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A Combined Lagrangian Method for Simulation of Axisymmetric Gas-Particle Vortex Flows

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Abstract—A combined fully Lagrangian approach for meshless modeling of unsteady axisymmetric vortex flows of a gas-particle mixture with an incompressible carrier phase is developed. The approach proposed is based on the combination of a meshless vortex method for calculating axisymmetric flows of the carrier phase described by the Navier–Stokes (or Euler) equations and the full Lagrangian method for calculating the parameters of the dispersed phase. The combination of these methods reduces the problem of modeling the two phase flows to the solution of a high-order system of ordinary differential equations for the coordinates of toroidal vortex elements in the carrier phase and the particle trajectories, the velocity components, and the components of the Jacobian of transformation from the Eulerian to the Lagrangian variables in the dispersed phase. The application of the method is illustrated by modeling the behavior of an admixture of inertial Stokes particles with a small mass concentration in unsteady flows like solitary vortex rings in a viscous carrier phase and groups of vortex rings in an effectively inviscid carrier phase.

Keywords: full Lagrangian approach, vortex method, inertial particles, concentration, diffusion velocity, axisymmetric flow, Navier–Stokes equation.

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The development of models and methods for calculating unsteady vortex flows of ‘gas (fluid)—particle’ mixtures is important because such flows are widely encountered in nature and engineering. The examples include gas-droplet flows in internal-combustion engines, where, after the fuel injection, vortex rings are formed [1]; dust rise during the helicopter landing in a desert [2]; gas-particle tornadoes and ‘dust devils’ [3–4]; the motion of gas-particle mixtures in vortex separators [5]; swirling bubbly flows [6], and others [7].

In many important applications, the two-phase medium is fairly dilute, the volume fraction of the dispersed inclusions is very small (of the order of 10^{-4}–$10-6$). In this case, the inertial dispersed admixture is usually modeled by the equations of a continuum devoid of self-stresses (pressureless continuum). This continuum is extremely compressible, in it there may arise the ‘folds’, i.e. the regions of crossing particle trajectories (where formally several ‘cold’ dispersed continua co-exist), the localized particle accumulation regions with singular values of the particle number density on their boundaries (caustics) [8], as well as strong discontinuities of the dispersed-phase parameters and the zones devoid of particles. The boundaries of the regions of crossing particle trajectories are unknown beforehand, that makes the standard Euler approach practically inapplicable for the quantitative description of complex unsteady gas-particle flows. All mentioned above features of the particle density fields are evident in unsteady flows with localized vorticity regions. A detailed study of the dispersed-phase concentration fields in such flows (in the presence of discontinuities, integrable singularities in the particle number density, caustics, and folds of the dispersed-phase volume) requires the development of efficient methods for the accurate calculation of the carrier-phase hydrodynamic fields and special methods of computing the concentration fields of the inclusions.

This work is devoted to the development of a combined Lagrangian method (CLM) for modeling unsteady axially symmetric gas-particle vortex flows with a viscous (or efficiently inviscid) carrier phase. Earlier, a combined fully Lagrangian approach was proposed by the authors in [9,10] for the case of 2D plane-parallel flows. The method is based on the modification and combination of two Lagrangian approaches, namely, a vortex method for solving the Navier-Stokes (or Euler) equations for the carrier phase and the full Lagrangian method for calculating the parameters of the dispersed phase. The full Lagrangian approach, proposed in [11] and then developed for various classes of dispersed flows (see, for example, [3, 12–17]), is a highly reliable and efficient method which enables one, in addition to the calculation of the dispersed-phase velocity field, to calculate correctly the concentration fields of the dispersed admixture in the flows with intersecting particle trajectories. This method is based on the use of the dispersed-phase continuity equation in the Lagrangian form and the derivation of additional equations for the components of the Jacobian of the transformation from the Eulerian to the Lagrangian variables.

In recent years, different Lagrangian approaches have been developed also for the numerical simulation of pure-fluid flows. Among them are the SPH ('smoothed particle hydrodynamics') methods [18, 19], which use the standard 'velocity–pressure' variables, and various variants of vortex methods, using the 'velocity–vorticity' variables, which generalize the methods of discrete vortices to the case of viscous flows (see, for example, books [20, 21], papers [22–25] and the literature therein). The vortex methods have proven themselves as handy tools for the solution of plane and axisymmetric problems in which the vorticity is localized in space, which allows to compute economically unsteady velocity fields in viscous and efficiently inviscid flows.

In the combined Lagrangian method developed in the present work, for calculating the carrier-phase parameters we use a variant of a vortex method based on the introduction of the so-called 'diffusion velocity' of the vorticity and the divergence form of the transport equation for the vorticity in a viscous fluid [22, 23]. The parameters of the inertial dispersed phase are calculated using the full Lagrangian method [11]. In contrast to the standard Lagrangian-Eulerian methods, widely used in commercial software packages, the combined Lagrangian method proposed makes it possible to avoid the cumbersome procedure of the recalculation of the carrier-phase parameters from the Lagrangian to the Eulerian grid.

It should be noted that recently several publications appeared in the literature (e.g., [26, 27] and the papers cited therein), which were devoted to the development of methods that combine different variants of vortex methods for the carrier phase and a Lagrangian approach for the admixture. However, in all these studies (except [9, 10]) the Lagrangian approach was used only for calculating the particle trajectories, and no accurate calculation of the particle concentration field with account for the possibility of crossing particle trajectories was performed.

In the present work, the combined Lagrangian method is developed for the case of axially symmetric gas-particle flows without a swirl, and the application of the method is illustrated by several examples of simulating the behavior of an admixture of inertial Stokes particles in unsteady vortex flows like vortex rings and their groups.

1. BASIC IDEAS OF THE COMBINED LAGRANGIAN METHOD

We consider unsteady vortex flows (without a swirl) of a two-phase gas-particle mixture in an unbounded space. As the basis for the mathematical modeling we use the two-fluid 'dusty-gas' model [28], according to which at each point of space there are two continua, namely, the carrier and the dispersed phase. The carrier phase is incompressible and (in general case) viscous fluid with a constant density ρ, viscosity μ, velocity $\mathbf{v}$, and pressure p. The dispersed phase consists of identical spherical particles of radius σ and mass m, moving with the velocity $\mathbf{v}_s$ and having the number density n_s. The random motion of the particles is neglected, which leads to the absence of self-stresses (pressure) in the continuum of particles. The particle volume fraction is negligibly small, which makes it possible to use the known expressions for the force acting on the individual particle in a viscous flow. The mass concentration of the admixture is also assumed to be small,

and the effect of the particles on the carrier flow is neglected. These assumptions make it possible to solve the problem of calculating the vortex flow of the carrier phase independently of the dispersed-phase motion.

Description of the Carrier Phase

To calculate the parameters of the carrier phase in unsteady axially symmetric flows without a swirl, we use a variant of the vortex method based on the diffusion velocity of the vorticity [22, 23]. In the case considered, there is only one non-zero component of the velocity curl (referred to as the vorticity ω), directed along the variation of the azimuthal angle φ. In the 'velocity–ω' variables, the Navier–Stokes equations are rewritten in divergence form in the cylindrical coordinates $\mathbf{r} = (r,\ \varphi,\ z)$, where the symmetry axis coincides with Oz:

$$\frac{\partial(u_r r)}{\partial r} + \frac{\partial(u_r z)}{\partial z} = 0, \qquad \frac{\partial \omega}{\partial t} + \frac{\partial}{\partial r}\big[(u_r + u_{dr})\omega\big] + \frac{\partial}{\partial z}\big[(u_z + u_{dz})\omega\big] = 0. \tag{1.1}$$

Here, $\mathbf{v} = (u_r,\ 0,\ u_z)$ is the convective velocity of the carrier phase in the cylindrical coordinates, $\mathbf{v}_d = (u_{dr},\ 0,\ u_{dz})$ is the vorticity diffusion velocity which, in the case of axisymmetric flows without a swirl, in the cylindrical coordinates takes the form:

$$\mathbf{v}_d = -\frac{1}{\mathrm{Re}}\frac{\nabla(r\omega)}{r\omega}, \quad \omega = \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right), \quad \nabla(r\omega) = \left(\frac{\partial(r\omega)}{\partial r},\ 0,\ \frac{\partial(r\omega)}{\partial z}\right). \tag{1.2}$$

Here, all equations are written in dimensionless form. For the scaling, we used certain characteristic values of velocity U, length L, time L/U, vorticity U/L, and circulation UL. The Reynolds number is $\mathrm{Re} = \rho UL/\mu$. For particular examples, the choice of the characteristic scales of length and velocity will be explained below. In a viscous fluid, the vorticity is not 'frozen' in the medium: it is transported with the velocity equal to the sum of the flow velocity and the diffusion velocity. Accordingly, when considering the vorticity transport, the 'Lagrangian' vortex element means an element the points of which move with the total velocity $\mathbf{v} + \mathbf{v}_d$.

To find the carrier-phase parameters for a specific initial vorticity distributions, at first the vorticity field is calculated at each time step from the second of Eqs. (1.1). When the vorticity is found, the convective velocity $\mathbf{v} = (u_r,\ u_z)$ can be obtained from the Biot–Savart integral, which in the case of axisymmetric flow without a swirl gives the following expressions for the velocity components [29]:

$$\begin{aligned} u_r(\mathbf{r}) &= -\frac{1}{2\pi r}\int\!\!\int \omega(R,\ Z) f'(k)\frac{\partial k}{\partial z}(rR)^{1/2} dR\, dZ, \\ u_z(\mathbf{r}) &= \frac{1}{2\pi r}\int\!\!\int \omega(R,\ Z)\left(\frac{f(k)}{2r} + f'(k)\frac{\partial k}{\partial r}\right)(rR)^{1/2}\, dR\, dZ, \\ f(k) &= \left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k), \quad k(\mathbf{r},\ \mathbf{R}) = \left(\frac{4rR}{(r+R)^2 + (z-Z)^2}\right)^{1/2}. \end{aligned} \tag{1.3}$$

Here, the radius-vectors $\mathbf{r}(r,\ z)$ and $\mathbf{R}(R,\ Z)$ are considered in the plane. The integration over Z is performed from ∞ to $+\infty$, and over R from 0 to ∞; $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind:

$$K(k) = \int_0^{\pi/2} (1 - k^2\cos^2\theta)^{-1/2}\, d\theta, \quad E(k) = \int_0^{\pi/2} (1 - k^2\cos^2\theta)^{1/2}\, d\theta.$$

The Description Of The Dispersed-Phase Dynamics

For the class of unsteady vortex flows under study, the typical feature is the onset of regions of crossing particle trajectories. On the boundaries of these regions, the caustics can be formed, near which the particle number concentration sharply increases [8]. The boundaries of the regions of intersecting trajectories and

the number of trajectories intersecting at a fixed Eulerian point of space are unknown in advance, which makes inapplicable the Euler approach to the description of the dispersed phase. In this regard, below we use a Lagrangian description of the dispersed phase, and the parameters of the particulate continuum are found using the full Lagrangian method described in a more detail in [3, 10–15].

As the Lagrangian coordinates, we choose the values of the Euler cylindrical coordinates of dispersed particles at a certain instant of time taken as the initial point: $\mathbf{r}_{s0} = \mathbf{r}(t = 0)$. The Lagrangian equations of a 'cold' (pressureless) continuum of particles in dimensionless form (in nondimensionalization, the scales of the coordinates, velocity, and time are taken as in the previous section, and the particle concentration is scaled to a certain characteristic value n_{se}) are as follows:

$$\frac{\partial \mathbf{r}_s(t,\ \mathbf{r}_{s0})}{\partial t} = \mathbf{v}_s(t,\ \mathbf{r}_{s0}), \tag{1.4}$$

$$\frac{\partial \mathbf{v}_s(t,\ \mathbf{r}_{s0})}{\partial t} = \mathbf{f}_s(\mathbf{v}_s,\ \mathbf{v}(t,\ \mathbf{r}_s)), \tag{1.5}$$

$$n_s(t,\ \mathbf{r}_{s0})|\det J|r_s = n_{s0}(0,\ \mathbf{r}_{s0})r_{s0}. \tag{1.6}$$

Here, $\mathbf{f}_s$ is the dimensionless force exerted on the particle, $|\det J|$ is the modulus of the Jacobian of the transformation from the cylindrical Eulerian coordinates to the Lagrangian coordinates: $J_{pq} = \partial(\mathbf{r}_s)_p/\partial(\mathbf{r}_{s0})_q$ (p, q correspond to 1, 2, where 1 corresponds to r and 2 to z). Taking the modulus of the Jacobian in (1.6) automatically takes into account the change in the orientation of the Lagrangian contour (and ensures the positiveness of the particle concentration) with the formation of folds in the continuum of particles.

To calculate the concentration along particle trajectories using (1.6), we derive additional equations for the Jacobian components J_{pq}. These equations are obtained by the differentiation of Eqs. (1.4)–(1.5) with respect to the Lagrangian coordinates:

$$\frac{\partial J_{pq}(t,\ \mathbf{r}_{s0})}{\partial t} = \vartheta_{pq}(t,\ \mathbf{r}_{s0}), \tag{1.7}$$

$$\frac{\partial \vartheta_{pq}(t,\ \mathbf{r}_{s0})}{\partial t} = \frac{\partial(\mathbf{f}_s)_p}{\partial(\mathbf{r}_{s0})_q}. \tag{1.8}$$

Here, $\vartheta_{pq} = \partial(\mathbf{v}_s)_p/\partial(\mathbf{r}_{s0})_q$ are the derivatives of the velocity components with respect to the Lagrangian coordinates.

In the present study, as an example of the interphase momentum exchange we use the Stokes drag, which gives the main contribution in the case of heavy particles whose density is much greater than that of the carrier phase (gas-particle mixtures, aerosols). Examples of the application of the full Lagrangian method in the cases when the interphase momentum exchange includes the Archimedes, added mass, and lifting forces can be found, for instance, in [3, 9, 15].

In dimensional form, the expression for the Stokes drag is

$$\mathbf{f}_s^* = 6\pi\sigma\mu(\mathbf{v}^* - \mathbf{v}_s^*).$$

Taking this into account, dimensionless Eqs. (1.5) and (1.8) take to form:

$$\frac{\partial \mathbf{v}_s}{\partial t} = \beta\left[\mathbf{v}(t,\ \mathbf{r}_s) - \mathbf{v}_s\right], \quad \frac{\partial \vartheta_{pq}}{\partial t} = \beta\left[\frac{\partial u_p(t,\ \mathbf{r}_s)}{\partial r}J_{pq} + \frac{\partial u_p(t,\ \mathbf{r}_s)}{\partial z}J_{pq} - \vartheta_{pq}\right].$$

Here, each index "p" and "q" takes the values of r and z, $\beta = 6\pi\sigma\mu L/mU$ is the dimensionless parameter of particle inertia (sometimes in the literature the inverse value, called the Stokes number, is used).

For a fixed value $\mathbf{r}_{s0}$ (along a chosen trajectory $\mathbf{r}_s(\mathbf{r}_{s0},\ t)$), system (1.4)–(1.8) becomes a system of ordinary differential equations. As the initial conditions for system (1.4)–(1.8), we use the initial coordinates and velocity of the particles, the values $J_{rr} = J_{zz} = 1$, $J_{rz} = J_{zr} = 0$, and the values of ϑ_{pq} corresponding to

the initial conditions for the particle velocity field at the starting point of the considered trajectory. The calculations are performed for M chosen particle trajectories, which start in the region originally occupied by the dispersed phase at the points with the initial coordinates $\mathbf{r}_{s0j}$ $(j = 1, \ldots, M)$. The number of calculated trajectories M is determined by the required level of detailing the description of the dispersed-phase velocity and concentration fields.

Discrete-Vortex Method for Calculating the Carrier-Phase Parameters

In calculating the parameters of the dispersed phase, the values of the carrier-phase velocity components and their derivatives with respect to the Eulerian coordinates, entering in the right-hand sides of the equations, should be known at the points of the calculated particle trajectories. Below we describe a discrete version of the procedure of finding the carrier-phase parameters using the meshless vortex method. The domain of the initial non-zero vorticity Ω in the plane $(r,\ z)$ is split into a set of N small Lagrangian elements with areas ΔS_i and coordinates of their centers $(r_i,\ z_i)$ $(i = 1 \ldots, N)$. In the space, each such element corresponds to a toroidal vortex ring, characterized by the ring radius r_i, the coordinate along the symmetry axis z_i, a small radius of the ring section $a_i \ll r_i$, and the velocity circulation γ_i, equal to the vorticity flux through ΔS_i. The radius of the ring section satisfies the relation $\pi a_i^2 = \Delta S_i$. Due to the divergence form of the vorticity transport equation in (1.1), the Lagrangian vortex elements, moving with the total velocity $\mathbf{v} + \mathbf{v}_d$, preserve the vorticity flux, and hence their values of the velocity circulation. Hence, the circulations of the chosen elements γ_i are constant, and $\sum\limits_i \gamma_i = \Gamma$, where Γ is the total circulation of the vorticity domain Ω.

In the Lagrangian variables t and $\mathbf{r}_{i0} = \mathbf{r}_i(t = 0)$, the equations of motion of the centers of the chosen small vortex elements can be written in the form:

$$\frac{\partial \mathbf{r}_i(t,\ \mathbf{r}_{i0})}{\partial t} = \mathbf{v}(t,\ \mathbf{r}_i) \ +\ \mathbf{v}_d(t,\ \mathbf{r}_i), \quad \mathbf{r}_i = (\mathbf{r}_i,\ z_i). \tag{1.9}$$

The values of the flow velocity components at the points $\mathbf{r}_i$ are found from the discrete analog of the Biot–Savart integral (1.3):

$$\begin{gathered} u_r(\mathbf{r}_i) = -\frac{1}{2\pi r_i}\sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_i f'(k_{ij})\frac{\partial k_{ij}}{\partial z_i}\sqrt{r_i r_j}, \quad k_{ij} = \left(\frac{4r_i r_j}{(r_i + r_j)^2 + (z_i - z_j)^2}\right)^{1/2}, \\ u_z(\mathbf{r}_i) = \frac{\gamma_i}{4\pi r_i}\left(\ln\frac{8r_i}{a_i} - \frac{1}{4}\right) + \frac{1}{2\pi r_i}\sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_i\left(\frac{f(k_{ij})}{2r_i} + f'(k_{ij})\frac{\partial k_{ij}}{\partial r_i}\right)\sqrt{r_i r_j}. \end{gathered} \tag{1.10}$$

Here, the first term in the expression for $u_z(\mathbf{r}_i)$ is responsible for the self-induced velocity of the $i-$th vortex ring [30].

To find the vorticity and its gradient with respect to the Eulerian coordinates, required for calculating the diffusion velocity $\mathbf{v}_d$ (1.2) at an arbitrary point $\mathbf{r}$, we use the interpolation based on the method of smoothed particle hydrodynamics [18–20]:

$$\omega(\mathbf{r})\int \omega(\mathbf{R})\delta(\mathbf{r},\ \mathbf{R})\,d\mathbf{R} \approx \int \omega(\mathbf{R})\delta_\varepsilon(\mathbf{r},\ \mathbf{R})\,d\mathbf{R} \approx \sum_{i=1}^{N}\gamma_i\delta_\varepsilon(\mathbf{r},\ \mathbf{r}_i),$$

$$\nabla\omega(\mathbf{r}) \approx \sum_{i=1}^{N}\gamma_i\nabla\delta_\varepsilon(\mathbf{r},\ \mathbf{r}_i).$$

Here, δ is the delta-function, δ_ε is an approximation of the delta-function by a 'cut-off' function which ensures a weak convergence to the delta-function as $\varepsilon \to 0$. To improve the accuracy of the approximation, the values of the small parameter ε_i are chosen different for the neighborhoods of different vortex elements (with the subscript "i"), because they depend on the distance to the center of the nearest element. The form

of the cut-off function can be chosen fairly arbitrarily, with satisfying the necessary balance between the complexity of calculations and the required accuracy. In the earlier articles [10, 31], we used the cut-off functions of both the first order [21]

$$\delta_\varepsilon(\mathbf{r},\ \mathbf{r}_i) = \frac{1}{\varepsilon_i}\exp\left(-\frac{|\mathbf{r}-\mathbf{r}_i|}{\varepsilon_i}\right)$$

and higher orders, for example:

$$\delta_\varepsilon(\mathbf{r},\ \mathbf{r}_i) = \frac{1}{3\pi\varepsilon_i^2}\left(4-\frac{|\mathbf{r}-\mathbf{r}_i|^2}{\varepsilon_i^2}\right)\exp\left(-\frac{|\mathbf{r}-\mathbf{r}_i|^2}{\varepsilon_i^2}\right).$$

In most calculations presented below, we used the cut-off function of the first order. With account of the form of the cut-off function δ_ε the expressions for the vorticity and the components of its gradient take the form:

$$\begin{aligned}
\omega(r,\ z) &\approx \sum_{i=1}^{N}\frac{\gamma_i}{\varepsilon_i}\exp\left(-\frac{\sqrt{(r-r_i)^2+(z-z_i)^2}}{\varepsilon_i}\right),\\
\frac{\partial\omega(r,\ z)}{\partial r} &\approx \sum_{i=1}^{N}\frac{\gamma_i}{\varepsilon_i^2}\exp\left(-\frac{\sqrt{(r-r_i)^2+(z-z_i)^2}}{\varepsilon_i^2}\right)\left(\frac{r_i-r}{\sqrt{(r-r_i)^2-(z-z_i)^2}}\right),\\
\frac{\partial\omega(r,\ z)}{\partial z} &\approx \sum_{i=1}^{N}\frac{\gamma_i}{\varepsilon_i^2}\exp\left(-\frac{\sqrt{(r-r_i)^2+(z-z_i)^2}}{\varepsilon_i^2}\right)\left(\frac{z_i-z}{\sqrt{(r-r_i)^2-(z-z_i)^2}}\right).
\end{aligned} \tag{1.11}$$

In addition, we use the relations for the small variable radii of the cross-sections of the vortex elements $a_i(t)$:

$$a_i^2\omega_i = a_{i0}^2\omega_{i0} = \mathrm{const}_i. \tag{1.12}$$

These relations are obtained from the condition of the constancy of the vorticity flux through the cross-section of the Lagrangian vortex element whose center travels with the total velocity $\mathbf{v}+\mathbf{v}_d$.

In accordance with the recommendations of [21], the value of the 'support' of the cut-off function ε_i is taken proportional to the distance to the nearest vortex element $l_i(t)$: $\varepsilon_i(t) = Cl_i(t)$, where C is a constant greater than unity.

The initial discretization of the region of non-zero vorticity implies the choice of the appropriate values of $\mathbf{r}_{i0}$, γ_i, and a_i, which should ensure a satisfactory approximation of the initial vorticity field according to formula (1.11). The more detailed information about the possible ways of discretization of the initial vorticity can be found in [20].

For finding the interphase force in (1.5) and its derivatives in (1.8), it is necessary to calculate the carrier-phase velocity and its derivatives with respect to the Eulerian coordinates $(r,\ z)$ at the points $(t,\ \mathbf{r}_{sj0})$ (j is the number of the trajectory). For this purpose, we use the formulas similar to (1.10) and their derivatives with respect to the spatial coordinates. If the point at which the velocity is calculated does not fall in the region of non-zero vorticity, the contribution of the self-induced velocity of the vortex ring (the first term in the formula for u_z in (1.10)) is neglected.

To calculate the derivatives of the carrier-phase velocity $\partial u_{r,z}/\partial r$ and $\partial u_{r,z}/\partial z$, it is possible to differentiate analytically the expressions (1.10) over the Eulerian coordinates r and z or to use the second-order finite-difference approximations of the derivatives with a discretization step h:

$$\begin{aligned}
\left(\frac{\partial u_{r,z}}{\partial r}\right)_{\mathbf{r}=\mathbf{r}_s} &\approx \frac{u_{r,z}(r_s+h,\ z_s)-u_{r,z}(r_s-h,\ z_s)}{2h},\\
\left(\frac{\partial u_{r,z}}{\partial}\right)_{\mathbf{r}=\mathbf{r}_s} &\approx \frac{u_{r,z}(r_s,\ z_s+h)-u_{r,z}(r_s,\ z_s-h)}{2h}.
\end{aligned}$$

Here, the velocity values at several points are calculated also with the use of (1.10).

The Advantages of the Combined Lagrangian Method

We should note one important advantage of the combined fully Lagrangian method compared with the standard Lagrangian–Eulerian methods, in which the velocity field of the carrier phase is calculated at the fixed grid points of the Eulerian mesh. In the general case, the grid points of the Eulerian mesh do not coincide with the current positions of the dispersed particles $\mathbf{r}_s$, so that at the points $\mathbf{r} = \mathbf{r}_s$ it is necessary to interpolate the values of the requested functions from the Euler mesh. It is a cumbersome procedure resulting in a substantial error the order of which depends on the order of the interpolation polynomial. Usually, in practice, the quadratic interpolation is used (see, for example, [17]). The proposed combined approach makes it possible to avoid the procedure of remeshing from the Eulerian to the Lagrangian mesh, and formulas (1.10) make it possible to find the velocity values at any point without an additional error.

The combined approach described above makes it possible to find all two-phase flow parameters (the coordinates of the vortex element centers, the velocity and vorticity of the carrier phase, the particle trajectories, the velocity and the concentration of the dispersed phase) from the solution of the system of governing equations, which consists of $2N + 12M$ ordinary differential equations (1.4)–(1.5), (1.7)–(1.9), finite integral sums (1.10)–(1.11), and finite algebraic relations (1.6) and (1.12).

The method can be used for the description of disperse multiphase mixtures with an incompressible viscous and an effectively inviscid carrier phase. In the latter case, the diffusion velocity is set to be zero, and the method of calculating the carrier-phase parameters reduces to a known class of vortex methods for axisymmetric flows of inviscid fluids [29]. In this case, instead of relation (1.12), it is necessary to use the relation $a_i^2 r_i = a_{i0}^2 r_{i0}$, which follows from the fact that the toroidal vortex tubes are 'frozen' in the inviscid incompressible carrier phase.

2. THE NUMERICAL SOLUTION OF THE EQUATIONS OF THE COMBINED LAGRANGIAN METHOD

The solution of the system of governing Eqs. (1.4)–(1.12) is found numerically. At each time step, for solving the system of Eqs. (1.9) describing the motion of the vortex elements the second-order Runge–Kutta method was used. To calculate the convective and diffusion velocities, in the right-hand side of Eqs. (1.9), the current values of the vortex element coordinates are substituted into formulas (1.10)–(1.11), in which the radius of the elementary vortex ring, used in the calculation of its self-induced velocity, is found from Eq. (1.12), and the value of the cut-off function support is calculated as $\varepsilon_i(t) = Cl_i(t)$ ($l_i(t)$ is the distance to the nearest vortex element). In the calculations, the value of the coefficient $C = 1.15$ was chosen from the conditions of optimal approximation of the initial vorticity field in the examples considered below. The current coordinates of the chosen M particles, and the corresponding values of the particle velocity components, and the components of the Jacobian $J_{pqj}(t)$ are found from the numerical solution of system (1.4)–(1.5), (1.7)–(1.8) for each trajectory using the Runge–Kutta method. When the Jacobian components are found, the concentration of the dispersed impurity is calculated using formula (1.6). Thus, at each time step all two-phase flow parameters become known along the chosen M Lagrangian trajectories of the particles.

3. EXAMPLES OF THE APPLICATION OF THE COMBINED LAGRANGIAN METHOD

As the first example illustrating the capabilities of the combined Lagrangian method, we considered the evolution of a cloud of inertial particles against the background of the motion of a system of thin coaxial vortex rings of equal circulation in an inviscid fluid. Earlier, such system of vortex rings in a pure inviscid fluid was studied in [32]. It was shown that, depending on the intensity and the initial location of the rings, different regimes of their motion along the symmetry axis can be realized, in particular, the so-called 'leapfrogging' regime, in which the vortex rings periodically change their radius and pass through each other. Against the background of this flow, a finite cloud of inertial particles, moving with a velocity slip with the carrier phase, is subjected to significant deformations. In the calculations, as the length scale L the maximum vortex ring radius is taken, and the velocity scale is Γ^*/L, where Γ^* is the dimensional velocity

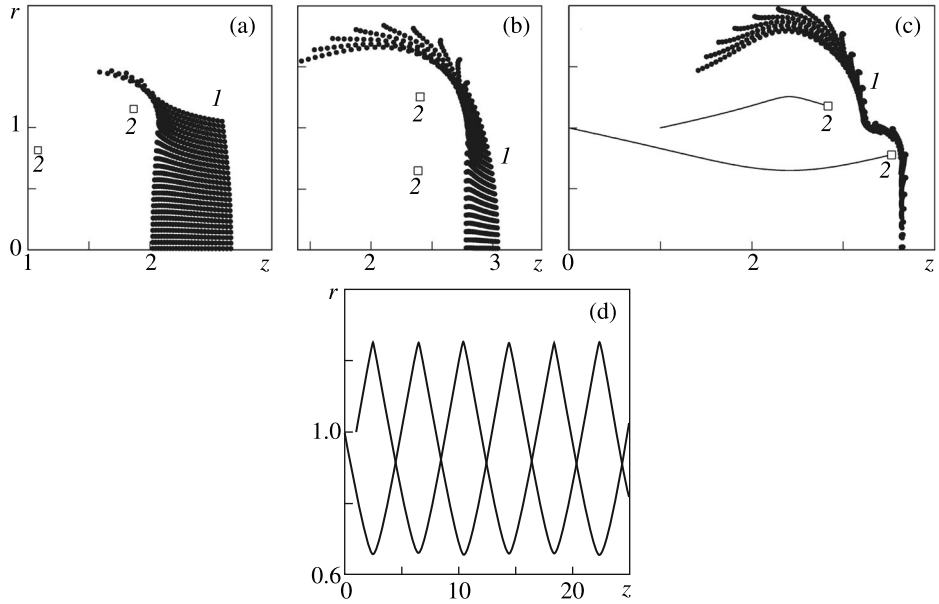

Fig. 1. Locations of the cloud of inertial particles (*1*) and two vortex rings (*2*) at subsequent instants of time $t = 1.72$, 3.12, 4.28 (a–c); (d) trajectories of the two vortex rings in inviscid fluid, corresponding to the periodic 'leapfrogging' regime.

circulation around a single vortex ring. Below, we present the examples of calculations corresponding to two and four vortex rings. The initial values of the parameters were as follows:

$$r_{i0} = 1, \quad z_{10} = 0, \quad z_{20} = 1, \quad a_{i0} = 0.01, \quad \gamma_i = 1, \quad N = 2,$$
$$r_{10} = r_{30} = 1, \quad r_{20} = r_{40} = 0.5, \quad z_{i0} = 0.5(i-1), \quad a_{i0} = 0.01, \quad \gamma_i = 1, \quad N = 4.$$

The particle cloud consists of M chosen elements, with the particle inertia parameter $\beta = 1$ (the velocity relaxation length is equal to the characteristic length scale of the flow). At the initial instant, the cloud was at rest and uniformly occupied the domain $\Omega_p = \{(r,\, z):\ r \in (0,\ 1],\ z \in [1.5,\ 2.5]\}$, which corresponds to the initial conditions (the subscript "j" is omitted):

$$\mathbf{r}_{s0} \in \Omega_p, \quad \mathbf{v}_{s0} = 0. \tag{3.1}$$

All calculations presented were performed with a constant time step, which was smaller than $\Delta t = 10^{-4}$, that ensured the satisfactory accuracy. In the example considered, only Eqs. (1.4)–(1.5) allowing to determine the positions of the dispersed particles were solved; accordingly, it was sufficient to use only initial conditions (3.1). Figure 1 shows the calculations of the evolution of the particle cloud at successive instants of time (a–c) in the case of two vortex rings moving in the leapfrogging regime. In the course of motion, the volume of the dispersed phase is subjected to a significant deformation: it is elongated along the axis of vortex motion and displaced from the near-axial region. In Fig. 1, the trajectories of the two vortex rings for the leapfrogging regime considered are shown on a larger time scale. Within the graphical accuracy, the trajectories of the vortices coincide with the results of [32], which serves as the confirmation of the correctness of the algorithm and the accuracy of the presented numerical calculations.

In the case of the flow with four vortex rings (Fig. 2), the fragmentation of the particle cloud occurs. At the initial stage, the particle cloud is first deformed, and then a part of the cloud is separated and entrained by the two fastest vortices. The calculations show that with increase in the number of the localized and distant vortex rings the degree of the fragmentation of the particle cloud increases. In all calculations, multiple intersections of the particle trajectories are observed, which can be successfully tracked by the proposed algorithm of the combined Lagrangian method.

As the next example, we consider a gas-particle flow induced by the motion of a vortex ring of finite thickness in a viscous fluid at moderate Reynolds numbers. The motion of solitary vortex rings in pure

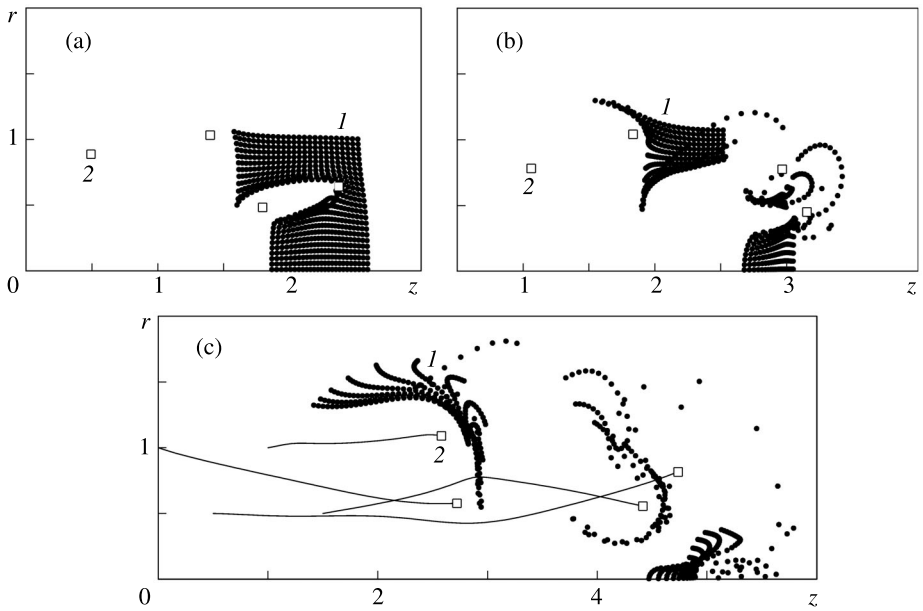

Fig. 2. Locations of the cloud of particles (*1*) and four vortex rings (*2*) at subsequent instants of time $t = 0.75$, 1.5, 3 (a–c).

viscous fluids was discussed in many publications (for example, see [33–35] and the literature therein). The initial stage of the evolution of a laminar vortex ring in a viscous fluid depends significantly on the initial distribution of vorticity, but then, over time, some integral parameters of the flow go on the asymptotics, obtained by different authors (see, a review [33]), for the purely diffusion regime of vorticity transport (when the convective terms in the vorticity transport equation can be neglected). The velocity of the vortex centroid is described by the asymptotic formula [33] (the dimensional variables are marked by the asterisks):

$$V_c^* = 0.0037038\frac{I^*/\rho^*}{\nu^*(t^* - t_0^*)^{3/2}}. \tag{3.2}$$

Here, ρ^* and ν^* are the density and kinematic viscosity of the fluid, t_0^* is a certain time value depending on the initial distribution of vorticity, the Reynolds number, and the details of the vortex ring evolution at the initial stage, V_c^* and I^* are the vortex-centroid velocity and the vortex impulse, which are given by the formulas:

$$V_c^* = \frac{dz_c^*}{dt^*}, \quad z_c^* = \frac{\int \omega^* r^{*2} z\, dr^*\, dz^*}{\int \omega^* r^{*2}\, dr^*\, dz^*}, \quad I^* = \pi\rho^*\Gamma^{*2}R_0^{*2}.$$

Here, the integrals are taken over the entire space, R_0^* is the initial toroidal radius of the vortex ring. In this problem, it is convenient to take the length scale equal to R_0^* and the velocity scale equal to Γ^*/R_0^*, then the Reynolds number is $\mathrm{Re} = \Gamma^*/\nu^*$ and the global dimensionless circulation is equal to unity. The solution in dimensionless form depends only on the Reynolds number and the initial vorticity distribution corresponding to the unit total circulation in the ring the toroidal radius of which is equal to unity.

In our dimensionless variables, asymptotics (3.2) take the form:

$$V_c = 0.0037038\pi\left(\frac{\mathrm{Re}}{t - t_0}\right)^{3/2}. \tag{3.3}$$

To verify the numerical algorithm of calculating the viscous carrier-phase flow using the above method of discrete vortex elements, we calculated the evolution of a vortex ring in a pure viscous fluid for different

Reynolds numbers. For simplicity, the initial (at a small $t_1 = 0.01$) vorticity distribution in the ring was taken as in the Oseen vortex:

$$\omega_0 = \frac{\mathrm{Re}}{4t_1}\exp\left[-\frac{\mathrm{Re}}{4t_1}\left(z^2 + (r-1)^2\right)\right]. \tag{3.4}$$

The circular domain of the initial non-zero vorticity was split into N vortex elements, which were constructed by splitting the equidistant annular elements by a given number of radial elements. The maximum number of the vortex elements in the calculations was as large as several thousand. The calculated values of the vortex centroid velocity for three values of the Reynolds number are presented in Fig. 3. Clearly, with increase in time the calculated velocity of the centroid approaches asymptotics (3.3) with a satisfactory accuracy, which confirms the efficiency of the numerical algorithm for calculating the parameters of the carrier phase. It turned out that for three considered values of $\mathrm{Re} = 50$, 100 and 200, the value of the free parameter t_0 in Eq. (3.3) can be chosen almost the same: $t_0 \approx -1.14$. Figure 4 shows a typical calculation of the velocity pattern formed in the vortex with time.

Further, we calculated the evolution of a finite cloud of inertial particles in the vortex flow under study. At the initial instant, the cloud occupied a disk-shaped region $\Omega_p = \{(r,\ z) : r \in [0,\ 1.5],\ z \in [0,\ 0.4]\}$. The initial number concentration of the particles in the cloud was assumed to be uniform $n_{s0}(r,\ z) = 1$. Two variants of the initial conditions for the particle velocity were considered: in the first variant, the velocity was equal to zero, and in the second variant the particle velocity was equal to that of the carrier phase. In calculating the particle concentration along the chosen trajectories, Eqs. (1.7)–(1.8) were solved with the initial conditions:

$$J_{rr} = J_{zz} = 1, \quad J_{rz} = J_{zr} = 0.$$

In the case of zero initial velocity of the particles, $\vartheta_{rr} = \vartheta_{zz} = \vartheta_{rz} = \vartheta_{zr} = 0$. In the second variant, at the initial instant we have:

$$\vartheta_{rr} = \frac{\partial u_r}{\partial r}, \quad \vartheta_{rz} = \frac{\partial u_r}{\partial z}, \quad \vartheta_{zr} = \frac{\partial u_z}{\partial r}, \quad \vartheta_{zz} = \frac{\partial u_z}{\partial z}.$$

For discretizing the initial admixture distribution, we used M values of initial coordinates of the particles, uniformly distributed in Ω_p.

The vortex ring under study, specified by the initial vorticity distribution (3.4), travels along the z-axis with a gradually decreasing velocity. In Fig. 5, for the Reynolds number equal to 100 and three subsequent instants of time, we present the distribution patterns of the vortex element centers *2* and the chosen particles of the dispersed phase (*1*, the dot color intensity corresponds to the value of the particle number density at the considered point of space). It is clear that, with time, the particle cloud is elongated and 'coiled' onto the vortex. Therewith, ahead of and behind the vortex localized high particle concentration zones are formed. It should be noted that for fairly inertial particles the formation of 'folds' in the dispersed continuum is typical. It is associated with the intersections of particle trajectories. This effect is illustrated in Fig. 6, where several Lagrangian surfaces are shown, which at the initial instant were flat surfaces $z = \mathrm{const}$ (a), (c). In Figs. 6b and 6d, we show the calculated distributions of the particle concentration at the leading and the trailing Lagrangian surface. Clearly, the local particle concentration sharply increases at the points of the onset of a fold. These features of the behavior of the inertial dispersed phase make difficult the use of the standard Eulerian or Eulerian–Lagrangian approaches for the accurate calculation of the particle concentration distribution in vortex flows. However, the full Lagrangian approach developed in this paper makes it possible to cope successfully with the above mentioned problems.

Summary. A combined fully Lagrangian method for modeling axisymmetric (without swirl) vortex flows of dilute gas-particle mixtures in an unbounded space is proposed. The approach developed is based on the combination of a vortex blob method for the viscous (or effectively inviscid) incompressible carrier phase and the full Lagrangian method for calculating the parameters of the dispersed phase. The combined method has a clear advantage as compared to the standard Eulerian-Lagrangian approaches: it makes possible to

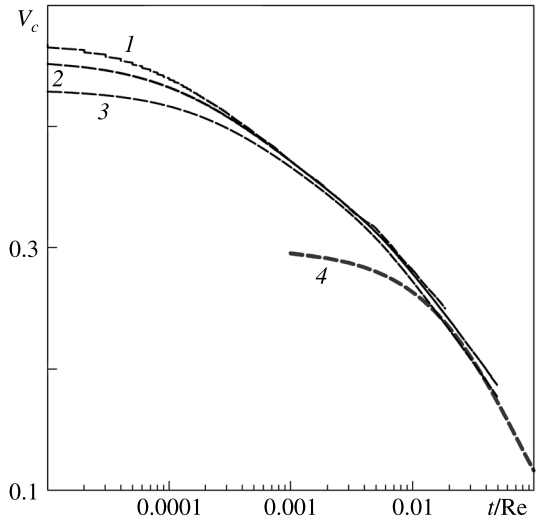

Fig. 3.

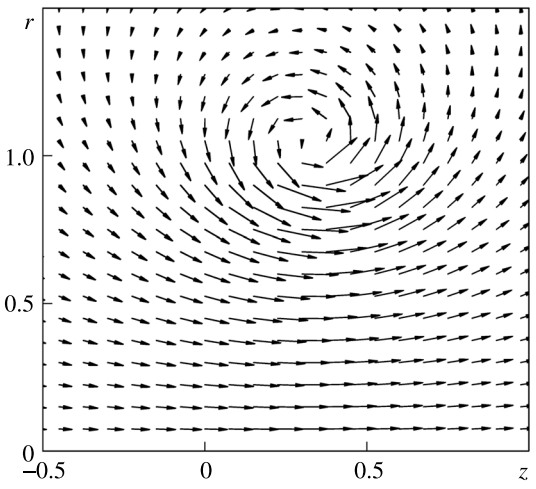

Fig. 4.

Fig. 3. The approach of the vortex centroid velocity to the self-similar asymptotics for Re = 200, 100, 50 (*1–3*); (*4*) asymptotics (3.3); the logarithmic scale is used in the *x*-axis.

Fig. 4. The velocity vector pattern in the vortex ring in the plane (r, z) for Re = 100, $t = 0.962$, $N = 2601$.

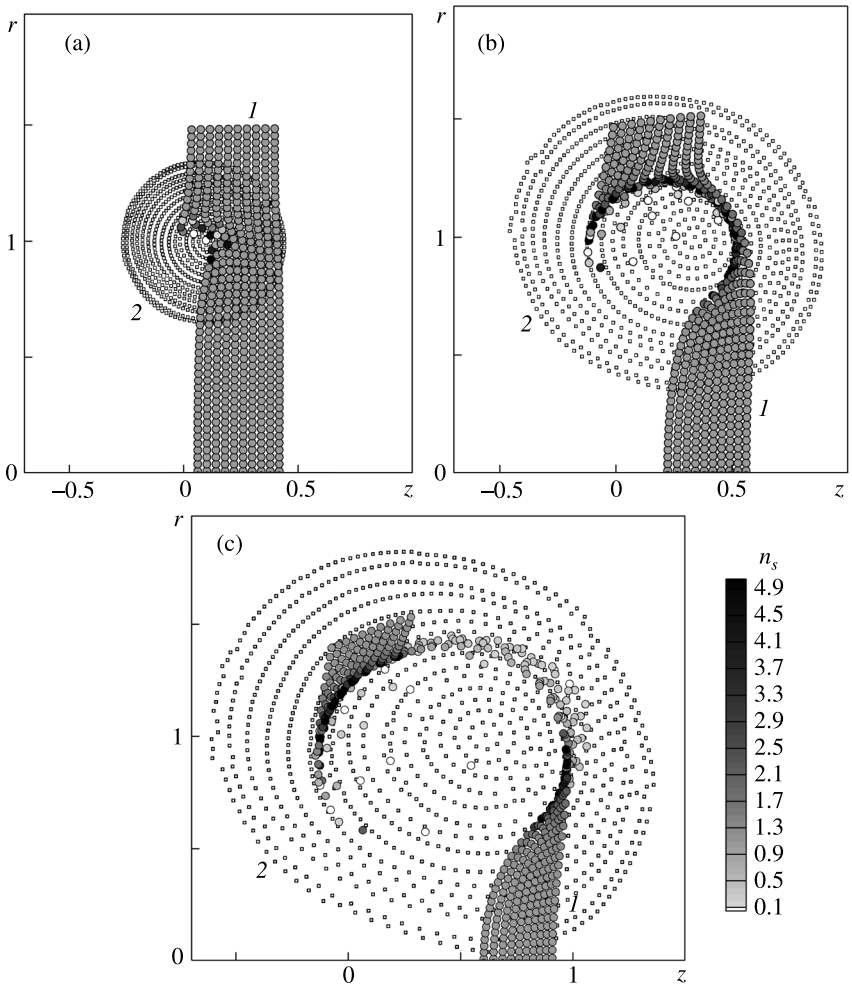

Fig. 5. Locations of the cloud of inertial particles (*1*, the color corresponds to the particle number concentration) and vortex elements (*2*) at subsequent instants of time $t = 0.2998$, 0.9998, 2 (a–c) for Re = 100, *beta* = 1; the number of vortices $N = 961$, the number of tracked particles $M = 500$.

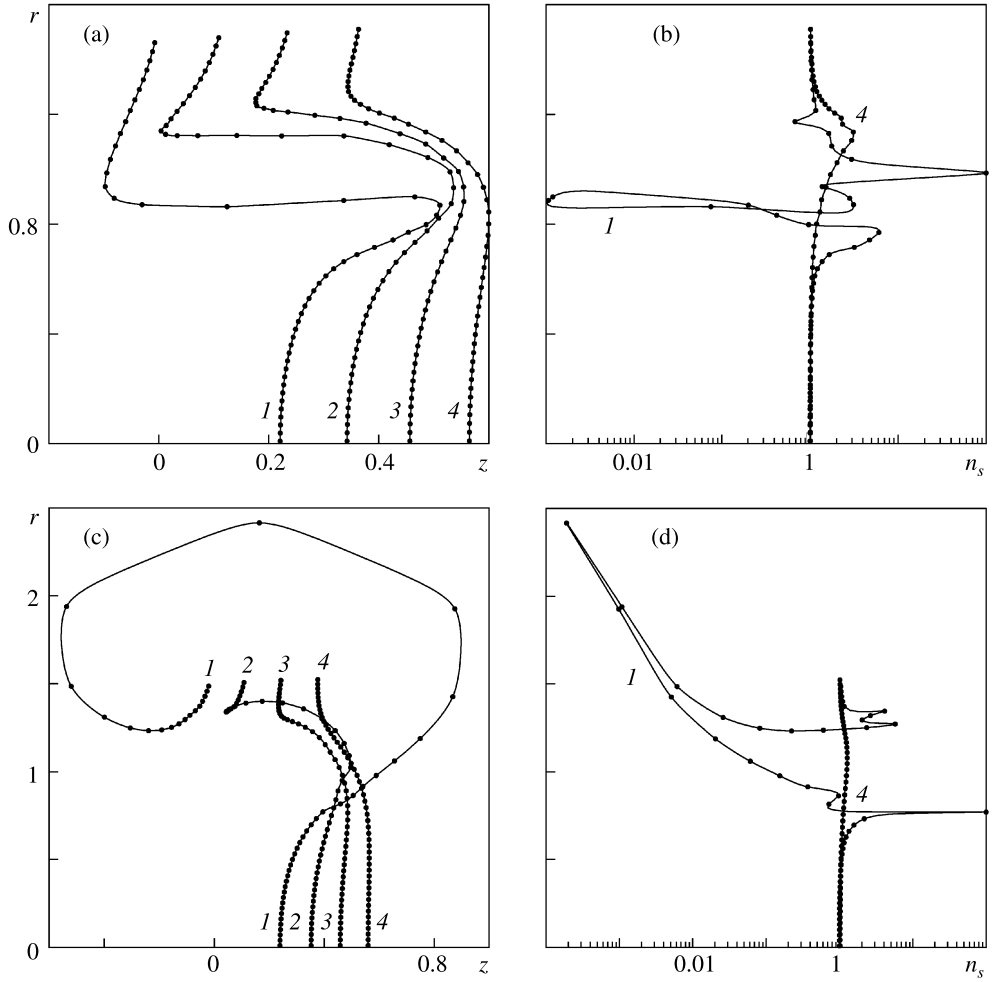

Fig. 6. Locations of the Lagrangian surfaces *1–4* and the distribution of the particle concentration on the leading (*1*) and trailing (*4*) Lagrangian surface for $\mathrm{Re} = 100$, $\beta = 0.5$, $N = 961$, $M = 500$: (a, b) $t = 1.3478$, at the initial instant the particles are at rest; (c, d) $t = 0.3998$, at the initial instant the particles move with the gas.

avoid the cumbersome procedure of the recalculation of the parameters from the Eulerian to the Lagrangian grid. The method is particularly convenient for accurate calculations of the dispersed phase concentration fields in vortex flows with crossing particle trajectories.

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