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On the Hyperbolicity of One-Dimensional Models for Transient Two-Phase Flow in a Pipeline

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Abstract—Characteristic properties of one-dimensional models of transient gas-liquid two-phase flows in long pipelines are investigated. The methods for studying the hyperbolicity of the systems of equations of multi-fluid and drift-flux models are developed. On the basis of analytical and numerical studies, the limits of the hyperbolicity domains in the space of governing dimensionless parameters are found, and the impact of the closure relations on the characteristic properties of the models is analyzed. The methods of ensuring the global unconditional hyperbolicity are proposed. Explicit formulas for the eigenvelocities of the system of the drift-flux model equations are obtained and the conclusions about their sign-definiteness are drawn.

Keywords: hyperbolicity, multiphase flows, multi-fluid approach, drift flux model, pipe flow.

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The development of models for multiphase flows is an important fundamental problem, which has a wide range of practical applications. In particular, an urgent problem is to construct a closed self-consistent model for a transient one-dimensional gas-liquid flow in a well or a pipeline. These flows are encountered in many technological processes in oil/gas and nuclear industries. Despite a large number of different applications, no commonly accepted model describing such flows has been developed so far.

In the oil and gas industry, two models are most widely used in studying multiphase flows in pipes [1], namely, a multi-fluid model and a more simplified drift-flux model. The systems of equations contain the conservation laws averaged over the cross section of the pipe and supplemented with a number of simplifying assumptions. The drift-flux model includes two mass balance equations for the gas and liquid phases and one momentum equation for the mixture. The multi-fluid model is based on the multi- continuum approach. It contains the mass and momentum balance equations for each phase. Different implementations of these approaches are the basis for commercial multiphase flow simulators used for oil and gas applications (PIPESIM, ECLIPSE, OLGA, LedaFlow, MAST, etc.). For the closure of equations, a number of additional relations are used, which lead to the fact that the system loses hyperbolicity, and the mathematical formulation of the initial-boundary-value problem becomes ill-posed. In this case, the model ceases to describe real physical phenomena, and a non-physical instability is observed in the numerical solution [2]. Thus, an urgent problem is to find the criteria of hyperbolicity of the existing systems of equations and to construct new unconditionally hyperbolic models of two-phase flow. The development of such approaches is necessary, in particular, for the design of industrial simulators of transient multiphase one-dimensional flows [3].

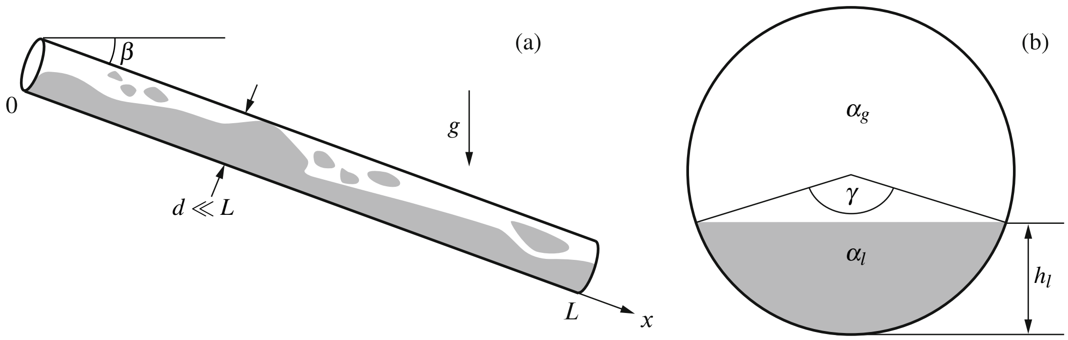

Fig. 1. Scheme of two-phase flow in the pipeline (a) and the pipe cross-section (b).

Despite decades of research, the problem of the loss of hyperbolicity of the multi-fluid model remains unsolved [4]. The often used assumption for the closure of the model is the condition, convenient from a practical point of view, of equality of the pressures in the phases. However, this simplification significantly restricts the region of hyperbolicity of the model [1, 5, 6]. Some publications deal with different modifications of the classical two-fluid model with a single pressure, which make the system of equations hyperbolic over a wide range of governing parameters (see e.g. [4, 7–9]). However, there is no generally accepted formulation developed so far for the system of governing equations for the two-fluid model.

The properties of the drift-flux model equations were studied by many authors (see, e.g. [10–13]). The characteristic equation of the system is of third order, it can be written in explicit form and formally solved. However, the resulting expressions for the roots are not suitable for interpretation, and no simple and quite general criterion of hyperbolicity is known. For studies, additional assumptions are usually employed, and a simplified form of the system is analyzed. This work is devoted to the development of models of gas-liquid flows in pipelines, the design of the methods for studying the hyperbolicity of these models, and finding the conditions of hyperbolicity for the systems of constitutive equations in a wide range of governing parameters, typical of various gas-liquid flows. In the first part of the article, the class of the flows under study is described and the corresponding system of equations is presented. The goal of finding the roots of the characteristic equation of the system is formulated. In the second part, in the framework of the two-fluid approach we consider two modifications of the classical two-fluid model, which allow us to expand the region of hyperbolicity: (i) a model with account for the liquid level gradient and (ii) a model with account for the interfacial pressure forces. On the basis of analytical and numerical studies, the boundaries of the regions of hyperbolicity of the model in the space of governing parameters are found. The third part is devoted to the analysis of the characteristics of the drift-flux model in two formulations, differing by the form of the momentum equation for the mixture. The conclusions are drawn about the sign-definiteness of the eigenvalues, which make it possible to determine the number of incoming and outgoing characteristics at the boundaries of the calculation domain and to give the correct formulation of the initial-boundary-value problem [14].

1. FORMULATION OF THE PROBLEM

We consider a transient isothermal gas-liquid flow in a straight circular pipe inclined to the horizontal at an arbitrary angle. The pipe diameter is much smaller than its longitudinal dimension: $d \ll L$ (Fig. 1a). When modeling this kind of flows, instead of the full equations for each phase and tracking the phase interface, it is common to use one-dimensional non-stationary equations of balance laws for the effective flow parameters obtained by averaging over the cross section of the pipe. The equations of the conservation laws are written in the form:

$$\mathbf{A}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial x} = \mathbf{C}, \tag{1.1}$$

where $\mathbf{U}(x, t) = (u_1(x, t), \ldots, u_n(x, t))^{\mathrm{T}}$ is the vector of unknown functions, $\mathbf{A}(\mathbf{U}(x, t), x, t)$ and $\mathbf{B}(\mathbf{U}(x, t), x, t)$ are $n \times n$ matrices, and $\mathbf{C}(\mathbf{U}(x, t), x, t)$ is the vector of right sides of the system of equations. Usually, the components of the vector $\mathbf{U}$ are velocities, densities and volume fractions of the phases, and also the pressure. The assumption of equal phase pressures is often used, which makes it possible to reduce the number of unknowns. Such system lies, in particular, in the basis of two widely used models of gas-liquid pipe flows, namely, the drift-flux and multi-fluid models, which are considered in this study.

In the case of a non-degenerate matrix $\mathbf{A}$, system (1.1) can be reduced to the form:

$$\frac{\partial \mathbf{U}}{\partial t} + \tilde{\mathbf{B}} \frac{\partial \mathbf{U}}{\partial x} = \tilde{\mathbf{C}}. \tag{1.2}$$

According to [15], system (1.2) is called hyperbolic, if there exists a non-singular matrix $\mathbf{\Omega}$, diagonalizing $\tilde{\mathbf{B}}$, so that

$$\mathbf{\Omega}^{-1}\tilde{\mathbf{B}}\mathbf{\Omega} = \mathbf{\Lambda} = \mathrm{diag}\,[\lambda_1, \ldots, \lambda_n]$$

and all the eigenvalues λ_k of the matrix $\tilde{\mathbf{B}}$ are real-valued. If the values of λ_k are different, the system is called strictly hyperbolic.

For the solution of Eqs. (1.1), it is required to formulate additionally the sufficient number of initial and boundary conditions, corresponding to the type of the equations. An initial-boundary-value problem for a hyperbolic system may be incorrect if the equations change type.

The loss of hyperbolicity may indicate that the model ceases to describe the considered phenomenon. In this case, non-physical oscillations develop in the numerical calculations, which does not make it possible to obtain a grid-convergent solution, stable with respect to the initial data. In the general case, for system (1.1) the hyperbolicity and stability criteria do not coincide, and the hyperbolic system can exhibit hydrodynamic instability [2]. However, these problems become equivalent when studying the stability of a uniform steady-state solution of system (1.1) with the right-hand side independent of $\mathbf{U}$, and when considering high-frequency disturbances. The stability conditions for different models of one-dimensional two-phase flows were investigated, for example, in [2, 16, 17] and the literature cited therein. In [16, 17], it was shown, in particular, that, for the multi-fluid model with account for the liquid level gradient, the development of instability of the phase interface in a stratified flow is associated with the change of regime, namely, the transition from the stratified two-layer flow to the slug or annular flow.

In the present work, we investigate the limits of the domains of hyperbolicity of the models that determine the range of parameters in which a well-posed (in Hadamard) mathematical formulation of the initial-boundary-value problem exists. The investigation of the stability of solutions of the hyperbolic systems formulated is beyond the scope of this study and requires further considerations.

The first step in establishing the type of the system of equations is the solution of the characteristic equation. If its roots are real-valued and distinct, the set of eigenvectors of the matrix of system (1.2) is complete, the matrix $\tilde{\mathbf{B}}$ can be diagonalized, and the system is strictly hyperbolic. In the case of multiple real-valued roots, it is necessary to find the eigenvectors directly and to check their completeness [14]. Since in the general case it is impossible to write the solution of the characteristic equation explicitly, in addition to the analytical study, numerical methods are also used to establish the hyperbolicity of the system.

2. THE MULTI-FLUID MODEL

Within the multi-continua approach [6], one-dimensional transient mass and momentum equations for each phase, averaged over the cross section of a long tube ($d \ll L$, see Fig. 1a) and supplemented with the relation for the volume fractions and the equations of state of the phases, take the form [1]:

$$\frac{\partial(\alpha_i\rho_i)}{\partial t} + \frac{\partial(\alpha_i\rho_i u_i)}{\partial x} = 0, \qquad (i = l,\ g), \tag{2.1}$$

$$\frac{\partial(\alpha_i\rho_i u_i)}{\partial t} + \frac{\partial(\alpha_i\rho_i u_i^2)}{\partial x} = -\alpha_i\frac{\partial p}{\partial x} + \alpha_i\rho_i g\sin\beta + F_i^p + F_i^\tau, \tag{2.2}$$

$$\alpha_l + \alpha_g = 1, \tag{2.3}$$

$$\rho_i = \rho_i(p). \tag{2.4}$$

Here, x is the coordinate measured along the pipe; t is the time; the indices "l", "g" refer to the liquid and gas phases, respectively; α_i, ρ_i, and u_i are the averaged values of volume fractions, densities, and velocities of the phases; p is the averaged pressure, assumed to be identical in both phases; g is the gravity force acceleration; β is the angle of pipe inclination to the horizontal; F_i^p are the phase interaction forces due to the pressure (hereinafter, the interfacial pressure forces) for i-th phase; F_i^τ are the friction forces, including the interfacial and wall friction.

The assumption of the equality of average phase pressures in the cross-section can be violated due to capillary effects or fast processes, when the motion of the phases on the pipe diameter scale is important [6]. We note that there is a class of unconditionally hyperbolic models, actively developed in recent years, which are based on the assumption of different pressures in the phases [18–20]. However, these models contain a larger number of equations and require additional closure relations. This is why they are not used so far in industrial simulations of gas-liquid pipe flows.

In many engineering applications, to make the systems closed, it is assumed that $F_i^p = 0$ and F_i^τ are specified as algebraic functions of the parameters α_i, ρ_i, and u_i, approximating experimental measurements of the friction for different flow regimes (see, for example, [1, 21, 22]). Thus, depending on the flow regime, different forms of closure relations for F_i^τ are used.

Under the assumptions formulated, system (2.1)–(2.4) turns out to be non-hyperbolic [5, 6, 14], and its uniform steady-state solution is unstable [2]. In this paper, we consider two modifications of original system (2.1)–(2.2), which make it possible to preserve the classic model conditionally hyperbolic, namely, the account for the gradient of the liquid phase level [22] and the interfacial pressure forces [7].

2.1. The model taking into account the liquid level gradient. We will restrict the range of applicability of the original model [1] by the stratified two-layer near-horizontal flows, when the light-weight gaseous phase moves above the heavy liquid layer (see Fig. 1b). In this case, it is necessary to take into account the additional pressure gradient associated with the presence of a gradient of the liquid level h_l. According to the equations of the thin-layer theory [23], the transverse pressure distribution in the liquid layer is determined by the hydrostatic relation $\partial p_l/\partial y = -\rho_l g\cos\beta$.

Integrating this equation with account of the conditions on the interface $p_l|_{y=h_l} = p$, we obtain $p_l = p + (h_l - y)\rho_l g\cos\beta$. As a result, under the assumption that the fluid is incompressible or weakly compressible, in the right side of the momentum equation for the liquid phase (2.2) an additional term arises [16, 17, 22]: $-\alpha_l\rho_l g\cos\beta\,\partial h_l/\partial x$.

When both phases are incompressible, the characteristic equation takes the form:

$$(\alpha_l\rho_g + \alpha_g\rho_l)\lambda^2 - 2(\alpha_l\rho_g u_g + \alpha_g\rho_l u_l)\lambda$$

$$+ \alpha_l\rho_g u_g^2 + \alpha_g\rho_l u_l^2 - \alpha_l\alpha_g\rho_l g\cos\beta\,\frac{dh_l}{d\alpha_l} = 0.$$

From the condition of non-negativity of the discriminant, we obtain the condition of hyperbolicity of the system:

$$u_s \le \phi\sqrt{\left(\alpha_l + \frac{\alpha_g}{\eta}\right)\frac{d\tilde{h}_l}{d\alpha_l}\cos\beta}. \qquad (2.5)$$

Here

$$u_s = \frac{|u_l - u_g|}{c_g}, \quad \phi = \frac{1}{\mathrm{Fr}} = \frac{\sqrt{gd}}{c_g}, \quad \eta = \frac{\rho_g}{\rho_l}, \quad \frac{d\tilde{h}_l}{d\alpha_l} = \frac{d\tilde{h}_l}{d\gamma}\frac{d\gamma}{d\alpha_l}$$

is the derivative of the dimensionless liquid level $\tilde{h}_l = h_l/d$ (see Fig. 1b), which can be determined using the value of the volume fraction α_l and the following geometrical relations:

$$h_l = R - R\cos\frac{\gamma}{2}, \qquad \alpha_l = \frac{\gamma - \sin\gamma}{2\pi}.$$

From (2.5), it is clear that, as the pipe inclination angle increases, the hyperbolicity region reduces. Figure 2 shows the limits of the hyperbolicity region defined by inequality (2.5) for the flow of oil-gas type (see table). For water–air flow, the hyperbolicity region will be qualitatively similar. Note that $\lim\limits_{\alpha_l\to 0} d\tilde{h}_l/d\alpha_l = \lim\limits_{\alpha_l\to 1} d\tilde{h}_l/d\alpha_l = \infty$, and for intermediate values of α_l the derivative $d\tilde{h}_l/d\alpha_l$ is finite. Clearly, the system remains hyperbolic on the entire range of values of the volume fraction α_g only for sufficiently low slip velocities u_s. The highest slip velocity is denoted as $u_s^{\max}$. In a stratified two-layer flow in the pipeline, the slip velocity can exceed $u_s^{\max}$, so the limits of the hyperbolicity region do not cover the entire range of applicability of the model.

We note that Eq. (2.5) coincides with the stability condition for the phase interface in the inviscid-flow case ($F_i^\tau = 0$) for long wavelength disturbances and determines the transition from the stratified two-layer flow to the slug or annular flow regime [16, 17].

2.2. The model with account for the pressure forces at the interface. We will now consider the modification of model (2.1)–(2.4) proposed in [7] and taking into account the interphase pressure force in the form:

$$F_i^p = -p_I\partial\alpha_i/\partial x, \qquad p_I = p - p^*.$$

Here, p^* is the pressure at the interface, different from the pressure p. For simplification, in [7] it was assumed that the gaseous phase is distributed in the fluid in the form of spherical bubbles, which defines a reasonably simple shape of the interface, and subsequently allows one to obtain a closure relation for p_I. In the general case of gas-liquid flows with arbitrary shape of the phase interface, it is not possible to formulate the closure relation for p_I. However, the addition of differential terms associated with the pressure difference across the interface makes a positive effect on the expansion of the limits of hyperbolicity of the model.

If we assume that $p^* = \mathrm{const}$, then, as shown in [7] for incompressible media, the hyperbolicity condition takes the form:

$$p_I = p - p^* \ge \frac{\alpha_l\alpha_g\rho_l\rho_g}{\alpha_l\rho_g + \alpha_g\rho_l}(u_l - u_g)^2.$$

The case of compressible fluids is investigated in [8], where the conditions of hyperbolicity are established for some specific closure relations for p_I.

We will now consider the general case of compressible media and assume that

$$p_I = \chi\frac{\alpha_l\alpha_g\rho_l\rho_g}{\alpha_l\rho_g + \alpha_g\rho_l}(u_l - u_g)^2, \qquad (2.6)$$

where χ is a constant, and a proper selection of this constant ensures that the system of the governing equations is hyperbolic on the widest possible range of parameters.

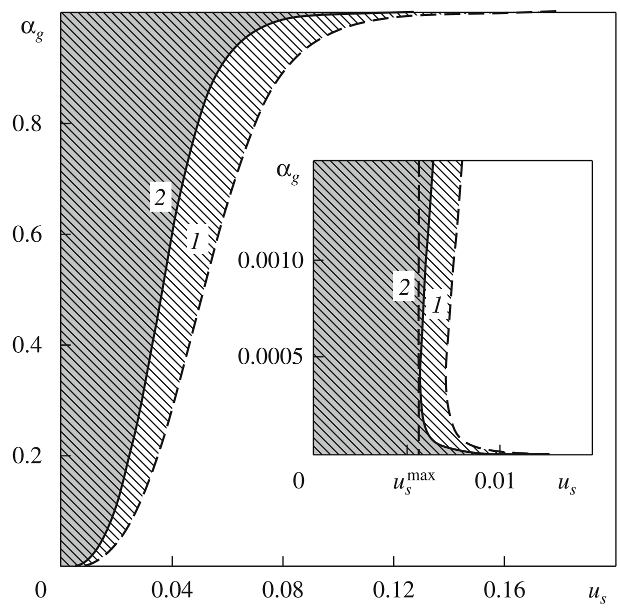

Fig. 2. Regions of hyperbolicity of system (2.1)–(2.4) with account of the liquid-level gradient for the flow type oil-gas flow type (see table) and for $\Phi = 614.3$ ($d = 0.1$ m): (*1, 2*) $\beta = 0°, 60°$.

With account of Eq. (2.6), the characteristic equation for system (2.1)–(2.4) in dimensionless form reads:

$$\begin{gathered}\big(\alpha_g + \eta(1 - \alpha_g)K^2\big)\big(u_l - \lambda\big)^2\big(u_g - \lambda\big)^2 \\ - \left(\frac{\alpha_g^2\chi\eta(1 - \alpha_g)u_s^2}{(1 - \alpha_g)\eta + \alpha_g} + \eta(1 - \alpha_g)\right)(u_g - \lambda)^2 \\ - \left(\frac{K^2(1 - \alpha_g)^2\chi\eta\alpha_g u_s^2}{(1 - \alpha_g)\eta + \alpha_g} + \alpha_g\right)(u_l - \lambda)^2 + \frac{\alpha_g\chi\eta(1 - \alpha_g)u_s^2}{(1 - \alpha_g)\eta + \alpha_g} = 0.\end{gathered} \tag{2.7}$$

Here, $K = c_g/c_l$, and λ, u_l, and u_g are scaled to the velocity c_g.

Consider a special case, where the density of the gas is negligibly small compared to the density of the liquid phase: $\eta \to 0$. In this case, the characteristic equation is simplified significantly, and its roots can be found explicitly:

$$\lambda_{1,2} = u_l, \qquad \lambda_{3,4} = u_g \pm 1.$$

In the general case, the problem is reduced to finding the number of real-valued roots of the 4-th order characteristic equation with variable coefficients. This was done using a numerical implementation of the Jenkins–Traub algorithm [24]. In a wide range of values of the problem parameters (volume fractions, velocities, densities), typical of different flows (including mixtures of oil-gas and water-air types, see table), we investigated the number of the real-valued roots of the characteristic equation. The possible variants of the hyperbolicity region are shown in Fig. 3. As a result of variation of the governing dimensionless parameters η and K, the optimal values of χ_{opt} were found, for which the system of equations is hyperbolic over the entire range of values of the volume fraction $\alpha_g \in (0,\ 1)$ and over the maximum possible range of the slip velocity $u_s < u_s^{\max} = \max_\chi|u_l - u_g|/c_g$ (see Fig. 3). In the case when one volume fraction is zero, the matrix $\mathbf{A}$ becomes singular, and the definition of hyperbolicity for system (1.1) is not applicable.

Figure 4 and the table present the calculated values of χ_{opt} and $u_s^{\max}$ as the functions of the dimensionless parameters η and K. It is clear that for small η the influence of K on χ_{opt} and $u_s^{\max}$ is small. This is because the parameter K enters in Eq. (2.7) only as the product ηK^2. It has been also found that for a given η with increase in K ($K \to 0$ corresponds to the case of incompressible liquid) the parameters χ_{opt} and $u_s^{\max}$ increase up to a certain value and then remain almost constant.

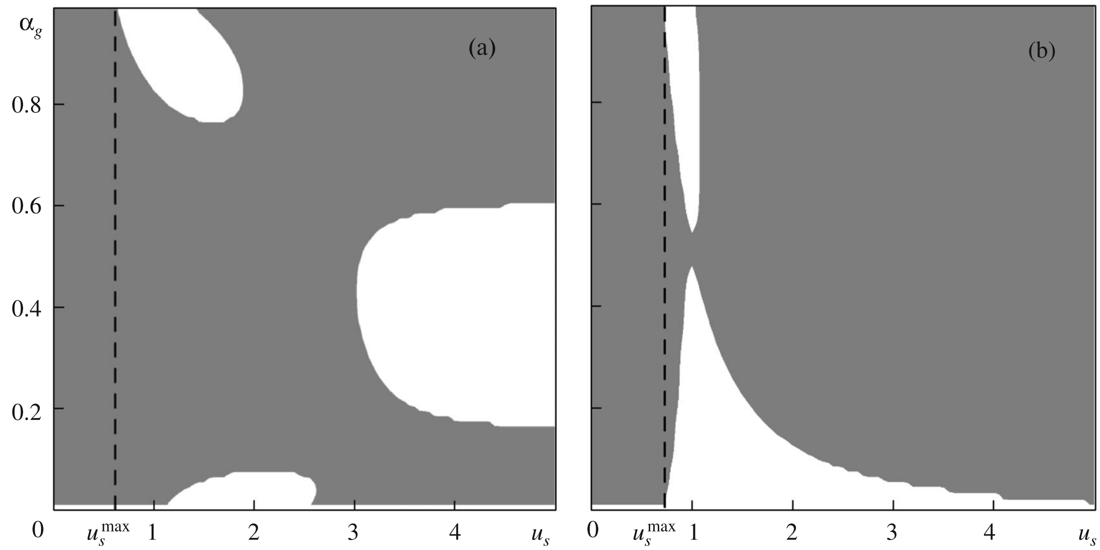

Fig. 3. Region of hyperbolicity for $\eta = 0.6$, $K = 0.6$, $\chi = 1.6$ (a) and parameters of the oil-gas flow type (table) (b).

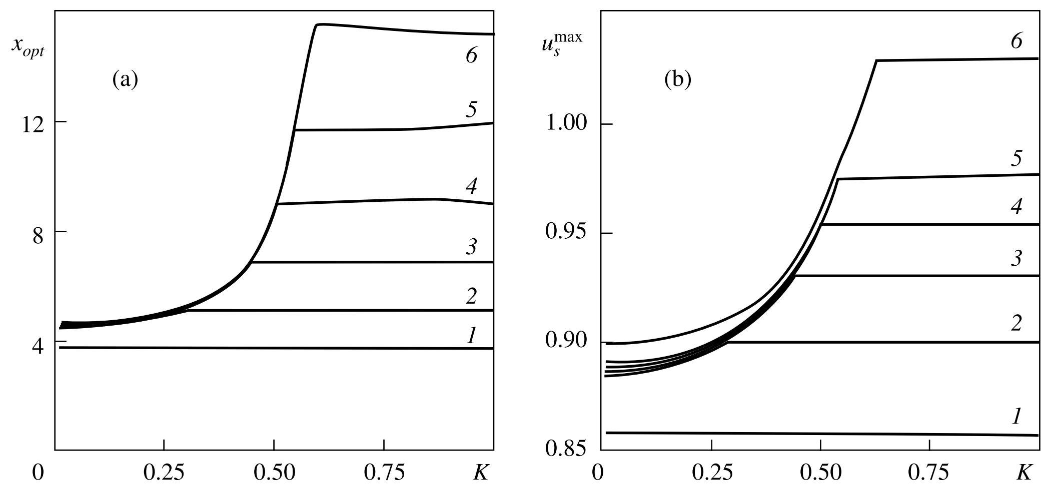

Fig. 4. The dependences $\chi_{opt}(K)$ (a) and $u_s^{\max}(K)$ (b): (*1*–*6*) $\eta = 0.1, 0.15, 0.2, 0.25, 0.3, 0.35$.

Using closure relation (2.6), we managed to establish the hyperbolicity of the system of equations (2.1)–(2.4) up to a sufficiently high slip velocity, generally not achieved in oil and gas wells and pipelines. It makes it possible to consider the model proposed as hyperbolic over the entire range of parameters, typical of practical applications.

3. THE DRIFT-FLUX MODEL

For modeling the multiphase flows in long pipelines, the so-called drift-flux model is often used. In the isothermal formulation, the model involves the mass balance laws for each phase, only one momentum equation for the mixture, and an algebraic relation between the phase velocities.

Under the assumption of absence of sources and constant cross-sectional area, the mass balance equations of the phases averaged over the cross section of the pipe have the same form as for multi-fluid model (2.1).

The algebraic formula relating the velocities of the phases is often written as [25]:

$$u_g = C_0 u_m + u_d, \tag{3.1}$$

where $u_m = \alpha_g u_g + \alpha_l u_l$ is the mean-volume velocity of the mixture, $C_0 = C_0(\alpha_g, u_m, p)$ is the profile

Typical values of parameters for two types of gas-liquid flows

Flow type	ρ_l, kg/m^3	ρ_g, kg/m^3	c_l, m/s	c_g, m/s	η	K	χ_{opt}	$u_s^{\max}$
Water-air	1000	1.23	1500	331	0.00123	0.22	2.08	0.72
Oil-gas	900	0.7	1470	430	0.00078	0.29	2.07	0.72

parameter which takes into account the cross-sectional distributions of the gas volume fraction and the velocities, and $u_d = u_d(\alpha_g,\ u_m,\ p)$ is the drift velocity. The relation can also be specified in the form of a slip law [13]:

$$u_g - u_l = \Phi(\alpha_g,\ u_g,\ p). \tag{3.2}$$

Equation (3.2) supplemented with the definition of the mean-volume velocity of the mixture can be written in the form (3.1). It is assumed that the system of differential equations of the drift-flux model is supplemented by the algebraic relation between the velocities in the form (3.1), and the profile parameter C_0 and the drift velocity v_d are regarded as given functions of the variables α_g, u_m, and p. In this case, the velocities of the phases are the functions of the same variables. Additional physical restrictions are imposed on the closure relation (3.1). In the pure-gas flow, the effects of nonuniform distributions of the volume fraction and the velocity across the cross-section, as well as the buoyancy effects, are absent, hence it is expected that

$$C_0(1,\ u_m,\ p) = 1, \qquad u_d(1,\ u_m,\ p) = 0. \tag{3.3}$$

In the literature, there are various formulations of the momentum equation for the mixture. In [13], this equation is obtained by summing up the momentum equations of the multi-fluid model (2.2) for each phase:

$$\frac{\partial}{\partial t}\big(\alpha_g\rho_g u_g + \alpha_l\rho_l u_l\big) + \frac{\partial}{\partial x}\big(\alpha_g\rho_g u_g^2 + \alpha_l\rho_l u_l^2 + p\big) = Q_l + Q_g. \tag{3.4}$$

Here, Q_i are the algebraic source terms for each phase.

In the model for transient well flow of [26], which is used, in particular, in the commercial reservoir-flow simulator ECLIPSE (Schlumberger), the momentum equation for the mixture is written in a non-conservative form in terms of the volume-averaged velocity of the mixture. In the applications, a quasi-steady-state version of the momentum equation is used, in which, in the momentum equation for the mixture, either the time derivative of the velocity or the full acceleration terms are neglected. In the literature, the last modification is called the no-pressure-wave model, since it does not take into account the propagation of rapid pressure waves in the space and describes the propagation of disturbances with the mass transfer velocity [27]. The total pressure difference is expressed as the sum of the terms responsible for gravity, friction and, in the quasi-stationary case, acceleration [26]. In this study, we use the following form of the momentum equation for the mixture:

$$\varepsilon_1\rho_m\frac{\partial u_m}{\partial t} + \varepsilon_2\rho_m u_m\frac{\partial u_m}{\partial x} + \frac{\partial p}{\partial x} = Q_m, \tag{3.5}$$

where $\rho_m = \alpha_l\rho_l + \alpha_g\rho_g$ is the density of the mixture and Q_m are the algebraic source terms. The coefficients ε_1 and ε_2, which take values 0 and 1, are introduced to consider the different variants of the model.

System of Eqs. (2.1), (3.4), and (3.5), supplemented with identity (2.3), the relations for the phase velocities, and dependencies (2.4) are closed systems of three differential equations for three unknown functions $\alpha_g(x,\ t)$, $u_m(x,\ t)$, and $p(x,\ t)$.

The characteristic properties of systems (2.1), (3.4), and (2.1), (3.5) in the general case are determined, among others, by the form of relation (3.1). The implementation of the above condition (3.3) ensures the reduction of both systems to the equations of one-dimensional motion of compressible gas with $\alpha_g = 1$.

The questions of the limits of applicability of each modification of the drift-flux model and its relation with the original multi-fluid model are considered in [28]. It is shown that the drift-flux model for disperse flow in the classical formulation (2.1), (3.4) follows from the balance laws, when imposing the condition that the characteristic scale of the problem is much greater than the relaxation length of the phase velocities. To derive the equations of the drift-flux model in the formulation (2.1), (3.5), it is necessary to assume additionally that either the volume fraction of the dispersed phase is small or velocity slip of the phases can be neglected, or the acceleration of the mixture as a whole can be neglected.

3.1. The classical formulation. In [10–12], the assumptions of the constancy of C_0 and u_d, and the incompressibility of the liquid were used. In [10, 11], the authors used additional assumptions of negligibly small terms corresponding to the gaseous phase in comparison with the similar terms for the liquid phase in (3.4) and the validity of the perfect gas law $\rho_g(p)$, which allowed them to obtain the explicit expressions for the characteristic velocities:

$$\lambda_1 = u_g, \qquad \lambda_{2,3} = u_l \pm \sqrt{\frac{p}{\alpha_g \rho_l (1 - \alpha_g C_0)}}. \tag{3.6}$$

From (3.6), we obtain the hyperbolicity condition for the system under study:

$$\alpha_g C_0 < 1. \tag{3.7}$$

The authors of [12] used the above mentioned assumptions and reduced the system to an equivalent form in Lagrangian coordinates, with one of the equations taking the form of a transport equation. Under additional assumptions $u_d = 0$ and $Q_g + Q_l = 0$, they also obtained that system (2.1), (3.4) is similar to the system of equations of one-dimensional gas dynamics in Lagrangian coordinates, and calculated its characteristic velocities.

In [13], for the class of closure relations (3.1) restricted by a differential condition on the slip velocity and including the case of constant C_0 and u_d, the authors obtained the first terms of the expansion of the characteristic velocities of the system in power series in small parameters which have the meaning of the fluid compressibility, the Mach numbers based on the slip velocity and the sound velocity, and the ratio of the gas and liquid densities.

We note that the algebraic relation between the velocities with $C_0 = \text{const} \neq 1$ and $u_d = \text{const} \neq 1$, considered in [10–12], does not ensure the reduction of system (2.1), (3.4) to the equations of single-phase flow with $\alpha_g = 1$. Hyperbolicity condition (3.7) is violated also for sufficiently high gas volume fractions. In [29], from physical considerations it is also shown that the violation of condition (3.7) results in the incorrect behavior of the system: from the expression for the surface density of the liquid flow $\alpha_l u_l = (1 - \alpha_g C_0)u_m - \alpha_g u_d$, it follows that if $\alpha_g C_0 \geq 1$ the liquid flow is negative for any, arbitrarily large, positive volume flow of the mixture.

We will now consider the closure relations $C_0 = C_0(\alpha_g)$ and $u_d = u_d(\alpha_g)$, which satisfy (3.3). The characteristic equation of the system in this case cannot be written explicitly since it is too cumbersome. The coefficients of the characteristic equation for the dimensionless characteristic velocities $\tilde{\lambda} = \lambda / c_l$ are the functions of five dimensionless parameters, namely, the volume fraction α_g, the density ratio of the phases η, the ratio of the sound velocities K, the Mach number based on mixture velocity $\mathrm{M}_m = u_m / c_l$, and the Mach number based on the drift velocity $\mathrm{M}_d = u_d(0)/c_l$. Over the range of parameters, typical of the wellbore flows, $(\mathrm{M}_m, \mathrm{M}_d) \ll 1$. Following the procedure of [13], we can obtain the zero terms of the expansions of the characteristic velocities in powers of small M_m and M_d:

$$\tilde{\lambda}_1 = O\big(\mathrm{M}_m, \mathrm{M}_d\big),$$

$$\tilde{\lambda}_{2,3} = \pm\left[\left(\frac{\alpha_g}{K^2 \eta} + 1 - \alpha_g\right)(1 - (1 - \eta)\alpha_g C_0(\alpha_g))\right]^{-1/2} + O(\mathrm{M}_m, \mathrm{M}_d).$$

The condition of non-negativity of the expression under the square-root sign is necessary for global hyperbolicity of system (2.1), (3.4). Numerical experiments show that for sufficiently small values of M_m and M_d this condition is necessary and sufficient. Regardless of the particular type of relations $C_0 = C_0(\alpha_g)$ and $u_d = u_d(\alpha_g)$ satisfying (3.3), for all sufficiently small M_m and M_d the roots of the characteristic equation of system (2.1), (3.4) are real-valued and different when and only when

$$(1 - \eta)\alpha_g C_0(\alpha_g) < 1. \tag{3.8}$$

Figures 5 and 6 illustrate this statement. In Fig. 5, we have plotted the closing relations used. For the profile parameter, we used the dependence:

$$C_0(\alpha_g) = C_0^0 \begin{cases} 1, & \alpha_g < b, \\ \left(1 + (C_0^0 - 1)\left(\dfrac{\alpha_g - b}{1 - b}\right)^2\right)^{-1}, & b \le \alpha_g \le 1, \end{cases} \tag{3.9}$$

where $C_0^0 \ge 1$, $0 \le b \le 1$.

The dependence for the drift velocity reads:

$$\frac{u_d(\alpha_g)}{c_l} = \mathrm{M}_d \begin{cases} 1, & \alpha_g < b_1, \\ 1 - \left(\dfrac{\alpha_g - b_1}{1 - b_1}\right)^2, & b_1 \le \alpha_g \le 1. \end{cases} \tag{3.10}$$

Figure 6 shows the region of hyperbolicity of system (2.1), (3.4) for the closure relations $C_0(\alpha_g) = C_0^0$ and (3.9), (3.10). The values of η and K are used as for the oil-gas flow in table, $\mathrm{M}_d = 1.5 \times 10^{-4}$, $C_0^0 = 1.2$, $b = 0.6$, and $b_1 = 0.9$. According to the calculation results shown in Fig. 6a, the limits of the region of hyperbolicity of the system at low Mach numbers coincide with those obtained from (3.8). From Fig. 6b, it follows that the use of closure relation (3.9) ensures the hyperbolicity of the system for all values of α_g and M_m lesser then the threshold value of 0.38, which is never achieved in the applications.

Thus, the correction of the non-physical behavior of closure relation (3.2) at high gas volume fractions also has a positive effect on the characteristic properties of the system, expanding the hyperbolicity region of (2.1), (3.4) in the range of parameters important for practical applications. In this case, condition (3.8) is less strict than condition (3.8), imposed by physical reasoning to ensure the hyperbolicity of the simplified system.

3.2. The formulation of ECLIPSE. The model without pressure waves ($\varepsilon_1 = \varepsilon_2 = 0$) was considered earlier in [27]. The type of the system was classified as mixed hyperbolically-parabolic, with the characteristic velocity equal to infinity of multiplicity two. The work also contains an expression for a single finite characteristic velocity. The properties of the generalized drift-flux model for three-phase flow with account for mass transfer between the phases were analyzed in [30].

We introduce the following notation:

$$\frac{1}{\rho_m c_m^2} = \sum_{i=g,l} \frac{\alpha_i}{\rho_i c_i^2}, \qquad \omega_i = \alpha_i \frac{\rho_m c_m^2}{\rho_i c_i^2}, \qquad \Psi_g = \frac{\partial}{\partial \alpha_g} \alpha_g u_g,$$

where c_i and c_m are sound velocities in the phases and in the mixture, and $\omega_g + \omega_l = 1$.

The characteristic equation for system (2.1), (3.5) takes the form:

$$(\lambda - \Psi_g)(\varepsilon_2 u_m - \varepsilon_1 \lambda)(\omega_g u_g + \omega_l u_l - \lambda) - c_m^2) = 0. \tag{3.11}$$

The root of the equation $\lambda_1 = \Psi_g$ does not depend on the form of the momentum equation for the mixture. The characteristic velocity λ_1 is not related to the compressibility of the phases and thus describes

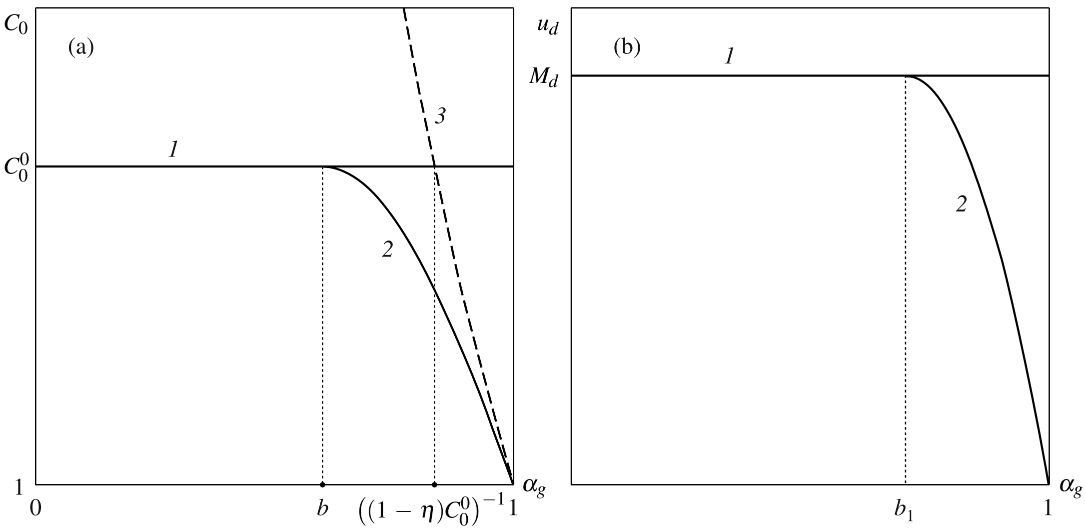

Fig. 5. (a) The dependence of the profile parameter C_0 on the gas volume fraction α_g: (*1*) $C_0 = C_0^0$; (*2*) dependence (3.9); (*3*) condition (3.8). (b) Dependence of the dimensionless drift velocity u_d on α_g: (*1*) $u_d = \mathrm{M}_d$; (*2*) dependence (3.10).

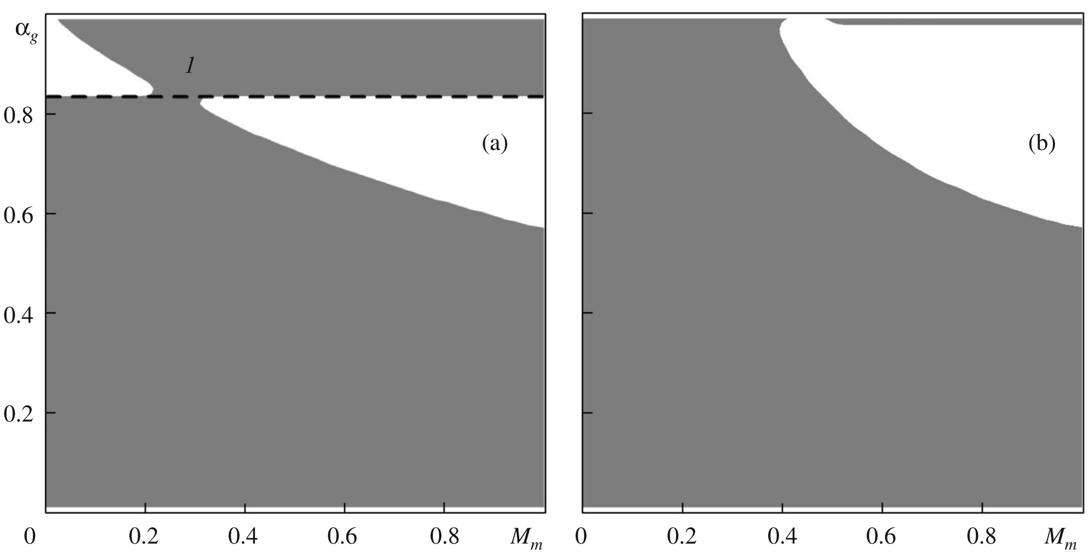

Fig. 6. Regions of hyperbolicity for the closure relations $C_0 = C_0^0$, (3.10) (a) and (3.9), (3.10) (b): (*1*) limits of the hyperbolicity region obtained from condition (3.8).

the propagation of slow waves of the volume fraction with the mass transfer velocity. In the general case, λ_1 is the function with alternating signs, and the corresponding characteristic can be incoming on both boundaries of the region in which the solution is constructed.

Solving Eq. (3.11) for the case of the non-stationary momentum equation for the mixture ($\varepsilon_1 = \varepsilon_2 = 1$), we obtain the two remaining roots:

$$\lambda_{2,3} = \frac{1}{2}\left(u_m + \omega_g u_g + \omega_l u_l \pm \sqrt{D}\right), \quad D = \left(u_m - \omega_g u_g - \omega_l u_l\right)^2 + 4c_m^2 > 0.$$

Thus, in the general case of the transient momentum equation for the mixture all eigenvalues are real-valued and distinct, and system (2.1), (3.5) is strictly hyperbolic. When there is no velocity slip $u_m = u_g = u_l$, the expression for the characteristic velocity becomes $\lambda_{2,3} = u_m \pm c_m$, similar to the characteristics of the equations of one-dimensional motion of a compressible gas, which justifies the name of the speed of sound

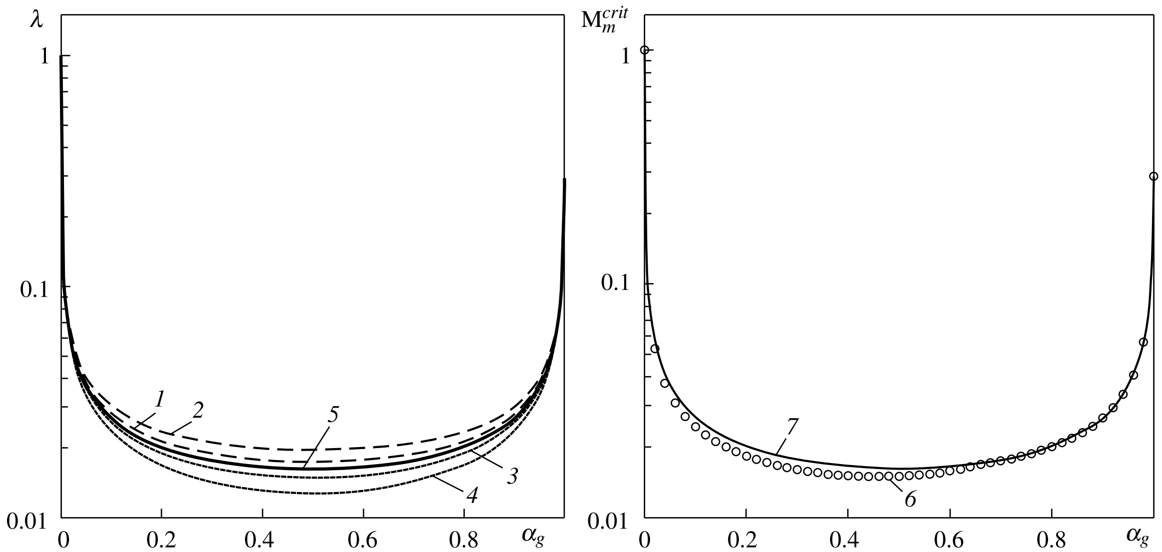

Fig. 7. (a) Dependence of the dimensionless characteristic velocities $\lambda_{2,3}$ and the sonic velocity c_m in the mixture on the gas volume fraction α_g: (*1*, *2*) λ_2 for M_m = 0.001, 0.003; (*3*, *4*) $|\lambda|_3$ for M_m = 0.001, 0.003; (*5*) c_m. (b) Dependence of the critical Mach number on the gas volume fraction: (*6*) numerical solution of (3.12); (*7*) approximating dependence $u_m = c_m$.

in the mixture. The dependence of c_m on α_g for $K < 1$ and $\eta \ll 1$ has a minimum:

$$\frac{c_m^{\min}}{c_l} = 2K\big(\eta^{1/2} + O(\eta^{3/2})\big), \qquad \alpha_g^{\min} = \frac{1}{2}\big(1 + (1 - K^2)\eta\big) + O(\eta^2).$$

Therefore, over the range of parameters $u_m \sim u_i \ll c_m^{\min}$, typical of wellbore flows, the characteristics $\lambda_{2,3}$ have different signs and describe the perturbations propagating up- and downstream.

In the case of the stationary momentum equation for the mixture ($\varepsilon_1 = 0$, $\varepsilon_2 = 1$), Eq. (3.11) becomes quadratic. The second root is

$$\lambda_2 = \omega_g u_g + \omega_l u_l - \frac{c_m^2}{u_m}.$$

For the systems with the transient and steady-state momentum equation for the mixture, the conditions of the change of sign of one of the characteristic velocities, which determine an analog of a sonic surface, coincide:

$$u_m\big(\omega_g u_g + \omega_l u_l\big) = c_m^2. \tag{3.12}$$

Figure 7 is plotted for the closure relations from [31], obtained by the calibration with respect to a large set of experimental data used in the simulation of wellbore flows. For this type of closure relations, the phase velocities and the characteristics are the functions of six dimensionless parameters, namely, α_g, η, K, M_m, $M_d = u_d^0/c_l$ (where $u_d^0 = u_d^0(p) = u_d(0,\ u_m,\ p)$), and the dimensionless diameter of the pipe $\hat{d}$ (defined in [31]). For η and K, we used the values as for the oil-gas flow in the table, $M_d = 1.5 \times 10^{-4}$ and $\hat{d} = 47$, which are typical of wellbore flows. Figure 7a shows the dependences of the sonic characteristic velocities $\tilde{\lambda}_{2,3} = \lambda_{2,3}/c_l$ on α_g. In Fig. 7b, we have plotted the dependence of the critical Mach number M_m^{crit} on the gas volume fraction, obtained from the numerical solution of Eq. (3.12), and the approximating dependence $M_m^{crit} = c_m/c_l$.

Eliminating the spatial derivative of the mixture velocity using Eq. (3.5), from the sum of continuity equations (2.1) in the case of a stationary momentum equation for the mixture we obtain:

$$\frac{\partial p}{\partial t} + \lambda_2 \frac{\partial p}{\partial x} = C,$$

where C is the sum of the algebraic terms.

Accordingly, one of Eqs. (2.1), (3.5) can be reduced to the form of the transport equation for pressure. Thus, in this case the pressure is a Riemann invariant, which is transported along the characteristic $dx/dt = \lambda_2$. Since for the characteristic velocities of the flow the signs of λ_2 and the mixture velocity do not coincide, for the system with a stationary momentum equation for the mixture the formulation of the boundary condition for the pressure at the outlet of the pipe is uniquely determined.

Finally, in the case of the non-inertial momentum equation for the mixture ($\varepsilon_1 = \varepsilon_2 = 0$) the root λ_1 is the only one, which agrees with the results of [27].

Summary. Two modifications of the classical two-fluid model describing a transient two-phase flow in a long pipeline are considered: (i) a model with account for the liquid level gradient and (ii) a model, which takes the interfacial pressure forces into account. In the case of incompressible media, with account of the gradient of the liquid phase level a dimensionless criterion is found analytically, which ensures the hyperbolicity of the original system of equations. For flows of the oil-gas type and for different pipe inclination angles, the regions of hyperbolicity of the system in the space of governing parameters of the flow are plotted. It is shown that this modification of the classical two-fluid model is hyperbolic only in a narrow range of values of the governing parameters. A dimensionless characteristic analysis of the model with account for the interfacial pressure forces is performed. In the case of compressible media, the closure relation for the pressure at the interface is proposed. To find the number of real roots of the characteristic equation, a numerical realization of the Jenkins–Traub algorithm is employed. Over a wide range of values of the problem parameters, typical of different flows (including oil and gas flows), the regions of real-valued roots of the characteristic equation are constructed. For a given type of flow, the optimal closure relation and the maximum slip velocity are given, below which the characteristic equation has only real-valued roots, and the system remains hyperbolic.

A characteristic analysis is performed for the system of equations of the drift-flux model in the classical formulation, with the closure relations using the profile parameter and the drift velocity, dependent of the gas volume fraction and ensuring the correct reduction of the system to the equations of one-dimensional motion of a compressible gas with the gas volume fraction tending to unity. The terms of the zero order are calculated in the expansions of the characteristic velocities in powers of the Mach numbers based on the mixture velocity and the drift velocity. The necessary condition for global hyperbolicity of the system is found, which is also the sufficient condition at low Mach numbers, as follows from numerical experiments. A characteristic analysis of the drift-flux model in the formulation of ECLIPSE is performed for arbitrary closure relations. Analytical expressions for the characteristic velocities are obtained, and it is shown that the system is strictly hyperbolic. One of the characteristics does not depend on the compressibility of the phases and is associated with slow perturbations of the gas volume fraction, propagating with mass transfer velocity, whereas the two other characteristics correspond to rapid pressure perturbations. An analog of the sonic surface for the system is determined, and the conclusions are drawn about the sign-definiteness of the eigenvalues and the correct formulation of the initial-boundary-value problem.

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REFERENCES

1. O. Bratland, Pipe Flow 2: Multi-Phase Flow Assurance. 2009. drbratland.com.
2. J.D. Ramshaw and J.A. Trapp, Characteristics, Stability, and Short-Wavelength Two-Phase Flows in Systems of Equations," Nuclear Sc. Eng. **66** (1), 93–102 (1978).
3. K.H. Bendiksen, D. Maines, R. Moe, and S. Nuland, "The Dynamic Two-Fluid Model OLGA; Theory and Applications," SPE Technology **6** (2), (1991).
4. D. Lhuillier, C.H. Chang, and T.G. Theofanous, "On the Quest for a Hyperbolic Effective-Field Model of Disperse Flows," J. Fluid Mech. **731**, 184–194 (2013).
5. Kh.A. Rakhmatulin, "Fundamentals of Gas Dynamics of Interpenetrating Continua," Prikl. Matem. Mekh. **20** (2), 184–195 (1956).
6. R.I. Nigmatulin, *Dynamics of Multiphase Media. V.1.* (Hemisphere, New York, 1989).

7. H. Stuhmiller, "The Influence of Interfacial Pressure Forces on the Character of Two-Phase Flow Model Equations," Int. J. Multiphase Flow **3** (6), 551–560 (1977).
8. M. Ndjinga, "Influence of Interfacial Pressure on the Hyperbolicity of the Two-Fluid Model," Comptes Rendus Mathematique **344** (6), 407–412 (2007).
9. A. Kumbaro and M. Ndjinga, "Influence of Interfacial Pressure Term on the Hyperbolicity of a General Multifluid Model," J. Comput. Multiphase Flows **3** (3), 177–196 (2011).
10. B. Théron, *Ecoulements Diphasiques Instationnaires en Conduite Horizontale*, Thése INP Toulouse, France, 1989.
11. S. Benzoni-Gavage, *Analyse Numérique des Modéles Hydrodynamiques d'Écoulements Diphasiques Instationnaires dans les Réseaux de Production Pétroliére*, Thése ENS Lyon, France, 1991.
12. S.L. Gavrilyuk and J. Fabre, "Lagrangian Coordinates for a Drift-Flux Model of a Gas-Liquid Mixture," Int. J. Multiphase Flow **22** (3) 453–460 (1996).
13. S. Evje and T. Flåtten, "On the Wave Structure of Two-Phase Models," Siam J. Appl. Math. **67** (2), 487–511 (2007).
14. B.L. Rozhdestvenskii and N.N. Yanenko, *Systems of Quasi-Linear Equations and Their Applications to Gas Dynamics* [in Russian] (Nauka, Moscow, 1978).
15. A.G. Kulikovskii, N.V. Pogorelov, and A.Yu. Semenov, *Mathematical Problems of Numerical Solution of Hyperbolic Systems of Equations* [in Russian] (Fizmatlit, Moscow, 2001).
16. D. Barnea and Y. Taitel, "Kelvin–Helmholtz Stability Criteria for Stratified Flow: Viscous Versus Non-Viscous (Inviscid) Approaches," Int. J. Multiphase Flow **19**, 639–649 (1993).
17. D. Barnea and Y. Taitel, "Interfacial and Structural Stability of Separated Flow," Int. J. Multiphase Flow **20** 387–414 (1994).
18. M.R. Baer and J.W. Nunziato, "Two-Phase Mixture Theory for the Deflagration to Detonation Transition (DDT) in Reactive Granular Materials," Int. J. Multiphase Flow **12** (6), 861–889 (1986).
19. E. Romenski and E.F. Toro, "Compressible Two-Phase Flows: Two-Pressure Models and Numerical Methods," Comput. Fluid Dyn. J. **13**, 403–416 (2004).
20. T. Gallouët, J.M. Hérard, and N. Seguin, "Numerical Simulation of Two-Phase Flows Using Two-Fluid Two-Pressure Approach," Mathem. Model. Meth. Appl. Sci. **14** (5), 663–700 (2004).
21. M. Bonizzi, P. Andreussi, and S. Banerjee, "Flow Regime Independent, High Resolution Multi-Field Modelling of Near-Horizontal Gas-Liquid Flows in Pipelines," Int. J. Multiphase Flow **35** (1), 34–46 (2009).
22. U. Kadri, R.F. Mudde, R.V.A. Oliemans, et al., "Prediction of the Transition from Stratified to Slug Flow or Roll-Waves in Gas-Liquid Horizontal Pipes," Int. J. Multiphase Flow **35** (11) 1001–1010 (2009).
23. A.A. Osiptsov, "A Self-Similar Solution to the Problem of Lava Dome Growth on an Arbitrary Conical Surface," Fluid Dynamics **39** (1), 53–68 (2004).
24. M.A. Jenkins and J.F. Traub, "A Three-Stage Algorithm for Real Polynomials Using Quadratic Iteration," SIAM J. Numer. Analysis **7** (4), 545–566 (1970).
25. N. Zuber and J.A. Findlay, 'Average Volumetric Concentration in Two-Phase Flow Systems," Trans. ASME. Ser. C. J. Heat Transfer **87** (4), 453–468 (1965).
26. A.R. Hasan and C.S. Kabir, *Fluid Flow and Heat Transfer in Wellbores* (Soc. Petrol. Eng., Richardson, Texas, 2002).
27. J.M. Masella, Q.H. Tran, D. Ferre, and C. Pauchon, "Transient Simulation of Two-Phase Flows in Pipes," Int. J. Multiphase Flow **24** (5), 739–755 (1998).
28. A.A. Osiptsov, K.F. Sin'kov, and P.E. Spesivtsev, "Justification of the Drift-Flux Model for Two-Phase Flow in a Circular Pipe," Fluid Dynamics **49** (5), 614–626 (2014).
29. P.A. Varadarajan and P.S. Hammond, "Numerical Scheme for Accurately Capturing Gas Migration Described by 1D Multiphase Drift Flux Model," Int. J. Multiphase Flow **73** 57–70 (2015).
30. P. Spesivtsev, K. Sinkov, and A.A. Osiptsov, "The Hyperbolic Nature of a System of Equations Describing Three-Phase Flows in Wellbores," in: *14th European Conference on the Mathematics of Oil Recovery, Catania, Italy, 2014*.
31. H. Shi, J.A. Holmes, L.J. Durlofsky, et al. "Drift-Flux Modeling of Two-Phase Flow in Wellbores," SPE Journal **10** (1), 24–33 (2005).