tag{1.4}$$

Here $\mathbf{e}_z$ represents cylindrical coordinate $(r,\ \theta,\ z)$ unit vector in the direction of z-axis. Furthermore, when the fluid starts to move, we have

$$\mathbf{V}(r,0) = 0, \quad \mathbf{S}(r,0) = 0. \tag{1.5}$$

This suggests that $S_{rr} = S_{zz} = S_{\theta z} = 0$ and only non-trivial stress is $\zeta(r,t) = S_{r\theta}(r,t)$ [19]

$$(1 + \lambda^\beta D_t^\alpha)\zeta(r,t) = \mu\frac{\partial g(r,t)}{\partial r}. \tag{1.6}$$

Table 1.

r	$g(r,t)$ (MATLAB) [28]	$g(r,t)$ (Stehfest's) [29]	$g(r,t)$ (Tzou's) [30]
0	0.029041	0.028421	0.029542
0.05	0.332540	0.032639	0.033758
0.1	0.046589	0.045986	0.471070
0.15	0.071124	0.070525	0.071663
0.2	0.110263	0.109648	0.110833
0.25	0.168674	0.168010	0.169284
0.3	0.252228	0.251481	0.252888
0.35	0.367964	0.367105	0.368685
0.4	0.524078	0.523088	0.524869
0.45	0.729931	0.728808	0.730804
0.5	0.996080	0.994830	0.997047

The material parameter λ which has dimension t^{β} converts to relaxation time for $\beta \to 1$. The equations of fluid motion in the absence of pressure gradient and body forces in the axial direction lead to the following equation

$$\rho \frac{\partial g(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \zeta(r,t). \tag{1.7}$$

Solving Eqs. (1.6) and (1.7) to eliminate $\zeta(r,t)$, we obtain the following governing equation

$$(1 + \lambda^{\beta} D_t^{\beta}) \frac{\partial g(r,t)}{\partial t} = \nu \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \frac{\partial g(r,t)}{\partial r}. \tag{1.8}$$

Here $\nu = \frac{\mu}{\rho}$ denotes the kinematic viscosity of the fluid. The governing equations with fractional derivatives of Maxwell fluid are presented as Eqs. (1.6) and (1.8). The following governing equations with fractional derivatives are solved for velocity function and time dependent shear stress with some appropriate conditions for Maxwell fluid.

3. FORMULATION OF THE PROBLEM

Consider a Maxwell fluid with fractional derivatives moving in an infinitely long round cylinder of radius R. At $t = 0$, the fluid is at rest and after some time the cylinder starts to move due to tangential shear stress. As a result of tangential shear stress the fluid is gently moved. The fractional model of the problem can be written as

$$(1 + \lambda^{\beta} D_t^{\beta}) \zeta(r,t) = \mu \frac{\partial g(r,t)}{\partial r}, \tag{2.1}$$

$$(1 + \lambda^{\beta} D_t^{\beta}) \frac{\partial g(r,t)}{\partial t} = \nu \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \frac{\partial g(r,t)}{\partial r}. \tag{2.2}$$

For above mentioned governing Eqs. (2.1) and (2.2), the associated boundary and initial conditions for the flow are

$$g(r,0) = \left.\frac{\partial g(r,t)}{\partial t}\right|_{t=0} = 0, \quad \eta(r,0) = 0; \quad r \in [0,R], \tag{2.3}$$

$$\left.(1 + \lambda^{\beta} D_t^{\beta}) \zeta(r,t)\right|_{r=R} = \mu \frac{\partial g(r,t)}{\partial r} = f t^2, \quad f \text{ is a constant.} \tag{2.4}$$

To solve Eqs. (2.1) and (2.2) containing fractional derivatives, we have used Laplace transformation and modified Bessel equation.

4. CALCULATION OF THE VELOCITY FIELD

Applying the Laplace transformation of Eqs. (2.1) and (2.2), we have

$$\frac{\partial^2 \overline{g}(r,q')}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{g}(r,q')}{\partial r} - \frac{q'(1+\lambda^\beta q'^\beta)}{\nu}\overline{g}(r,q') = 0, \tag{3.1}$$

$$\left.\frac{\partial \overline{g}(r,q')}{\partial r}\right|_{r=R} = \frac{2f}{q'^3}, \tag{3.2}$$

where q' is the Laplace parameter. Eqs. (3.1) and (3.2) can be written as

$$\frac{\partial^2 \overline{g}(r,q')}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{g}(r,q')}{\partial r} - e(q')\overline{g}(r,q') = 0, \tag{3.3}$$

$$\left.\frac{\partial \overline{g}(r,q')}{\partial r}\right|_{r=R} = h(q'), \tag{3.4}$$

where

$$e(q') = \frac{q'(1+\lambda^\beta q'^\beta)}{\nu} \quad \text{and} \quad h(q') = \frac{2f}{q'^3}. \tag{3.5}$$

Now, using variable transformation $m = r\sqrt{e(q')}$ in Eq. (3.3), we get

$$m^2\frac{d^2\overline{g}}{dm^2} + m\frac{d\overline{g}}{dm} - (m^2 - 0^2)\overline{g} = 0. \tag{3.6}$$

Eq. (3.6) indicates the modified Bessel equation and its general solution is written as

$$\overline{g}(m,q') = C_1 I_0(m) + C_2 K_0(m), \tag{3.7}$$

where C_1 and C_2 are constants and $K_\alpha(m)$, $I_\alpha(m)$ express the modified Bessel functions. In order to have a finite solution at $m = 0$ ($r = 0$), C_2 must be zero. Then Eq. (3.7) becomes

$$\overline{g}(m,q') = C_1 I_0(m), \tag{3.8}$$

by using boundary condition given in Eq. (3.4) into Eq. (3.8), we have

$$C_1 = \frac{h(q')}{\sqrt{e(q')}I_1(R\sqrt{a})}, \tag{3.9}$$

substituting the value of C_1 in Eq. (3.8), we have

$$\overline{g}(r,q') = \frac{h(q')I_0(r\sqrt{e(q')})}{\sqrt{e(q')}I_1(R\sqrt{e(q')})}. \tag{3.10}$$

The expression in Eq. (3.10) is the complicated form of modified Bessel functions of first kind. For the solution of Eq. (3.10), it is tough to get the expression of inverse Laplace transform conventionally. To overcome this difficulty, we have used some numerical package. Here, numerical results of inverse Laplace transform is determined by using MATLAB.

5. CALCULATION OF THE SHEAR STRESS

Applying the Laplace transform to Eq. (2.1), we get

$$\overline{\zeta}(r,q') = \frac{\mu}{1+\lambda^\beta q'^\beta}\frac{\partial \overline{g}(r,q')}{\partial r}. \tag{4.1}$$

Taking derivative of Eq. (3.10), with respect to r, we get

$$\frac{\partial \overline{g}(r,q')}{\partial r} = h(q')\frac{I_1(r\sqrt{e(q')})}{I_1(R\sqrt{e(q')})}, \tag{4.2}$$

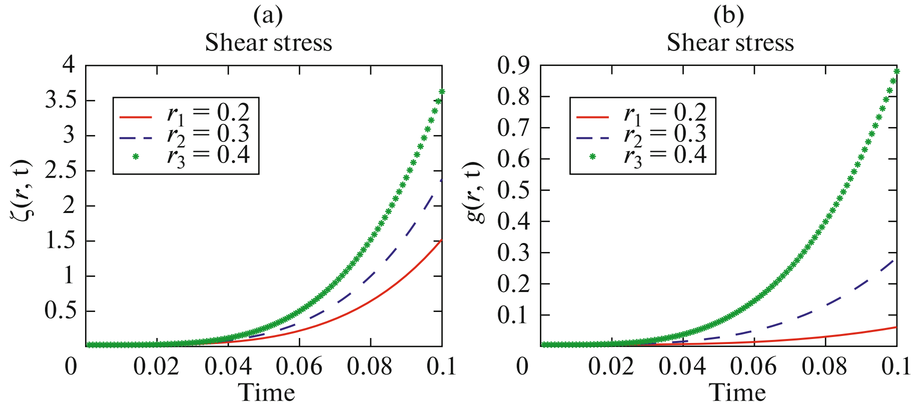

Fig. 1. Shear stress $\zeta(r,t)$ and velocity function $g(r,t)$ graphs of fluid by using physical parameters $R = 0.1$, $t = 0.1$, $\nu = 0.3575$, $\beta = 0.6$, $\mu = 15$, $\lambda = 5$ and having different values of r.

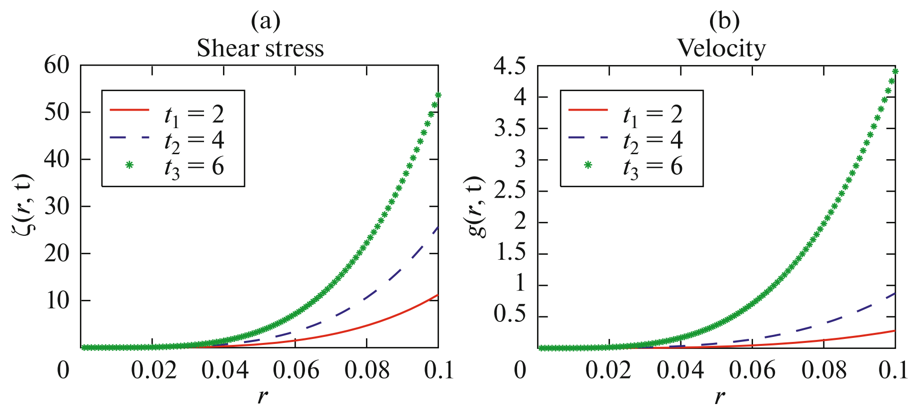

Fig. 2. Shear stress $\zeta(r,t)$ and velocity function $g(r,t)$ graphs of fluid by using physical parameters $R = 0.1$, $\nu = 0.3575$, $\beta = 0.6$, $\mu = 15$, $\lambda = 5$ and different values of t.

putting Eq. (4.2) into Eq. (4.1), we obtain

$$\overline{\zeta}(r, q') = \frac{\mu}{1 + \lambda^{\beta} q'^{\beta}} h(q') \frac{I_1(r\sqrt{e(q')})}{I_1(R\sqrt{e(q')})}. \tag{4.3}$$

For the solution of Eq. (4.3), we have found the inverse Laplace transform numerically through MATLAB.

6. LIMITING CASES

6.1. Ordinary Maxwell fluid

Taking $\beta \to 1$ into Eqs. (3.10) and (4.3), we procured results for velocity function and time dependent shear stress for ordinary Maxwell fluid executing the same motion.

6.2. Newtonian fluid

When we make $\lambda \to 0$, $\beta \to 1$ in Eqs. (3.10) and (4.3), the solutions for velocity function and time dependent shear stress corresponding to Newtonian fluid are determined.

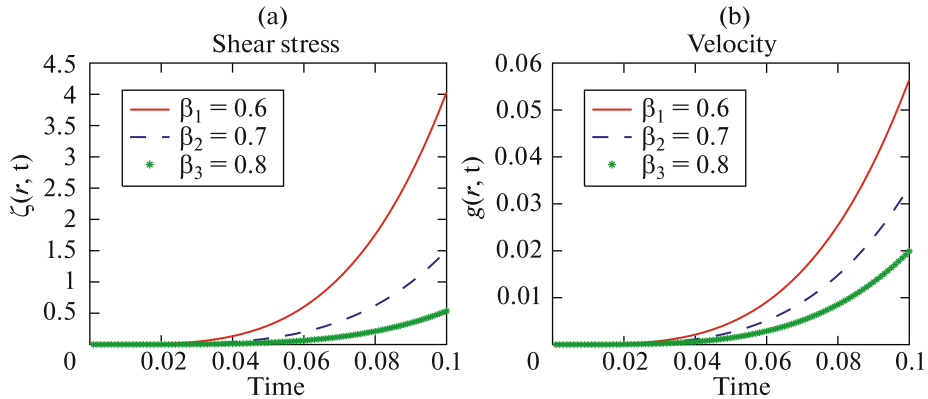

Fig. 3. Shear stress $\zeta(r,t)$ and velocity function $g(r,t)$ graphs of fluid by using physical parameters $R = 0.1$, $\nu = 0.3575$, $\mu = 15$, $\lambda = 5$, $t = 0.1$ and different values of β.

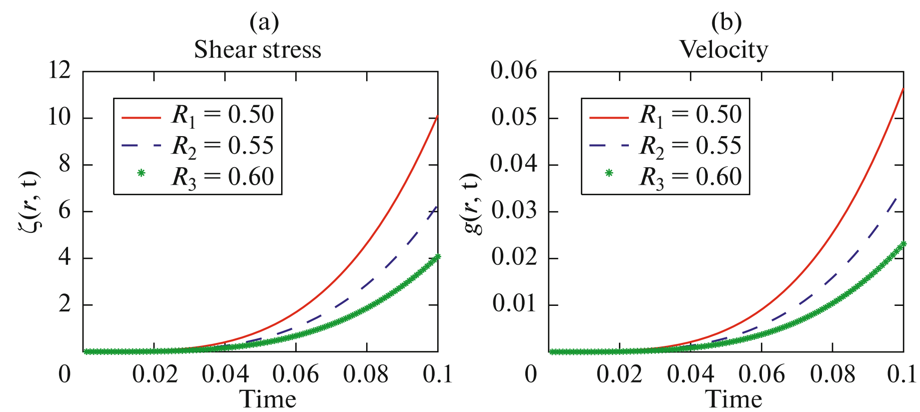

Fig. 4. Shear stress $\zeta(r,t)$ and velocity function $g(r,t)$ graphs of fluid by using physical parameters $\beta = 0.6$, $\nu = 0.3575$, $\mu = 15$, $\lambda = 5$, $t = 0.1$ and having different values of R.

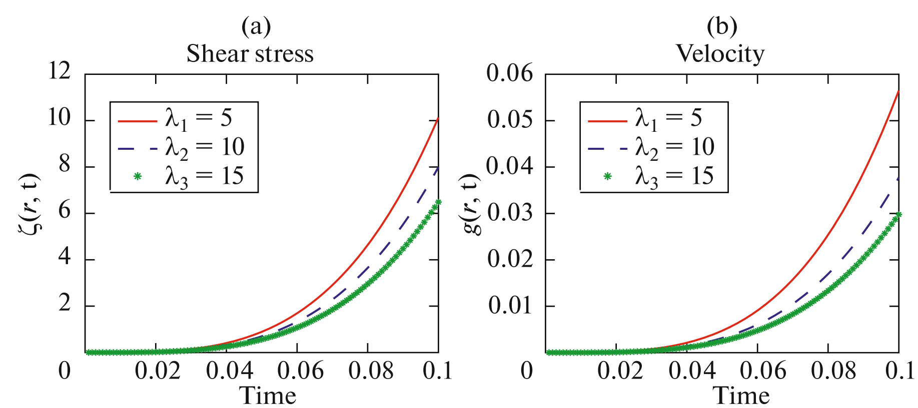

Fig. 5. Shear stress $\zeta(r,t)$ and velocity function $g(r,t)$ graphs of fluid by using physical parameters $\beta = 0.6$, $\nu = 0.3575$, $\mu = 15$, $R = 0.1$, $t = 0.1$ and different values of λ.

7. RESULTS AND DISCUSSION

Semianalytical solutions of velocity function and time dependent tangential shear stress for the unsteady motion of Maxwell fluid governed by fractional differential equations within an infinite long circular cylinder having radius R subject to translation motion have been obtained using modified Bessel functions and Laplace transforms. In this constitutive Maxwell model the fractional calculus approach

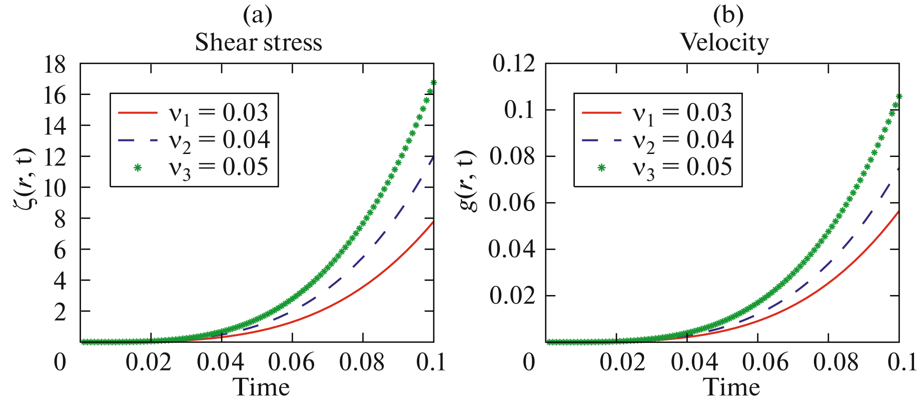

Fig. 6. Shear stress $\zeta(r,t)$ and velocity function $g(r,t)$ graphs of fluid by using physical parameters $\beta = 0.6$, $\lambda = 5$, $\mu = 15$, $R = 0.1$, $t = 0.1$ and different values of ν.

is applied. Our general results for velocity function and time dependent shear stress are represented in the series form being as modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$, which satisfy both the governing equations as well as initial and boundary conditions. The discrete inverse Laplace transform method has been used to minimize lengthy calculations of contour integrals and residues.

We present numerical results of fluid velocity obtained with MATLAB program and with other two numerical algorithms, namely the Stehfest's algorithm [29] and Tzou's algorithm [30] in order to provide the validation of results. According with Stehfest's algorithm, the inverse Laplace transform is given by

$$u(r,t) = \frac{\ln 2}{t}\sum_{k=1}^{N} V_k \overline{u}\left(r, \frac{\ln 2}{t}\right),$$

$$V_k = (-1)^{k+\frac{N}{2}} \sum_{j=\left[\frac{k+1}{2}\right]}^{\min(k,\frac{N}{2})} \frac{j^{\frac{N}{2}} 2j!}{(\frac{N}{2}-j)!j!(j-1)!(k-j)!(2j-k)!}, \tag{6.1}$$

where N is the number of the expansion terms and must be an even number ($N = 16$ tends to a good precision). The Tzou's algorithm is based on the Riemann-sum approximation. In this method the inverse Laplace is given by

$$u(r,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\overline{u}\left(r, \frac{4.7}{t}\right) + \mathrm{Re}\left(\sum_{k=1}^{N_1}(-1)^k \overline{u}\left(r, \frac{4.7 + k\pi i}{t}\right)\right)\right], \tag{6.2}$$

where $\mathrm{Re}(\cdot)$ is the real part, i is the imaginary unit and N_1 is a natural number. The values obtained with Eqs. (6.1), (6.2) and MATLAB programme are given in Table 1.

It is clear from Table 1 that, the results obtained with three algorithms are in a good agreement.

In the last section of paper the reaction of different physical parameters on the velocity function and tangential shear stress are delineated by graphs. Figs. 1(a) and 1(b) depict the effects of r on the velocity and the shear stress. Here, the velocity and time dependent shear stress are directly proportional to r. Figs. 2(a) and 2(b) illustrates that the velocity and the shear stress increased by increasing t. It is observed that the velocity and the tangential shear stress $\zeta(r,t)$ are inversely proportional to β as shown in Figs. 3(a) and 3(b). From Figs. 4(a) and 4(b), we conclude that the velocity and time-dependent adequate shear stress decrease as we enhance the value of R. Figs. 5(a) and 5(b) clearly show the opposite effect of λ on tangential shear stress and velocity. Figs. 6(a) and 6(b) represent the direct effect of ν on the velocity and time dependent adequate shear stress. Finally, the direct influence of μ on velocity and shear stress is represented by Fig. 7.

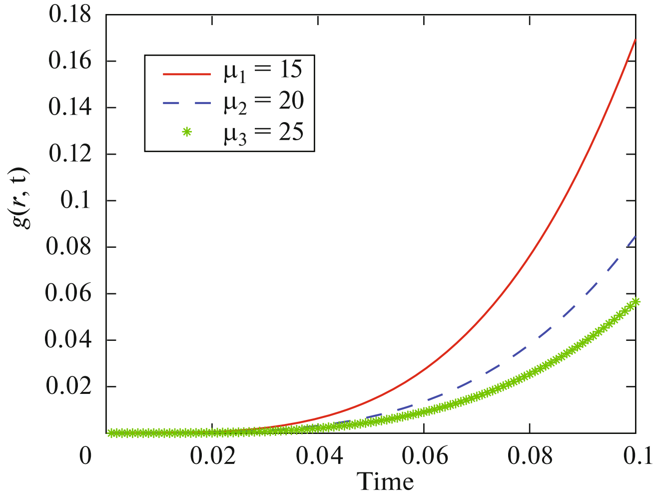

Fig. 7. Velocity function $g(r,t)$ graph of fluid by using physical parameters $\beta = 0.6$, $\lambda = 5$, $\nu = 0.3575$, $R = 0.1$, $t = 0.1$ and different values of μ.

8. CONCLUSIONS

This article describes the flow behavior of fractional Maxwell fluid model passing through a infinitely long cylinder of radius R considering to a longitudinal time dependent quadratic shear stress ft^2. By applying the Laplace transform to the time variable t, the approximated results for the fluid velocity function and quadratic shear stress are obtained in terms of the modified Bessel functions of first kind $I_0(\cdot)$ and $I_1(\cdot)$. Since the Laplace transforms for the velocity function and the adequate shear stress are enough complicated, we have obtained the inverse Laplace transforms by means of the numerical procedures. Firstly, we used a MATLAB numerical code to find the solution. In order to provide a validation of results, we have used other two numerical algorithms, namely the Stehfest's algorithm and Tzou's algorithm. As shown in Table 1, we found a good agreement between results obtained with three numerical methods. It is important to observe that the fluid layers situated close cylinder surface have a significant motion, while the fluid situated in the central area of the cylinder has a very slow motion. The fluid modeled with fractional derivatives flow faster than the ordinary fluid. When the fractional parameter decreases, the fluid velocity increases. The shear stress has the behavior similar with velocity, therefore it is increasing when the fractional parameter decreases. The semianalytical solutions of velocity function and tangential shear stress are procured by modified Bessel functions and the Laplace transforms and can reduce to known results for Newtonian and ordinary Maxwell fluids. The main results are mentioned as:

- The ordinary Maxwell fluid model and Maxwell model with fractional derivatives have some differences in describing the axial flow in a cylinder as models with fractional derivatives can describe the rate type fluids effectively than the classical models.
- When a fractional model of Maxwell fluid compared with classical model it shows a more stable behavior even through for small times.
- The velocity function and time dependent shear stress increase, when r, t and ν are increased.
- With the enhancement of β, R, λ and μ, the velocity and time dependent shear stress are decreased.
- The similar solutions related to Newtonian and classical Maxwell fluid are attained being limiting cases by taking $\lambda \to 0$ and $\beta \to 1$.

REFERENCES

1. S. F. Han, *Constitutive Equation and Computational Analytical Theory of Non-Newtonian Fluids* (Science Press, Beijing, 2000).
2. R. S. Rivlin and J. L. Ericksen, "Stress deformation relations for isotropic materials," J. Rational Mech. Anal. **4** (5), 323–425 (1955).
3. C. Truesdell and W. Noll, *The Non-Linear Field Theories of Mechanics* (Springer, Verlag Berlin Heidelberg, 1992).
4. A. M. Benharbit and A. M. Siddiqui, "Certain solutions of planar motion of a second grade fluid for steady and unsteady cases," Acta Mech. **94** (1), 85–95 (1992).
5. T. Hayat, S. Asghar, and A. M. Siddiqui, "Periodic unsteady flows of a non-Newtonian fluid," Acta Mech. **131** (3), 169–175 (1998).
6. S. Panda, S. C. Martha, and Chakrabarti, "Three layer fluid flow over a small obstruction on the bottom of a channel," ANZIAM J. **56**, 248–274 (2015).
7. K. R. Rajagopal, "A note on unsteady unidirectional flows of a non-Newtonian fluid," Int. J. Non-Linear Mech. **17** (5), 369–382 (1982).
8. K. R. Rajagopal and A. S. Gupta, "An exact solution for the flows of non-Newtonian fluid past an infinite porous plate," Meccanica **19** (2), 158–160 (1984).
9. K. R. Rajagopal, "On the creeping flow of second order fluid", J. Non-Newtonian Fluid Mech. **48** (2), 239–246 (1984).
10. T. E. Stokes, G. C. Hocking, and L. K. Frobes, "Unsteady flow induced by a withdrawal point beneath a free surface," ANZIAM J. **47**, 185–202 (2005).
11. T. W. Ting, "Certain non-steady flows of second-order fluids," Arch. Rational Mech. **14** (1), 1–23 (1963).
12. N. D. Waters and M. J. King, "The Unsteady flow of an elastico-viscous liquid in a straight pipe of circular cross-section," J. Phys. D: Appl. Phys. **4** (2), 204–211 (1971).
13. P. H. Srivastava, "Non-steady helical flow of a visco-elastic liquid," Arch. Mech. Stos. **18** (1), 145–150 (1966).
14. J. J. Choi, Z. Rusak, and J. A. Tichy, "Maxwell fluid suction flow in a channel," J. Non-Newtonian Fluid Mech. **85** (2), 165–187 (1999).
15. N. Makris and M. C. Constantinou, "Fractional derivative model for viscous dampers," J. Struct. Eng. **117** (9), 2708–2724 (1991).
16. K. D. Rahaman and H. Ramkissoon, "Unsteady axial viscoelastic pipe flows," J. Non-Newtonian Fluid Mech. **57** (1), 27–38 (1995).
17. Y. A. Andrienko, D. A. Siginer, and Y. G. Yanovsky, "Resonance behavior of viscoelastic fluids in Poiseuille flow and application to flow enhancement," Int. J. Non-Linear Mech. **35** (1), 95–102 (2000).
18. J. A. D. Rio, M. L. D. Haro, and S. Whitaker, "Enhancement in the dynamic response of a viscoelastic fluid flowing in a tube," Phys. Rev. E **58**, 6323–6327 (1998).
19. W. Akhtar, C. Fetecau, and A. U. Awan, "Exact solutions for the poiseuille flow of a generalized Maxwell fluid Induced by time-dependent shear stress," ANZIAM J. **51**, 416–429 (2010).
20. M. Khan, K. Maqbool, and T. Hayat, "Influence of Hall current on the flows of a generalized Oldroyd-B fluid in a porous space," Acta Mech. **184** (1), 1–13 (2006).
21. M. Khan, S. Nadeem, T. Hayat, and A. M. Siddiqui, "Unsteady motions of a generalized second grade fluid," Math. Comput. Model. **41** (6), 629–637 (2005).
22. M. Khan, T. Hayat, and S. Asghar, "Exact solution for MHD flow of a generalized Oldroyd-B fluid with modified Darcy's law," Int. J. Eng. Sci. **44** (5), 333–339 (2006).
23. H. T. Qi and H. Jin, "Unsteady rotating flows of viscoelastic fluid with the fractional Maxwell model between coaxial cylinders," Acta Mech. Sin. **22** (4), 301–305 (2006).
24. H. T. Qi and M. Y. Xu, "Stokes' first problem for a viscoelastic fluid with the generalized Oldroyd-B model," Acta Mech. Sin. **23** (5), 463–469 (2007).
25. W.C. Tan, C. Fu, W. Xie, and H. Cheng, "An anomalous sub diffusion model for calcium spark in cardiac myocytes," Appl. Phys. Lett. **91** (18), 883–901 (2007).
26. W. C. Tan and M. Y. Xu, "The impulsive motion of flat plate in a generalized second order fluids," Mech. Res. Commun. **29** (1), 3–9 (2002).
27. W. C. Tan, W. X. Pan, and M. Y. Xu, "A note on unsteady flows of a viscoelastic fluid with the fractional Maxwell model between two parallel plates," Int. J. Nonlinear Mech. **38** (5), 615–620 (2003).
28. A. Talbot, "The accurate numerical inversion of Laplace transforms," IMA J. Appl. Math. **23** (1), 97–120 (1979).
29. H. Stehfest, "Algorithm 368: Numerical inversion of Laplace transforms," Commun. ACM **13** (1), 47–49 (1970).
30. D. Y. Tzou, *Macro to Microscale Heat Transfer: The Lagging Behaviour* (Taylor and Francis, Washington, 1997).