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Exact Solutions to the Problem of Deep-Bed Filtration with Retardation of a Jump in Concentration within the Framework of the Nonlinear Two-Velocity Model

N. E. Leont'ev

Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Leninskiye Gory 1, Moscow, 119991 Russia

e-mail: leontiev_n@mail.ru

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Abstract—Exact solutions with plane and cylindrical waves are obtained for one-dimensional problems of injection of a suspension into a porous reservoir when lagging of the suspended particles behind the carrier fluid is taken into account in the case of large change in the porosity. It is shown that taking lagging of the particles behind the fluid into account can lead to slowing-down the motion of jump in concentration. This is in agreement with the results of a series of experiments. It is also noted that, in principle, models in which the particles pass in average ahead of the carrier fluid are possible in the problems of deep bed filtration.

Keywords: flow through porous medium, deep bed filtration, two-velocity suspension, exact solution, strong discontinuity, jump in concentration.

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The problems of suspension flow through porous media with allowance for deposition of suspended particles onto the porous skeleton and gradual decrease in the porosity and permeability of the medium are of interest to series of practical applications, for example, purification of polluted fluids or clogging the near-wall zone in flooding or well drilling in oil-producing industry [1]. As a rule, the laboratory investigations of deep bed filtration represent experiments in which one-dimensional (with plane waves) suspension flows with a constant volume flow rate are implemented along the axis of a cylindrical sample of porous material. The flow rate is maintained due to an appropriate variation in the pressure drop between the inlet and outlet sample cross-sections.

Lagging of the leading boundary of the zone, into which particles penetrate, behind the leading boundary of the zone occupied by the injected carrier fluid was observed in a series of experiments on injection of a low-concentration suspension with fine particles (of the order of 1 μm) into an initially non-polluted sample saturated with a pure fluid at the initial instant [2, 3]. This result is not in agreement with the frequently used assumption [4, 5] on coincidence of the average suspended-particle velocity with the average carrier-fluid velocity which makes it possible to use the one-velocity model in the continual approach.

A series of approaches were proposed to explain this disagreement, in particular, the models in which convective diffusion in the kinetic and continuity equations [3] or finite jump in the porosity on the leading front [6] were taken into account. In the present study another possible explanation of retardation of the front is considered within the framework of the two-velocity model of suspension flow which takes into account averaged retardation of suspended particles from the carrier fluid flow.

1. FORMULATION OF THE PROBLEM

Assuming that the carrier fluid, the suspended particles, and the porous skeleton are incompressible and that the fluid is not accumulated on the skeleton (only the particles settle on the inner surface), we can write the continuity equations for particles and carrier fluid in the divergent form [7]:

$$\frac{\partial}{\partial t}(m\alpha) + \operatorname{div}(m\alpha\mathbf{v}) = \frac{\partial m}{\partial t}, \qquad \frac{\partial}{\partial t}((1-\alpha)m) + \operatorname{div}(m(1-\alpha)\mathbf{w}) = 0, \tag{1.1}$$

where $\mathbf{v}$ and $\mathbf{w}$ are the average particle and carrier fluid velocities which, generally speaking, are different, m is the porosity, α is the volume concentration of the suspended particles, and t is time.

Addition of these equations gives the volume flux balance equation for the entire suspension

$$\operatorname{div}(m\alpha\mathbf{v} + m(1-\alpha)\mathbf{w}) = 0,$$

which can replace one of Eqs. (1.1). We note that in Eqs. (1.1) it is not assumed, as is frequently made, that the porosity deviates only slightly from the initial value and the concentration is low; therefore, the system obtained is nonlinear.

On the surface of discontinuity traveling with a velocity D, the mass flux conservation for particles and fluid gives the relations

$$[(\alpha - 1)m]D - [\alpha m v_n] = 0, \qquad [(1-\alpha)m]D - [(1-\alpha)m w_n] = 0, \tag{1.2}$$

in which subscript "n" denotes the normal component of the vectors and the jump in quantities across the surface of discontinuity is denoted by square brackets. In this case one of these relations can be replaced by the total balance of the volume fluxes on the discontinuity

$$[\alpha m v_n + (1-\alpha)m w_n] = 0.$$

At each of the points of the medium the clogging rate is determined by the kinetic equation of particle deposition (or entrainment) with the known right-hand side

$$\partial m/\partial t = f(m,\ \alpha,\ \operatorname{grad}\alpha,\ \mathbf{v},\ \mathbf{w},\ldots).$$

In the present study we will use the very simple model in which the colmatage (porosity variation) rate is proportional to the particle flux through a given point

$$\partial m/\partial t = -\gamma\alpha m|\mathbf{v}|, \tag{1.3}$$

where γ is a given coefficient.

Generally speaking, the relation between the particle and fluid velocities can be given by a known functional dependence of the form:

$$\mathbf{v} = g(\mathbf{w},\ \alpha,\ m,\ldots)$$

or in a more complex way. In the present study we will use the very simple linear relation

$$\mathbf{v} = k\cdot\mathbf{w} \tag{1.4}$$

with a constant proportionality coefficient k (when $\alpha = 0$, i.e., there is a non-polluted fluid without particles inside the porous medium, the value of k can formally be set the same as in suspension); the case of variable k which depends on the concentration will be discussed in Section 4.

The frequently used one-velocity models [4, 5] correspond to $k = 1$; more rarely used models with retardation of the particles from the carrier fluid, for example [8], correspond to the case $k < 1$ (values of $k = 0.80$–0.85 were obtained in certain experiments [9]).

We especially note that the case $k > 1$ is possible in principle, for example, as a result of displacement of the particles across the pore channel under the impact of some physicochemical or hydrodynamic factors. In particular, a similar phenomenon in which the mean erythrocyte velocity is higher than the mean plasma velocity owing to the erythrocyte concentration in the neighborhood of the axis of blood vessel is well known in blood circulation mechanics [10].

Four equations (1.1), (1.3), and (1.4) are sufficient to determine four scalar quantities v, w, α, and m in one-dimensional flows considered below when the total suspension flow rate is given. In this case we can, in fact, split off the problem of determination of the suspension pressure from Darcy's law (with the viscosity depending on the concentration and the permeability depending on the porosity) or its generalizations. In the generic case the vector equation of motion of the suspension is solved simultaneously with the equations mentioned above. We draw attention to the fact that in the case of finite particle concentration it is possible [11] to consider suspension as a medium with two pressures (pressure in the liquid phase and pressure in the phase of particles). In this case writing of the equations expressing the momentum balance is an individual problem.

2. FLOW WITH PLANE WAVES

One-dimensional flow (with plane waves) along the x axis can be described by the hyperbolic system of two equations

$$\frac{\partial \alpha}{\partial t} + \frac{qF'(\alpha)}{m}\frac{\partial \alpha}{\partial x} = -\frac{\gamma q(1-\alpha)F(\alpha)}{m}, \qquad \frac{\partial m}{\partial t} = -\gamma q F(\alpha), \tag{2.1}$$

where the function

$$F(\alpha) = \frac{k\alpha}{k\alpha + 1 - \alpha}$$

determines the fraction of the volume flow rate of the particles in the conserved total volume flow rate of the suspension

$$q = m\alpha v + m(1-\alpha)w = \text{const}, \quad q > 0$$

(the case of the time-dependent flow rate q can be reduced to the case of the constant flow rate by means of change of time [6]).

In the problem of injection of a suspension into an initially non-polluted ($m = m_0 = \text{const}$) porous sample saturated with a pure liquid ($\alpha = 0$) we can assume that a constant concentration α_0 is maintained at the inlet boundary $x = 0$ (inside the reservoir). We underline that if the suspension arrives at the sample from a tank, then in the general case the suspension concentration outside the porous medium α_e differs from α_0. In fact, for the particle and carrier fluid velocities $\mathbf{v}_e$ and $\mathbf{w}_e$ outside the porous medium from the relations on the discontinuity (1.2) we can obtain the relation

$$\frac{\alpha_e v_e}{(1-\alpha_e)w_e} = \frac{k\alpha_0}{1-\alpha_0}, \qquad \alpha_e v_e = qF(\alpha_0),$$

therefore, if in the pure liquid the proportionality coefficient k_1 differs from k in the relation between the velocities $\mathbf{v}_e = k_1\mathbf{w}_e$, then $\alpha_e \neq \alpha_0$.

After the beginning of injection, a higher-concentration wave separated from the non-polluted part by the surface of discontinuity begins to move through the sample with a velocity D. We will assume that an additional porosity continuity condition $[m] = 0$ must be satisfied on the discontinuity. The case of finite jump in porosity was analyzed in [6]. The sole relation following from the mass conservation law takes the form:

$$D = \frac{q}{m}\cdot\frac{[F(\alpha)]}{[\alpha]}. \tag{2.2}$$

The problem under consideration has an exact solution in which the concentration behind the jump is independent of time and determined by the implicit relation

$$\gamma x = G(\alpha(x)), \qquad G(\alpha) = \int_{\alpha}^{\alpha_0} \frac{F'(\alpha_1)\,d\alpha_1}{(1-\alpha_1)F(\alpha_1)} \tag{2.3}$$

or

$$\gamma x = -\ln\frac{\alpha}{\alpha_0} - \frac{1-k}{k}\ln\frac{1-(1-k)\alpha}{1-(1-k)\alpha_0} + \frac{1}{k}\ln\frac{1-\alpha}{1-\alpha_0}. \tag{2.4}$$

From the relation on the discontinuity (2.2) we can find the implicit time dependence of the concentration behind the jump $\alpha_f(t)$

$$\gamma q t = -\int_{\alpha_0}^{\alpha_f} \frac{m_0\alpha F'(\alpha)\,d\alpha}{(1-\alpha)F^2(\alpha)} \tag{2.5}$$

or

$$\gamma q t = \frac{m_0}{k}\ln\frac{\alpha_0(1-\alpha_f)}{\alpha_f(1-\alpha_0)}.$$

Finally, the porosity distribution behind the jump can be found after integrating the kinetic equation (1.3) (α_f plays the role of a parameter)

$$m(x,\ t) = m_0 - F(\alpha_f)(t - t(\alpha_f)), \qquad x = x(\alpha_f). \tag{2.6}$$

Relations (2.3), (2.5), and (2.6) give the complete solution to the problem which goes over, when $k = 1$, in the solution for the one-velocity model [6].

In the case of retardation of the particles from the carrier fluid, the evolutionarity conditions for jump in saturation [12, 13] are automatically fulfilled in the solution obtained. This statement can be verified using the following geometric interpretation: for the first of the equations of system (2.1) the velocity of propagation of the characteristics in the plane $(x,\ t)$ on both sides of the discontinuity is determined by the slope of a tangent to the graph of the function $F(\alpha)$: $m_0 dx/dt = qF'(\alpha)$, whereas the velocity of the discontinuity in the solution under consideration is determined by the angle of inclination of a chord that connects the points on the graph corresponding to the states ahead of and behind the jump $m_0 D = qF(\alpha_f)/\alpha_f$ (there is a similar interpretation in the well-known Buckley–Leverett theory [14]). When $k < 1$ the graph of the function $F(\alpha)$ is convex downwards; therefore, a single characteristic, as is required, departs in the plane $(x,\ t)$ from the discontinuity on which two conditions are specified.

From the same geometric interpretation it follows that in the two-velocity model when $k < 1$ the jump velocity turns out to be lower than that for the one-velocity model, the jump velocity decreasing with time. This statement is illustrated by the laws of motion of a jump reproduced in Fig. 1 for the cases of the one- and two-velocity models compared in present study.

3. FLOW WITH CYLINDRICAL WAVES

Similarly, we can obtain the exact solution to the problem of flow with cylindrical waves in a thin reservoir after the beginning of suspension injection through a well of finite radius r_0 with a constant volume flow rate $q > 0$ per unit length of the well. In this case flow can be described by the system

$$\frac{\partial \alpha}{\partial t} + \frac{qF'(\alpha)}{2\pi r m}\frac{\partial \alpha}{\partial r} = -\frac{\gamma q}{2\pi r m}(1 - \alpha)F(\alpha), \qquad \frac{\partial m}{\partial t} = -\frac{\gamma q}{2\pi r}F(\alpha),$$

where r is the radial coordinate. We note that in this problem at the initial instant the porosity is assumed to be constant; therefore, we cannot use the analogy [15] between flows with plane and cylindrical waves to obtain the solution on the basis of the results of the previous section.

Consideration of the solution with the time-independent concentration $\alpha(r)$ downstream of the front gives an expression which coincides with (2.4) with the change of the variable x on the left-hand side by the expression $r - r_0$.

Under the additional assumption on continuity of the porosity $[m] = 0$ on the strong discontinuity $r = r_f(t)$ from the mass flux conservation condition

$$\frac{d}{dt}\left(\frac{r_f^2}{2}\right) = \frac{q}{2\pi m_0}\frac{F(\alpha_f)}{\alpha_f}$$

we can find the time dependence of the concentration downstream of the front $\alpha_f(t)$ in the implicit form:

$$\frac{\gamma^2 q t}{2\pi m_0} = -\int_{\alpha_0}^{\alpha_f} \frac{(\gamma r_0 + G(\alpha))\alpha F'(\alpha)\,d\alpha}{(1 - \alpha)F^2(\alpha)}.$$

Taking the known dependence $\alpha(r)$ into account, this leads to the law of motion of the front in the parametric form and thereafter the porosity distribution downstream of the front can be calculated.

Hence, as a particular case, for the one-velocity suspension ($k = 1$) there follows the well-known result [6]

$$r_f^2 - r_0^2 = \frac{qt}{\pi m_0},$$

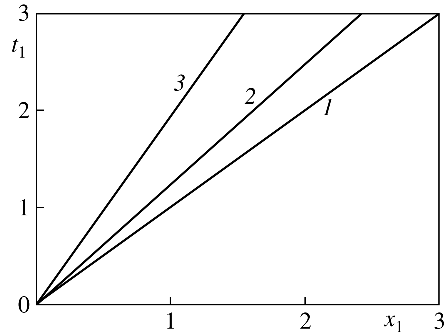

Fig. 1. Law of motion of a jump in concentration in the plane of dimensionless variables $(x_1;\ t_1)$ ($x_1 = \gamma x$ and $t_1 = \gamma qt/m_0$) for flow with plane waves when $\alpha_0 = 0.1$: curves (*1–3*) correspond to $k = 1$, 0.8, and 0.5, respectively.

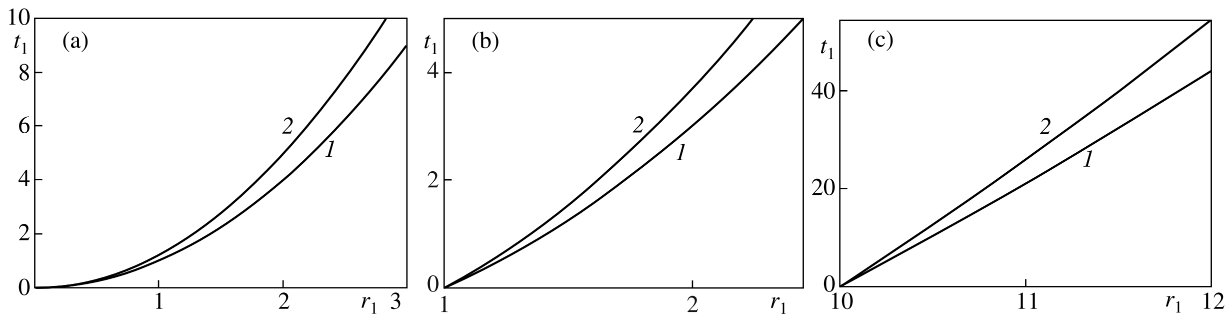

Fig. 2. Law of motion of a jump in concentration in the plane of dimensionless variables $(r_1;\ t_1)$ ($r_1 = \gamma r$ and $t_1 = \gamma^2 qt/(\pi m_0)$) for flow with cylindrical waves when $\alpha_0 = 0.1$: (a–c) correspond to $\gamma r_0 = 0.1$, 1, and 10, respectively; curves (*1*, *2*) correspond to $k = 1$ and 0.8, respectively.

which can also be obtained using the integral relation method. When $k < 1$ the evolutionarity of the discontinuity in the exact solution obtained can be verified using the geometric interpretation analogous to that given in the previous section.

As in the case of flow with plane waves, when $k < 1$ in the problem under consideration the velocity of the jump in concentration turns out to be lower than that for the one-velocity model, the velocity of the jump in concentration decreasing with time. In Fig. 2 we have compared the laws of motion of the front for the one- and two-velocity models in three typical cases of small, moderate, and large well radii r_0 as compared with the characteristic space scale $1/\gamma$. As it must be, in the case of the well of relatively large radius when clogging takes place in a thin near-well zone, the law of motion of the front is qualitatively similar to that in the problem with plane waves.

4. CASE OF THE VARIABLE COEFFICIENT k

From the formal point of view, the exact solutions obtained, which are determined, for example, in the plane case by the relations (2.3), (2.5), and (2.6), remain valid both when the fluid passes ahead of the suspended particles ($k > 1$, $k = \text{const}$) and in the presence of an arbitrary dependence of the disparity between the velocities and the concentration $k(\alpha)$; we will assume that the function $k(\alpha)$ decreases so that $k(0) = 1$.

However, for the special case of the solutions with constant concentration downstream of the front under consideration, if we do not impose additional conditions on discontinuities, the jumps turn out to be nonevolutionary.

In the first case $k > 1$ the graph of the function $F(\alpha)$ turns out to be convex upwards; therefore, still two additional conditions (in addition to two available conditions) must be satisfied on the jump.

In the second case of variable coefficient $k(\alpha)$, which is more interesting, the graph of the function $F(\alpha)$ has the inflection point; therefore, a single additional condition is necessary for the high concentrations downstream of the front and two additional conditions for the low concentrations.

The way out of this situation can be either specification of additional conditions from consideration of the jump structure [16] (examples of such structures with reference to jumps in concentration in porous media were discussed in [17]) or, evidently, treatment of the solution of the more general form with the time-dependent distribution $\alpha(x, t)$ of the concentration downstream of the front.

Summary. The model of deep bed filtration is considered taking finite variation in the porosity and the difference between the suspended particle and carrier fluid velocities into account. Exact solutions to one-dimensional problems on suspension injection into an initially non-polluted reservoir are found. Qualitative features of the motion of the leading boundary of the zone occupied by suspended particles are studied. The possibility of consideration of models with the overrunning motion of particles or with the concentration dependence of the discrepancy between the velocities is noted.

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