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Vertical Fluxes Induced by Weakly Nonlinear Internal Waves on a Shelf

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Abstract—Free internal waves in a vertically nonuniform stratified flow are considered with account of the coefficients of horizontal turbulent transport. The dispersion equation and the wave decay rate are derived in the linear approximation. The vertical component of the Stokes drift velocity and the wave fluxes of heat and salt are determined in the second order of the wave amplitude. It is shown that it is the vertical component of the Stokes drift velocity which is nonzero, when turbulent viscosity and diffusion are taken into account, that mainly contribute to the wave transport. The wave flux of salt is greater than the turbulent flux. Taking the flow into account leads to a decrease in the vertical wave fluxes but the wave flux of salt remains greater than the turbulent flux.

Keywords: internal waves, turbulence, Stokes drift.

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Internal waves are presented everywhere in the ocean, since the generating energy sources are always working; these are atmospheric pressure disturbances, wind stresses on the sea surface, the interaction of flows and tides with bottom relief inhomogeneities [1], flow instability [2], etc. The internal waves exist due to density stratification which is presented everywhere below the upper mixed layer. The vertical transfer in the sea medium is usually associated with small-scale turbulence which is alternate in nature and is accounted for by introducing "effective" turbulent transfer coefficients. The turbulent viscosity effect on the internal waves was studied in [1, 3], where it was shown that the internal waves are decaying.

Nonlinear effects that accur during internal wave propagation manifest themselves in the generation of mean (on wave scale) flows [4, 5]. The vertical velocity of the induced flow has opposite signs on the leading and backward fronts of the packets; for this reason, the integral vertical transfer is absent from the internal wave region. In the inviscid case the vertical component of the Stokes drift velocity for a fixed internal-wave mode is zero. In the presence of turbulent viscosity and diffusion the vertical component of the Stokes drift velocity and the wave fluxes of heat $\langle u_3 T\rangle$ and salt $\langle u_3 S\rangle$ are nonzero [6, 7] (here, u_3 is the vertical velocity component and T and S are the wave disturbances of the temperature and the salinity, while the angular brackets mean the averaging over the wave period).

Below we consider the mean flow effect on these wave fluxes, the vertical component of the Stokes drift velocity, and the total wave transfer. Under actual sea conditions the coefficients of the vertical turbulent transfer are 3 to 5 orders smaller than those of the horizontal transfer; because of this, here we study only the effect of the horizontal transfer coefficients on the internal waves with allowance for mean flows.

1. FORMULATION OF THE PROBLEM

We will consider free internal waves in a vertically nonuniform flow with account for horizontal turbulent viscosity and diffusion in the Boussinesq approximation. The vertical distribution of the internal wave amplitudes, the dispersion equation, and the wave decay rate are found in the linear approximation.

The vertical component of the Stokes drift velocity and the wave fluxes of heat and salt are determined in the second order of the wave amplitude.

The dimensionless variables are introduced following formulas given in [6, 7] (here, the primes denote the dimensional physical quantities)

$$x_i' = Hx_i, \quad (i = 1\text{–}3), \qquad t' = t/\omega_*, \qquad U_0' = H\omega_* U_0, \qquad V_0' = H\omega_* V_0,$$

$$u_i' = H\omega_* u_i, \qquad \rho' = \rho_0'(0)\rho, \qquad \rho_0(x_3) = \rho_0'(0)\rho_0(x_3),$$

$$P' = \rho_0'(0)H^2\omega_*^2 P, \qquad K' = K\mu, \qquad M' = M\mu, \qquad \zeta' = H\zeta,$$

where ω_* is the characteristic wave frequency, x_1 and x_2 are two horizontal coordinates, x_3 is the vertical coordinate, the x_3 axis is upward directed, u_i ($i = 1$–3) are two horizontal components and the vertical component of the wave flow velocity, ρ and P are the wave disturbances of the density and the pressure, ρ_0 is the undisturbed mean water density, H is the sea depth, K and M are the coefficients of horizontal turbulent viscosity and diffusion, $\mu = K'$, and ζ is the vertical displacement of the free surface of the sea. Two components of the mean flow velocity U_0 and V_0 are assumed to depend on x_3. The coefficients of horizontal turbulent transfer are constant. In the Boussinesq approximation the system of hydrodynamics equations for the wave disturbances takes the form:

$$\frac{\partial u_1}{\partial t} + u_i\frac{\partial u_1}{\partial x_i} + U_0\frac{\partial u_1}{\partial x_1} + V_0\frac{\partial u_1}{\partial x_2} + u_3\frac{dU_0}{dx_3} = -\frac{\partial P}{\partial x_1} + \varepsilon^2 K\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right), \tag{1.1}$$

$$\frac{\partial u_2}{\partial t} + u_i\frac{\partial u_2}{\partial x_i} + U_0\frac{\partial u_2}{\partial x_1} + V_0\frac{\partial u_2}{\partial x_2} + u_3\frac{dV_0}{dx_3} = -\frac{\partial P}{\partial x_2} + \varepsilon^2 K\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2}\right), \tag{1.2}$$

$$\frac{\partial u_3}{\partial t} + u_i\frac{\partial u_3}{\partial x_i} + U_0\frac{\partial u_3}{\partial x_1} + V_0\frac{\partial u_3}{\partial x_2} = -\frac{\partial P}{\partial x_3} + \varepsilon^2 K\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) - \rho, \tag{1.3}$$

$$\frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} + U_0\frac{\partial \rho}{\partial x_1} + V_0\frac{\partial \rho}{\partial x_2} = \varepsilon^2 M\left(\frac{\partial^2 \rho}{\partial x_1^2} + \frac{\partial^2 \rho}{\partial x_2^2}\right) - u_3\frac{\partial \rho_0}{\partial x_3}, \tag{1.4}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{1.5}$$

Here, $\varepsilon^2 = \mu/\omega_* H^2$ is a small parameter proportional to the horizontal turbulent viscosity μ.

The boundary conditions on the free surface ($x_3 = 0$) are as follows:

$$P - g_1\zeta = 0, \qquad g_1 = \frac{g}{\omega_*^2 H}, \tag{1.6}$$

$$K\frac{\partial u_3}{\partial x_1} = 0, \qquad K\frac{\partial u_3}{\partial x_2} = 0, \tag{1.7}$$

$$\frac{\partial \zeta}{\partial t} + U_0\frac{\partial \zeta}{\partial x_1} + V_0\frac{\partial \zeta}{\partial x_2} = u_3. \tag{1.8}$$

The dynamic conditions (1.6) and (1.7) determine the absence of the normal and tangential stresses, while Eq. (1.8) is the kinematic condition on the free surface [8]. In the case of the data used below the parameter g_1 is very large: $g_1 \sim 10^3$; hence it follows that in Eq. (1.6) $\zeta \approx 0$. The condition $\zeta = 0$, or the “solid top” condition for the internal waves, filters out the surface waves [9]. In view of Eq. (1.8), $u_3(0) = 0$. Precisely this condition on the surface will be used below.

The boundary conditions at the bottom are the solid top condition and the absence of tangential stresses (smooth slip condition [3])

$$u_3(-1) = 0, \tag{1.9}$$

$$K\frac{\partial u_3}{\partial x_1} = 0, \qquad K\frac{\partial u_3}{\partial x_2} = 0, \qquad x_3 = -1. \tag{1.10}$$

In view of the fact that the vertical transfer coefficients are neglected, the tangential stresses at the bottom are zero.

2. LINEAR APPROXIMATION

We will seek the solutions of the linear approximation in the form:

$$\begin{gathered} u_3^0 = u_{30}(x_3)Ae^{i\theta} + \text{c.c.}, \quad u_1^0 = u_{10}(x_3)Ae^{i\theta} + \text{c.c.}, \quad u_2^0 = u_{20}(x_3)Ae^{i\theta} + \text{c.c.}, \\ P_1 = P_{10}(x_3)Ae^{i\theta} + \text{c.c.}, \qquad \rho_1 = \rho_{10}(x_3)Ae^{i\theta} + \text{c.c.}, \end{gathered} \tag{2.1}$$

where c.c. denotes compex conjugate terms, A is the amplitude coefficient, θ is the wave phase, $\partial\theta/\partial x_1 = k$, and $\partial\theta/\partial t = -\omega$ (k is the wavenumber and ω is the frequency). It is assumed that the wave travels along the x_1 axis.

Substituting Eqs. (2.1) into system (1.1)–(1.5) yields the relation between the amplitude functions u_{10}, u_{20}, ρ_{10}, and P_{10} and u_{30}

$$u_{10} = \frac{i}{k}\frac{\partial u_{30}}{\partial x_3}, \qquad \Omega = \omega - k\cdot U_0,$$

$$P_{10} = (i\Omega - \varepsilon^2 Kk^2)\frac{1}{k^2}\frac{\partial u_{30}}{\partial x_3} + \frac{i}{k}\frac{\partial U_0}{\partial x_3}u_{30},$$

$$(i\Omega - \varepsilon^2 Mk^2)\rho_{10} = u_{30}\frac{d\rho_0}{dx_3},$$

$$(i\Omega - \varepsilon^2 Kk^2)u_{20} = u_{30}\frac{dV_0}{dx_3}.$$

The function u_{30} satisfies the equation

$$(i\Omega - \varepsilon^2 Mk^2)\left\{i\Omega u_{30} - \frac{d}{dx_3}\left[(i\Omega - \varepsilon^2 Kk^2)\frac{1}{k^2}\frac{\partial u_{30}}{\partial x_3} + \frac{i}{k}\frac{\partial U_0}{\partial x_3}u_{30}\right] - \varepsilon^2 Kk^2 u_{30}\right\} + N^2 u_{30} = 0. \tag{2.2}$$

The boundary conditions for u_{30} are as follows:

$$x_3 = 0: \quad u_{30} = 0, \tag{2.3}$$

$$x_3 = -1: \quad u_{30} = 0. \tag{2.4}$$

The boundary conditions (1.9) and (1.10) are fulfilled automatically.

Equation (2.2) includes the small parameter ε; following the method described in [8, 10] we will represent the solution u_{30} and the frequency ω as asymptotic series in ε

$$u_{30}(x_3,\ \varepsilon) = w_0(x_3) + \varepsilon w_1(x_3) + \varepsilon^2 w_2(x_3) + \ldots, \tag{2.5}$$

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots \tag{2.6}$$

After the substitution of expansions (2.5) and (2.6) into Eq. (2.2) we arrive at the following boundary value problem for w_0 in the zeroth approximation in ε

$$\frac{d^2 w_0}{dx_3^2} + k^2\frac{N^2 - \Omega_0^2}{\Omega_0^2}w_0 + \frac{k}{\Omega_0}\frac{d^2 U_0}{dx_3^2}w_0 = 0, \tag{2.7}$$

where $-d\rho_0/dx_3$ is the square of the Brunt–Wäisälä frequency and $\Omega_0 = \omega_0 - kU_0$ is the wave frequency with the Doppler shift.

The boundary conditions for w_0 are as follows:

$$w_0(0) = 0, \qquad w_0(-1) = 0. \tag{2.8}$$

In the no-flow case ($U_0 = 0$) the boundary value problem (2.7), (2.8) has a numerable set of eigenfunctions (modes). Any value of the wavenumber k is associated with a certain value of the frequency $\omega_0 < \max(N)$ corresponding to the given mode. At $U_0 \neq 0$ a discrete spectrum of real frequencies can no exist [11]. This is due to the singularity in Eq. (2.7) at $\Omega_0 = 0$ (only hydrodynamically stable solutions are considered). In the presence of this singularity there exists a critical layer in which the phase velocity of the wave is equal to the flow velocity. However, under the actual sea conditions on the scales of observable internal waves the phase wave velocity can often be two and three times greater than the flow velocity. For this reason, the dispersion curves change only slightly, when the flow is taken into account [12]. This is illustrated by the calculations of the dispersion curves of the two first modes presented below.

The next term in expansion (2.5) is determined from the equation

$$\frac{d^2 w_1}{dx_3^2} + k^2\frac{N^2 - \Omega_0^2}{\Omega_0^2}w_1 + \frac{k}{\Omega_0}\frac{d^2 U_0}{dx_3^2}w_1 = \frac{\omega_1}{\Omega_0}\left(2k^2 w_0 - 2\frac{d^2 w_0}{dx_3^2} - \frac{k w_0}{\Omega_0}\frac{d^2 U_0}{dx_3^2}\right) = f_1(x_3). \tag{2.9}$$

The boundary conditions for the function w_1 are as follows:

$$w_1(0) = 0, \qquad w_1(-1) = 0. \tag{2.10}$$

The condition of the solvability of the boundary value problem (2.9), (2.10) is as follows:

$$\int\limits_{-1}^{0} f_1\, w_0\, dx_3 = 0.$$

For $\omega_1 \neq 0$ this condition is not generally fulfilled and the boundary value problem (2.9), (2.10) has no solutions.

The next approximation w_2 in the parameter ε satisfies the equation

$$\begin{gathered}\Omega_0^2\left[-k^2 + \frac{d^2}{dx_3^2}\right]w_2 + N^2k^2w_2 + k\Omega_0\frac{d^2 U_0}{dx_3^2}w_2 \\ = (-\omega^2 - iMk^2)\left(\frac{d^2 w_0}{dx_3^2}\Omega_0 - k^2\Omega_0 w_0 + k\frac{d^2 U_0}{dx_3^2}w_0\right) \\ + \Omega_0\left[\omega_2\left(k^2 w_0 - \frac{d^2 w_0}{dx_3^2}\right) - iKk^2\frac{d^2 w_0}{dx_3^2} + iKk^4 w_0\right] = \Phi.\end{gathered} \tag{2.11}$$

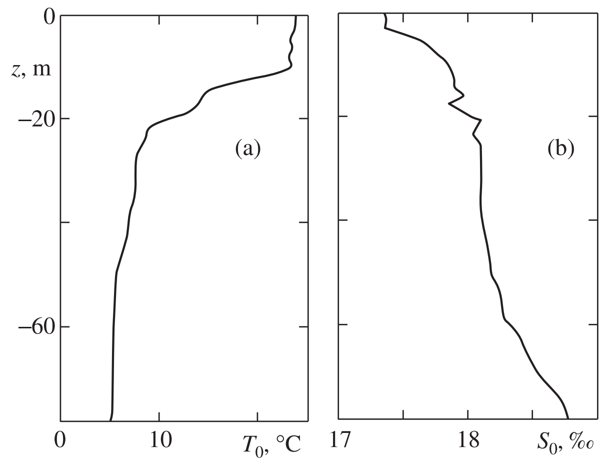

Fig. 1. Vertical temperature (a) and salinity (b) profiles.

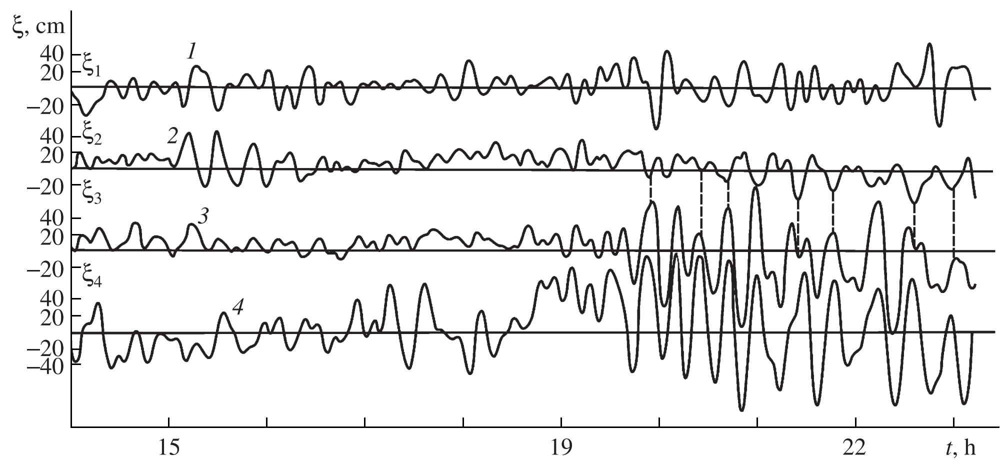

Fig. 2. Time dependence of the vertical displacements of the temperature contours; (*1*–*4*) relate to ξ_1 to ξ_4.

The boundary conditions for the function w_2 are as follows:

$$w_2(0) = 0, \qquad w_2(-1) = 0. \tag{2.12}$$

The condition of the solvability of the boundary value problem (2.11), (2.12) is as follows:

$$\int_{-1}^{0} \Phi w_0 \, dx_3 = 0.$$

Hence we derive the expression for ω_2

$$\omega_2 = -i\int_{-1}^{0}\left(\frac{M_1 k^4}{\Omega_0^2}N^2 w_0 - Kk^2\frac{d^2 w_0}{dx_3^2} + k^4 K w_0\right)\frac{w_0}{\Omega_0}dx_3\left[\int_{-1}^{0}\left(\frac{N^2}{\Omega_0^3}2k^2 + \frac{k}{\Omega_0^2}\frac{d^2 U_0}{dx_3^2}\right)w_0^2 dx_3\right]^{-1}. \tag{2.13}$$

The diffusion equation for the wave disturbances of salinity takes the form:

$$\frac{\partial s}{\partial t} + (u_1 + U_0)\frac{\partial s}{\partial x_1} + (u_2 + V_0)\frac{\partial s}{\partial x_2} + u_3\frac{\partial s}{\partial x_3} + u_3\frac{\partial S_0}{\partial x_3}$$
$$= \varepsilon^2\frac{\partial}{\partial x_1}\left(M\frac{\partial s}{\partial x_1}\right) + \varepsilon^2\frac{\partial}{\partial x_2}\left(M\frac{\partial s}{\partial x_2}\right),$$

where $S_0(x_3)$ is the mean salinity profile.

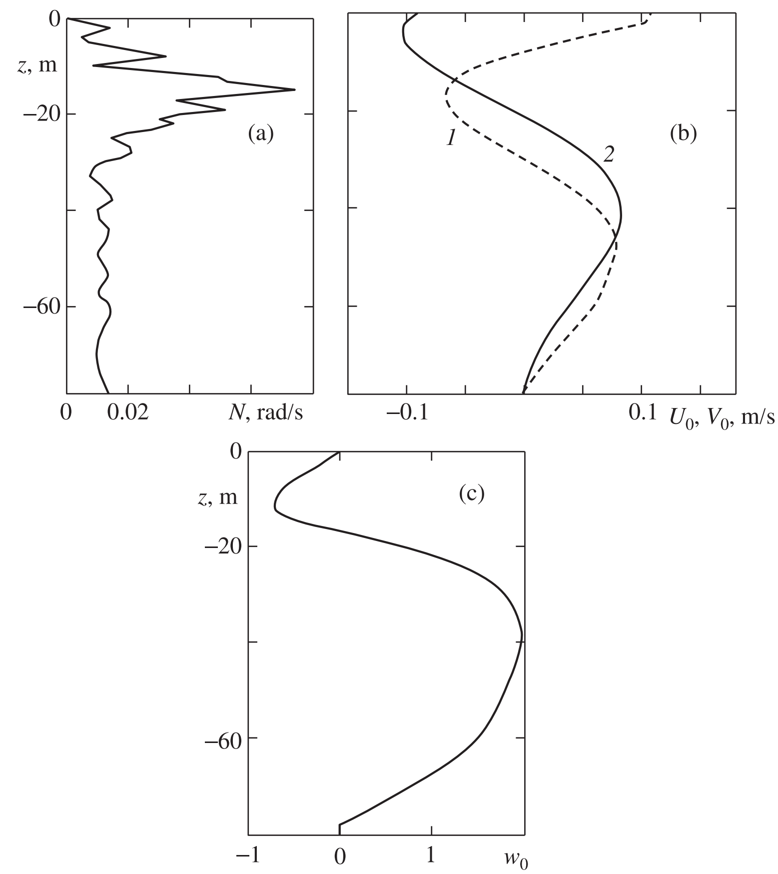

Fig. 3. Vertical profile of the Brunt–Väisälä frequency (a); vertical profiles (b) of the flow velocity components U_0 (*1*) and V_0 (*2*); and the eigenfunction (c) of the 15-minute internal waves.

We will seek the solutions of the linear approximation in the form:

$$s_1 = s_{10}(x_3)Ae^{i\theta} + \text{c.c.} \tag{2.14}$$

The function s_{10} is expressed in terms of u_{30}

$$s_{10} = -\frac{iu_{30}}{\Omega + i\varepsilon^2 k^2 M}\frac{dS_0}{dx_3}. \tag{2.15}$$

3. NONLINEAR EFFECTS

The velocity of the Stokes drift of the fluid particles is determined by the formula [13]

$$u_s = \left\langle \int_0^t (u\,d\tau \cdot \nabla)\,u \right\rangle,$$

where u is the Eulerian wave velocity field and the angular brackets mean the averaging over the wave period. On the second order of wave amplitude the vertical component of the Stokes velosity takes the form:

$$u_{3s} = \frac{2\delta\omega}{\omega_0^2}\frac{d}{dx_3}(w_0^2)A_1A_1^*. \tag{3.1}$$

Here, $\delta\omega = \varepsilon^2\omega_2/i$ is the wave decay rate due to turbulence and $A_1 = A\exp(\delta\omega \cdot t)$. When the coefficients of turbulent viscosity and diffusion are zero, $K = M = 0$, the decay rate and the vertical component of the Stokes drift velocity are also zero.

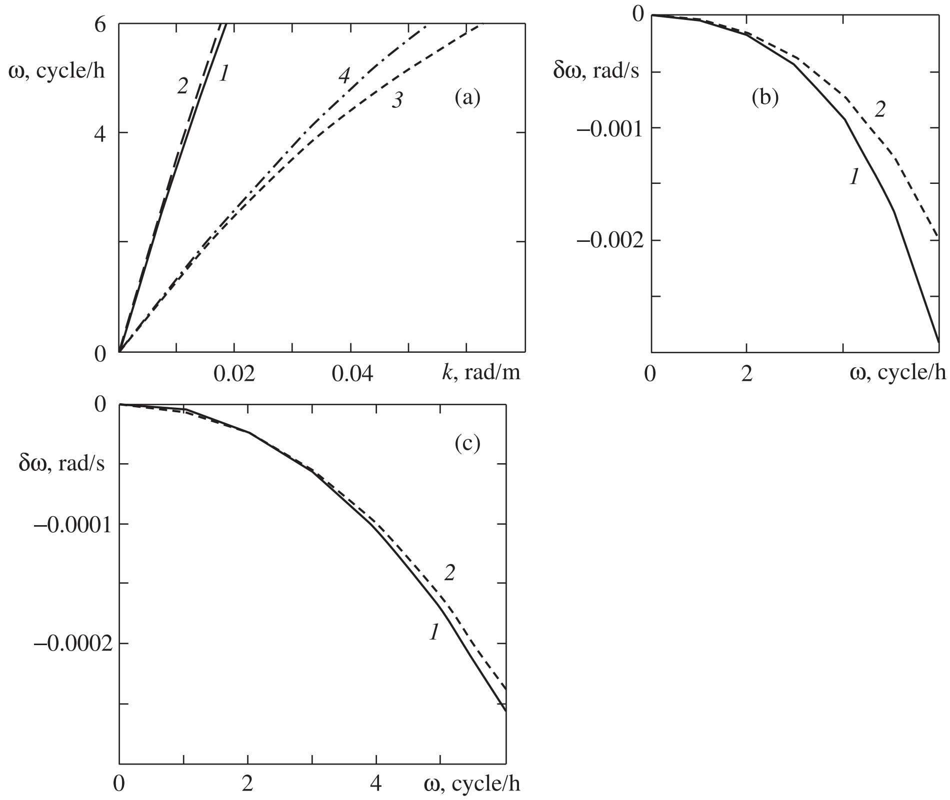

Fig. 4. Dispersion curves (a); mode I without (*1*) and with (*2*) flow and mode II without (*3*) and with (*4*) flow; frequency dependence of the wave decay rate (b) for mode II without (*1*) and with (*2*) flow; and same (c) for mode I.

We will determine the wave flux of salt $\langle u_3 S\rangle$ accurate to the terms of the order of ε^2 taking Eqs. (2.14) and (2.15) into account

$$\frac{\langle u_3 S\rangle}{|A_1^2|} = -2w_0^2\left(\delta\omega + \varepsilon^2 k^2 M\right)\Omega_0^{-2}\frac{dS_0}{dx_3}. \tag{3.2}$$

The wave flux of heat $\langle u_3 T\rangle$ is determined by the same Eq. (3.2) with the vertical salinity gradient dS_0/dx_3 replaced by the vertical temperature gradient dT_0/dx_3.

4. RESULTS OF THE CALCULATIONS

The vertical wave fluxes of heat and salt are calculated for the internal waves observed in the full-scale experiment south-west to the town of Eupatoria during the third stage of the 44th sailing of the research ship *Mikhail Lomonosov*. Figure 1 presents the vertical profiles of the temperature and the salinity in the measurement area. In Fig. 2 we have plotted four realizations of the temperature contour elevations calculated according to the data of the GRAD instruments, that is to say, the gradient-distributed temperature transducers [14]. The first instrument was placed in a 5 to 15 m layer, while the second to fourth ones were on the 15 to 25, 25 to 35, and 35 to 60 horizons. It can be readily seen that intense oscillations with the period of about 15 min in the 15 to 25 m layer are in antiphase with those in the 25 to 60 m layer, which indicates the second mode oscillations. The maximum elevation amplitude was 0.5 m. The vertical profiles of the Brunt–Väisälä frequency and two flow velocity components are presented in Fig. 3a and 3b. The boundary value problem (2.7), (2.8) for the internal waves is numerically solved using the implicit third-order Adams scheme. The wavenumber is determined by means of the shooting method using the fulfillment of the boundary conditions (2.8) as a criterion. The eigenfunction of the 15 minute internal second-mode waves are presented in Fig. 3c.

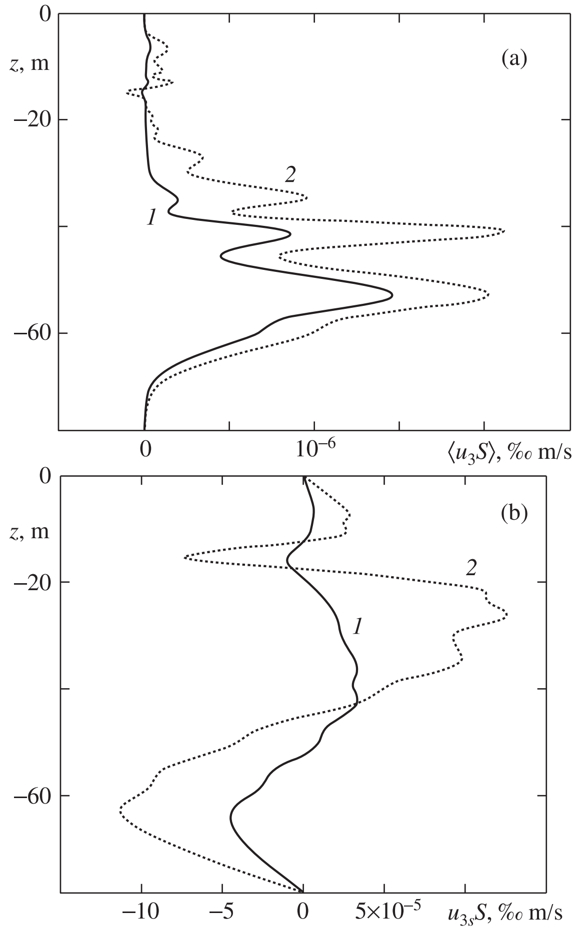

Fig. 5. Vertical fluxes of salt in the presence (*1*) and absence (*2*) of flow.

The wavenumber is 0.032 rad/m. The normalizing multiplier A_1 is determined from the known value of the maximum amplitude of vertical displacements ($\max\zeta_3 = 0.5$ m) using the relation $d\zeta_3/dt = u_3$

$$\zeta_3 = \frac{iw_0}{\Omega_0}A_1\exp(ikx - i\omega_0 t) + \text{c.c.}$$

Hence it follows that

$$A_1 = \frac{\Omega_0 \max\zeta_3}{2\max|w_0|}.$$

Thus, the vertical displacement amplitude is proportional to w_0. The function w_0 extrema correspond to the maximum vertical displacements according to the experimental data (Figs. 2 and 3c), since in the experiment the second mode was observable. The wavelength of the 15 minute internal second-mode waves is 196 m, while the typical value of the horizontal turbulent transfer coefficient is 1 m^2/s. The dispersion curves are plotted in Fig. 4a in the presence and the absence of a flow. Curves *1* and *2* correspond to the first mode with and without flow and curves *3* and *4* to the second mode with and without flow. In the presence of a flow the dispersion curves lie somewhat higher than in the case of its absence.

The boundary value problem of determining the function w_2 is numerically solved using the implicit third-order Adams scheme at $K = 2M$; the unique solution orthogonal to w_0 and the wave decay rate $\delta\omega$ are determined. The decay rate of the 15 minute internal second-mode waves $\delta\omega = -7.24\times10^{-4}$ rad/s. The dependence of the wave decay rate on the wave frequency is presented in Fig. 4c and 4b, for the first and second mode waves, in the presence and the absence of the flow. Curves *1* and *2* correspond to the cases of the absence and the presence of the flow. At a fixed wave frequency the absolute value of the second-mode

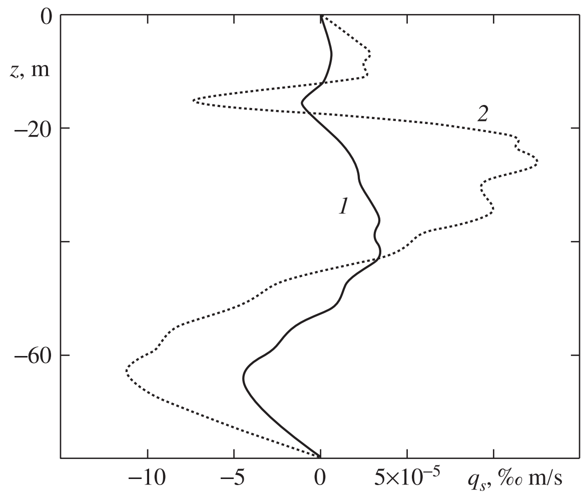

Fig. 6. Profiles of the overall wave fluxes of salt in the presence (*1*) and absence (*2*) of flow.

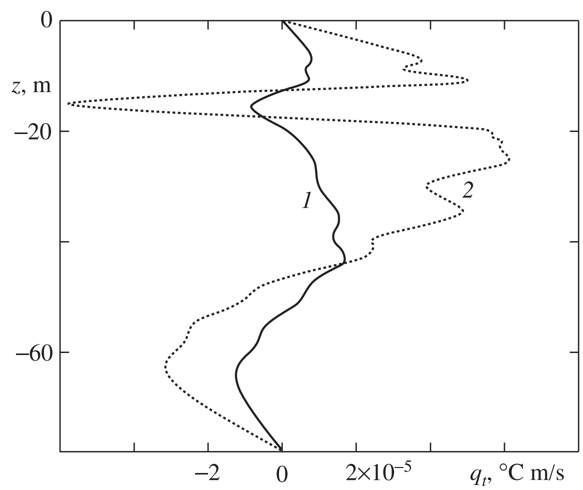

Fig. 7. Profiles of the overall wave fluxes of heat in the presence (*1*) and absence (*2*) of flow.

wave decay rate is greater than that of the first mode. Taking the flow into account leads to a decrease in the absolute value of the decay rate.

The wave fluxes of salt are calculated for the vertical salinity profile presented in Fig. 1b. The vertical wave fluxes of salt $\langle u_3 S\rangle$ (3.2) for the 15 minute internal second-mode waves are presented in Fig. 5a. In the presence of the flow the wave flux of salt is smaller than in its absence. The wave fluxes of salt $S_0 u_{3s}$ at the expense of the vertical component of the Stokes drift velocity are shown in Fig. 5. Generally, the absolute value of the wave flux is smaller in the presence of the flow than in its absence.

The overall wave flux $q_s = \langle u_3 S\rangle + S_0 u_{3s}$ is chiefly determined by the second term, that is, the vertical component of the Stokes drift velocity, and is smaller in the presence of the flow than in its absence (Fig. 6). The vertical wave flux of heat $q_T = \langle u_3 T\rangle + T_0 u_{3s}$ is also mainly determined by the vertical component of the Stokes drift velocity and is smaller in the presence of the flow than in its absence (Fig. 7). The vertical salinity gradient dS_0/dz is not greater than 0.1 ‰/m, while the typical value of the vertical turbulent diffusion coefficient in the stratified sea medium on the shelf area $M_3 \sim 10^{-4}$ m^2/s [15]; for this reason the turbulent flux of salt $s_f = -M_3 dS_0/dz$ is of the order of 10^{-5}‰ m/s and is smaller than the wave flux of salt (Fig. 6).

Summary. The vertical component of the Stokes drift velocity is nonzero, when turbulent viscosity and diffusion are taken into account. The vertical wave fluxes of salt $\langle u_3 S \rangle$ and heat $\langle u_3 T \rangle$ are also nonzero. The overall wave fluxes of heat and salt are mainly determined by the vertical component of the Stokes drift velocity and, when the flow is taken into account, are smaller in absolute value than the corresponding fluxes in the no-flow case. The wave fluxes of salt are greater than the turbulent ones.

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