ISSN 0012-2661, Differential Equations, 2022, Vol. 58, No. 5, pp. 631–648. © Pleiades Publishing, Ltd., 2022.
Russian Text © The Author(s), 2022, published in Differentsial'nye Uravneniya, 2022, Vol. 58, No. 5, pp. 638–655.

PARTIAL DIFFERENTIAL EQUATIONS

Rayleigh Waves for Elliptic Systems in Domains with Periodic Boundaries

S. A. Nazarov[1*]

[1] *Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, St. Petersburg, 199178 Russia*

e-mail: *srgnazarov@yahoo.co.uk*

Received October 26, 2021; revised January 21, 2022; accepted April 21, 2022

Abstract— We consider formally self-adjoint elliptic systems of partial differential equations generating formally positive operators and having the polynomial property. Sufficient conditions that ensure the existence of Rayleigh surface waves in the Neumann problem on a half-space with periodic boundary are found. We give examples of specific problems of mathematical physics in which our sufficient conditions are simplified or turn into a criterion and study problems in the theory of plates and piezoelectrics that are not covered by general results. The latter problem requires a major modification of the approach.

DOI: 10.1134/S0012266122050044

1. STATEMENT OF THE PROBLEM

Let Ω be a domain in the half-space $\mathbb{R}^d_+ = \{x = (y, z) : y \in \mathbb{R}^{d-1},\ z \in \mathbb{R}_+ = (0, +\infty)\}$, $d \geq 2$, invariant with respect to integer shifts along the axes $y_j = x_j$, $j = 1, \dots, d-1$, and infinite in the direction of the axis $z = x_d$; i.e.,

$$\Omega = \left\{x \in \mathbb{R}^d : (y + \alpha, z) \in \Omega\right\} \quad \text{for any multiindices} \quad \alpha = (\alpha_1, \dots, \alpha_{d-1}) \in \mathbb{Z}^{d-1}, \tag{1}$$

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and $\{x \in \mathbb{R}^d : z > R\} \subset \Omega$, $R > 0$.

The boundary Γ is assumed to be $(d-1)$-dimensional and Lipschitz. Let us introduce the semi-infinite prism $\Pi = \{x \in \Omega : |y_j| < 1/2,\ j = 1, \dots, d-1\}$ with section $\omega = (-1/2, 1/2)^{d-1}$ (a unit cube in $\mathbb{R}^{d-1}$) and curvilinear, not necessarily connected base $\varpi = \{x \in \Gamma : y \in \omega\}$. Consider the boundary value problem

$$\mathcal{L}(\nabla)u(x) = \lambda u(x), \quad x \in \Pi, \tag{2}$$

$$\mathcal{B}(x, \nabla)u(x) = 0, \quad x \in \varpi, \tag{3}$$

with the quasiperiodicity conditions

$$\begin{gathered} \partial_j^m u_k(x)|_{x_j = 1/2} = e^{i\theta_j} \partial_j^m u_k(x)|_{x_j = -1/2}, \quad x|_{x_j = \pm 1/2} \in \partial\Pi, \\ j = 1, \dots, d-1, \quad k = 1, \dots, K, \quad m = 0, \dots, 2t_k - 1. \end{gathered} \tag{4}$$

Let us explain our notation. First of all, $\nabla = (\partial_1, \dots, \partial_{d-1})^{\mathrm{T}}$, $\partial_j = \partial/\partial x_j$, and the matrix differential operator

$$\mathcal{L}(\nabla) = \overline{\mathcal{M}(-\nabla)}^{\mathrm{T}} \mathcal{A} \mathcal{M}(\nabla) \tag{5}$$

is formally self-adjoint. Here $^{\mathrm{T}}$ stands for transposition, the bar means complex conjugation, and $\mathcal{M}(\nabla)$ is a matrix of homogeneous differential operators with constant (complex) coefficients whose entries have the orders $\operatorname{ord} \mathcal{M}_{nk} = t_k \in \mathbb{N} = \{1, 2, 3, \dots\}$, where $k = 1, \dots, K$, $n = 1, \dots, N$, $K, N \in \mathbb{N}$, and $N \geq K$. The matrix $\mathcal{M}(\nabla)$ itself is algebraically complete [1, Ch. 3]; i.e., there exists a number $\rho_{\mathcal{M}} \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that for any row $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_K)$ whose components $\mathbf{p}_k$ are homogeneous polynomials of degrees $\operatorname{ord} \mathbf{p}_k = t_k + \rho$, $\rho \geq \rho_{\mathcal{M}}$, there exists a polynomial row $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ such that

$$\mathbf{p}(\xi) = \mathbf{q}(\xi)\mathcal{M}(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^d. \tag{6}$$

This condition ensures the Korn inequality [1, Sec. 3.7.4]

$$\begin{gathered}\left\|u;H^{\mathbf{t}}(\Xi)\right\|^2 \leq \mathcal{C}_{\Xi,\mathcal{M}}\Big(\left\|\mathcal{M}(\nabla)u;L^2(\Xi)\right\|^2+\left\|u;L^2(\Xi)\right\|^2\Big)\\ \text{for}\quad u=(u_1,\ldots,u_k)^{\mathrm{T}}\in H^{\mathbf{t}}(\Xi)=H^{t_1}(\Xi)\times\cdots\times H^{t_K}(\Xi),\end{gathered}\tag{7}$$

where the Lebesgue space $L^2(\Xi)$ with the natural inner product $(\,\cdot\,,\cdot\,)_\Xi$ and the Sobolev space $H^l(\Xi)$ of order $l\in\mathbb{N}$ equipped with the standard norm occur and $\Xi\in\mathbb{R}^d$ is an arbitrary domain with $(d-1)$-dimensional Lipschitz boundary $\partial\Xi$ and compact closure $\overline{\Xi}=\Xi\cup\partial\Xi$. Of course, the Korn constant $C_{\Xi,\mathcal{M}}$ in (7) is independent of the vector function $u\in H^{\mathbf{t}}(\Xi)$. One has the Green's formula

$$(\mathcal{L}u,v)_\Xi=a(u,v;\Xi)+(\mathcal{N}u,\mathcal{D}v)_{\partial\Xi},\tag{8}$$

where $v=(v_1,\ldots,v_K)^{\mathrm{T}}\in H^{\mathbf{t}}(\Xi)$, $u\in H^{t_\bullet+\mathbf{t}}(\Xi)$, and $t_\bullet=\max\{t_1,\ldots,t_K\}$. Moreover, $\mathcal{D}(x,\nabla)$ is a Dirichlet system on $\partial\Xi$, $\operatorname{ord}\mathcal{D}_{pk}\leq t_k-1$ (see [2, Ch. 2, Sec. 2]), and the $T\times K$ matrix $\mathcal{N}(x,\nabla)$ is the operator of Neumann boundary conditions (3) with $T=t_1+\ldots+t_K$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are Hermitian positive definite numerical matrices of size $N\times N$ and $K\times K$, respectively; then the sesquilinear form

$$a(u,v;\Xi)=\big(\mathcal{A}\mathcal{M}(\nabla)u,\mathcal{M}(\nabla)v\big)_\Xi\tag{9}$$

is Hermitian and positive, and the variational statement of problem (2)–(4) with the spectral parameter λ and the Floquet parameter $\theta=(\theta_1,\ldots,\theta_{d-1})\in[-\pi,\pi)^{d-1}$,

$$a(u,v;\Pi)=\lambda(\mathcal{B}u,v)_\Pi\quad\text{for all}\quad v\in H^{\mathbf{t}}_\theta(\Pi),\tag{10}$$

is realized on the space $H^{\mathbf{t}}_\theta(\Pi)$ of vector functions $u\in H^{\mathbf{t}}(\Pi)$ satisfying the stable $(p=0,\ldots,t_k-1)$ quasiperiodicity conditions (4) (we use the terminology in the monograph [2, Ch. 2]),

$$\begin{aligned}H^{\mathbf{t}}_\theta(\Pi)=\Big\{u\in H^{\mathbf{t}}(\Pi):\partial_j^m u_k(x)|_{x_j=1/2}=e^{i\theta_j}\partial_j^m u_k(x)|_{x_j=-1/2}\quad\text{for}\quad x|_{x_j=\pm1/2}\in\partial\Pi,\\ j=1,\ldots,d-1,\quad k=1,\ldots,K,\quad m=0,\ldots,t_k-1\Big\}.\end{aligned}\tag{11}$$

The Neumann boundary conditions (3) and the intrinsic $(p=t_k,\ldots,2t_k-1)$ quasiperiodicity conditions (3) for a smooth solution in $H^{t_\bullet+\mathbf{t}}_\theta(\Pi)$ can be reconstructed from the integral identity (10) using the Green's formula (8) (see, e.g., [2, Ch. 2; 3, Ch. 2]).

2. RAYLEIGH WAVES

Assume that problem (9) has a nontrivial solution $u\in H^{\mathbf{t}}(\Pi)$ for some $\lambda\in\mathbb{R}_+$ and $\theta\in[-\pi,\pi)^{d-1}$. In the usual manner (see, e.g., the monographs [4, 5]), we define the vector function

$$\Omega\ni x\mapsto\mathbf{u}(x)=e^{i(\theta_1[y_1]+\ldots+\theta_{d-1}[y_{d-1}])}u\big(y_1-[y_1],\ldots,y_{d-1}-[y_{d-1}],z\big),\tag{12}$$

where $[b]$ is the integer part of a number $b\in\mathbb{R}$. One can readily verify that, owing to the quasiperiodicity conditions incorporated in definition (11), the vector function (12) lies in the space $H^{\mathbf{t}}_{\mathrm{loc}}(\overline{\Pi})$. Moreover, it will be verified in Sec. 4 that $\mathbf{u}$ is a function infinitely differentiable in the domain Ω and exponentially decaying as $z\to+\infty$. Thus, under the additional assumption of smoothness of the surface Γ, this vector function satisfies the differential problem

$$\begin{aligned}\mathcal{L}(\nabla)\mathbf{u}(x)&=\lambda\mathbf{u}(x),\quad &&x\in\Omega,\\ \mathcal{B}(x,\nabla)\mathbf{u}(x)&=0,&&x\in\Gamma,\end{aligned}\tag{13}$$

and is localized near the boundary; i.e., it possesses all properties of classical Rayleigh waves.

Such peculiar "surface waves" in deformable media were first discovered by Lord Rayleigh [6] and then, in other versions, by Lamb [7] and Stoneley [8]. The associated physical phenomena have found various applications in seismology and seismic exploration, in methods of nondestructive testing of surface damage and joint strength, and in many other applied disciplines. Therefore, the number of published studies in this direction, carried out at various levels of rigor, is huge—let us mention several monographs [9–11] and papers [12–20] as well as the survey article [21].

Most of the results, especially for vector problems, for example, in the theory of elasticity, are obtained using analytical methods in the case of straight boundaries and computational methods in the case of oscillating ones. Further, as in [16–18, 20], variational methods of spectral analysis suitable for arbitrary periodic boundaries and a wide class of systems of differential equations are applied. At the same time, when proving the existence of a wave (12) exponentially decaying as $z \to +\infty$, the methods used do not provide any accurate information about its structure; i.e., the study of the spectral characteristics of the Rayleigh waves found is outside the scope of the present paper.

In the next two sections, we study problem (2)–(4) for $\lambda = 0$. For this problem, we establish Theorems 1 and 2, allowing one to draw conclusions about the continuous spectrum of problem (10). In Sec. 5 (Theorem 5), sufficient conditions are proved for the discrete spectrum to be nonempty in the case of $\theta \neq 0$ (for $\theta = 0$ it is certainly empty). In Sec. 6, the result obtained is applied to scalar and vector problems on acoustic and elastic media, and in Sec. 7, two mechanical problems are considered that are not covered by Theorem 5 and require modification of the approach; moreover, for the piezoelectric problem considered (Sec. 7, 5°) the result and the method of its derivation essentially differ, for example, from the problem of elasticity theory (Sec. 6, 2°).

3. POLYNOMIAL PROPERTY AND THE SPECTRUM

On the Sobolev space $H^{\mathbf{t}}_\theta(\Pi)$, we introduce the equivalent norm

$$|||u|||_\theta = \big(a(u,u;\Pi) + (\mathcal{B}u,u)_\Pi\big)^{1/2}. \tag{14}$$

The desired estimate

$$\big\|u; H^{\mathbf{t}}_\theta(\Pi)\big\| \le c_\theta |||u|||_\theta \tag{15}$$

is ensured by an application of the Korn inequality (7) in the truncated prism $\Pi(R) = \{x \in \Pi : z < R\}$ and the cubes $Q_{R+m} = (R+m, R+m+1) \times (-1/2, 1/2)$, $m \in \mathbb{N}_0$. Summing these inequalities, we arrive at the relation

$$\big\|u; H^{\mathbf{t}}_\theta(\Pi)\big\|^2 \le \max\{\mathcal{C}_{\Pi(R),\mathcal{M}}, \mathcal{C}_{Q_0,\mathcal{M}}\}\Big(\big\|\mathcal{M}(\nabla)u; L^2(\Pi)\big\|^2 + \big\|u; L^2(\Pi)\big\|^2\Big)$$

and finally take into account the positive definiteness of the matrices $\mathcal{A}$ and $\mathcal{B}$ in formulas (9) and (14). The inequality opposite to (15) is obvious.

We denote the Hilbert space (11) equipped with the inner product

$$\langle u, v\rangle_\theta = a(u,v;\Pi) + (\mathcal{B}u,v)_\Pi \tag{16}$$

by $\mathcal{H}_\theta$, and in this space, we introduce a positive, symmetric, continuous, and hence self-adjoint operator $\mathcal{T}_\theta$ using the formula

$$\langle \mathcal{T}_\theta u, v\rangle_\theta = (\mathcal{B}u,v)_\Pi \quad \text{for all} \quad u,v \in \mathcal{H}_\theta. \tag{17}$$

Its norm does not exceed one. Owing to definitions (14) and (16), (17), the variational problem (10) turns out to be equivalent to the abstract equation

$$\mathcal{T}_\theta u = \tau u \quad \text{in the space } \mathcal{H}_\theta$$

with the new spectral parameter

$$\tau = (1+\lambda)^{-1}. \tag{18}$$

Based on the spectrum $\Sigma_\theta \subset [0,1]$ of the operator $\mathcal{T}_\theta$, we determine the spectrum of problem (10) (or (2)–(4) in the case of a smooth boundary Γ),

$$\sigma_\theta = \big\{\lambda : (1+\lambda)^{-1} \in \Sigma_\theta\big\} \subset \overline{\mathbb{R}_+} = [0,+\infty). \tag{19}$$

Moreover, the relationship (18) between the spectral parameters conveys all the qualities (discreteness, continuity, etc.) of components of the spectrum $\mathcal{T}_\theta$ to the components of the spectrum σ_θ.

Let us study the spectrum (19) using information about problem (10) obtained on the basis of the theory of elliptic boundary value problems in domains with cylindrical outlets to infinity [22, Ch. 5; 23, Sec. 3] and the polynomial property [23–25] of the sesquilinear form (19),

$$a(u,u;\Xi)=0,\quad u\in H^{\mathbf{t}}(\Xi)\quad\Leftrightarrow\quad u\in\mathcal{P}|_{\Xi}. \tag{20}$$

Here $\mathcal{P}$ is a finite-dimensional subspace of vector polynomials. It follows from the algebraic completeness of the matrix $\mathcal{M}(\nabla)$ that we have the equality [23, Proposition 1.6]

$$\mathcal{P}=\big\{p=(p_1,\dots,p_K)^{\mathrm{T}}:\mathcal{M}(\nabla)p(x)=0,\quad x\in\mathbb{R}^d\big\}. \tag{21}$$

We point out that the linear space (21) can also contain polynomials $p=(p_1,\dots,p_K)^{\mathrm{T}}$ for which $\operatorname{ord}p_k\ge t_k$ (see Sec. 6, 2°).

The statement (21) provides much useful information about problem (10), including the following assertion [23, Theorems 1.9, 3.4, and 5.1], whose verification is explained in the next section.

Theorem 1. *For $\theta\in[-\pi,\pi)^{d-1}\setminus\{0\}$, the problem*

$$a(u,v;\Pi)=f(v)\quad\textit{for all}\quad v\in H^{\mathbf{t}}_\theta(\Pi) \tag{22}$$

with a continuous (anti)linear functional $f\in H^{\mathbf{t}}_\theta(\Pi)^*$ has a unique solution $u\in H^{\mathbf{t}}_\theta(\Pi)$, and one has the estimate

$$\big\|u;H^{\mathbf{t}}_\theta(\Pi)\big\|\le c_\theta\big\|f;H^{\mathbf{t}}_\theta(\Pi)^*\big\|,$$

where the factor c_θ is independent of f but increases unboundedly as $\theta\to0\in\mathbb{R}^{d-1}$.

Now let us derive some original information about the spectrum (19), which we represent as the disjoint union of the essential spectrum σ^e_θ and the discrete spectrum σ^d_θ.

Corollary 1.

1. *For $\theta\in[-\pi,\pi)^{d-1}\setminus\{0\}$, the continuous spectrum σ^c_θ coincides with the essential spectrum σ^e_θ and acquires a positive cutoff point $\lambda^\dagger_\theta$, so that the discrete spectrum σ^d_θ can only appear on the interval $(0,\lambda^\dagger_\theta)$.*
2. *For $\theta=0$, the spectrum $\sigma_0=\sigma^e_0=\sigma^c_0$ occupies the entire closed positive half-line $\overline{\mathbb{R}_+}$, and therefore, $\sigma^d_0=\varnothing$.*

Proof. Since the second term on the left-hand side in the integral identity

$$a(u,v;\Pi)-\lambda(\mathcal{B}u,v)_\Pi=f(v)\quad\text{for all}\quad v\in H^{\mathbf{t}}_\theta(\Pi) \tag{23}$$

generates a continuous perturbation vanishing as $\lambda\to0$ of the operator of problem (22), the unique solvability property in Theorem 1 is conveyed to problem (23) for $\lambda\in[0,\lambda^\#_\theta)$ and some $\lambda^\#_\theta>0$. According to the general results [26, 27] (see also [22, Ch. 1, Sec. 2 and Remark 3.1.5]), problem (10) does not have eigenvalues of infinite multiplicity; i.e., $\sigma^c_\theta=\sigma^e_\theta$, and moreover, σ^c_θ is the ray $[\lambda^\dagger_\theta,+\infty)$, i.e., a simply connected set, where, of course, $\lambda^\dagger_\theta\ge\lambda^\#_\theta>0$.

Regarding the second statement, which means that $\lambda^\dagger_0=0$, see Theorem 3.

4. EXPONENTIAL AND POLYNOMIAL SOLUTIONS; SOLVABILITY OF THE PROBLEM FOR $\theta=0$

Let the support of the functional f on the right-hand side in the integral identity (23) be compact, and let $\operatorname{supp}f\subset\overline{\Pi(R)}$. Then the solution $u\in H^{\mathbf{t}}_\theta(\Pi)$ is infinitely differentiable on the set $\overline{\Pi\setminus\Pi(R+\delta)}$ for each $\delta>0$. Indeed, smoothness inside the prism $\{x\in\Pi:z>R\}$ is ensured by the local estimates (see the monograph [2, Sec. 3] and the papers [28, 29]) for solutions of elliptic systems.[1] Choose an index $j\in\{1,\dots,d-1\}$, to be definite, $j=1$, and consider the vector function

$$\widehat{u}(x)=\begin{cases}u(x) & \text{for}\ \ x_1\in(-1/2,1/2)\\ e^{\pm i\theta_1}u(x_1\mp1,x') & \text{for}\ \ x_1\mp1\in(-1/2,1/2),\end{cases}$$

[1] The ellipticity of the operator $\mathcal{L}(\nabla)$ follows from the polynomial property (20); see [23, Ch. 5, Sec. 1].

where $x' = (x_2, \dots, x_{d-1}) \in (-1/2, 1/2)^{d-2}$ and $x_d > R$. By virtue of the quasiperiodicity conditions incorporated in the definition of the space (11), the vector function $\widehat{u}$ lies in the space $H^{\mathbf{t}}_{\mathrm{loc}}(\overline{\Pi_1(R)})$; i.e., the above-mentioned local estimates show that it is smooth inside the extended prism $\Pi_{\boxminus}(R) = (-3/2, 3/2) \times (-1/2, 1/2)^{d-2} \times (R, +\infty)$, and hence in $\Pi(R)$, up to the faces $\{\pm 1/2\} \times (-1/2, 1/2)^{d-2} \times (R, +\infty)$. Exhausting the rest of the indices j one by one, we arrive at the desired assertion about smoothness.

Let us give some information from the theory of elliptic boundary value problems in cylindrical domains [22, Ch. 5; 23, Sec. 3; 27]. With problem (2)–(4) in the infinite cylinder $\omega \times \mathbb{R}$ we associate the operator pencil

$$\mathbb{R} \ni \mu \mapsto \mathfrak{A}^{\lambda}_{\theta}(\mu) = \mathcal{L}(\nabla_y, \mu) - \lambda \mathcal{B} : H^{l+\mathbf{t}}_{\theta}(\omega) \to H^{l-\mathbf{t}}_{\theta}(\omega), \tag{24}$$

where $l \geq t_{\bullet}$ and $H^l_{\theta}(\omega)$ is the Sobolev space of functions satisfying the quasiperiodicity conditions on the opposite faces of the unit cube ω,

$$\begin{gathered} \partial^m_j U_k(y)|_{y_j=1/2} = e^{i\theta_j} \partial^m_j U_k(y)|_{y_j=-1/2}, \\ y|_{y_j=\pm 1/2} \in \partial\omega, \quad j = 1, \dots, d-1, \quad k = 1, \dots, K, \quad m = 0, \dots, l-1. \end{gathered} \tag{25}$$

By virtue of the ellipticity of the operator $\mathcal{L}(\nabla)$, the spectrum of the pencil (24) consists of normal eigenvalues (without finite accumulation points) located in the union of the strip $\{\mu \in \mathbb{C} : |\mathrm{Re}\,\mu| \leq \beta^0_{\lambda}\}$ and the double sector $\{\mu \in \mathbb{C} : |\mathrm{Re}\,\mu| \leq \beta^1_{\lambda} |\mathrm{Im}\,\mu|\}$, where β^0_{λ} and β^1_{λ} are positive numbers (see [26; 30, Ch. 1; 22, Ch. 1]). To each eigenvalue μ there corresponds a canonical system of Jordan chains

$$\left\{U^{p,q} : p = 1, \dots, \varkappa^g, \ q = 0, \dots, \varkappa^a_p - 1\right\}, \tag{26}$$

which consists of the eigenvectors ($q = 0$) and associated vectors ($q > 0$) satisfying the equations

$$\mathfrak{A}^{\lambda}_{\theta}(\mu) U^{p,q} = -\sum_{j=1}^{q} \frac{1}{j!} \frac{d^j \mathfrak{A}^{\lambda}_{\theta}}{d\mu^j}(\mu) U^{p,q-j}, \quad p = 1, \dots, \varkappa^g, \quad q = 0, \dots, \varkappa^a_p - 1. \tag{27}$$

Here $\varkappa^g$ is the geometric multiplicity and $\varkappa^a_1, \dots, \varkappa^a_{\varkappa^g}$ are the particular algebraic multiplicities of the eigenvalue μ; $\varkappa^a_1 + \ldots + \varkappa^a_{\varkappa^g}$ is the total multiplicity, and Eqs. (27) with $q = \varkappa^a_p$ have no solutions (the chains are nonextendable). Based on (26), we construct exponential solutions of problem (2)–(4) in the cylinder $\omega \times \mathbb{R}$,

$$\mathcal{U}^{p,q}(y, z) = e^{\mu z} \sum_{j=0}^{q} \frac{z^j}{j!} U^{p,q-j}(y), \quad p = 1, \dots, \varkappa^g, \quad q = 0, \dots, \varkappa^a_p - 1. \tag{28}$$

Let us study exponential solutions for various values of the Floquet parameter $\theta \in [-\pi, \pi)^{d-1}$. We start from the case of $\theta = 0$ and note that among the polynomials in the linear space (20), only the ones independent of the variables $y_1, \dots, y_{d-1}$ satisfy the "pure" periodicity conditions on the opposite faces of the prism Π into which conditions (25) turn for $\theta = 0$.

Lemma 1. *In the linear space*

$$\mathcal{P}^0 = \{p \in \mathcal{P} : \textit{the polynomial } p \textit{ depends only on the variable } z\}, \tag{29}$$

one can introduce the basis

$$p^{k,q}(z) = \mathbf{e}_k \frac{z^q}{q!}, \quad k = 1, \dots, K, \quad q = 0, \dots, t_k - 1, \tag{30}$$

where $\mathbf{e}_k = (\delta_{1,k}, \dots, \delta_{K,k})^{\mathrm{T}}$ *is the unit coordinate vector in the space* $\mathbb{R}^K$ *and* $\delta_{j,k}$ *is the Kronecker delta.*

Proof. Note the following important property: $\partial_z p \in \mathcal{P}$ for any vector polynomial $p \in \mathcal{P}$ in view of the invariance of the form (9) under shifts along the z-axis. Owing to the structure of the matrix differential operator $\mathcal{M}(\nabla)$ (see Sec. 1), the linear span of the polynomials (30) is contained in $\mathcal{P}^0$. Assume that in the linear space (29) there exists a polynomial

$$P(z) = \left(a_1\frac{z^{t_1}}{t_1!}, \ldots, a_K\frac{z^{t_K}}{t_K!}\right)^{\mathrm{T}}, \quad a = (a_1, \ldots, a_K)^{\mathrm{T}} \in \mathbb{C}^K, \quad |a| = 1. \tag{31}$$

Obviously, $\mathcal{M}(\mathbf{e}_d\partial_z) = \mathcal{M}(\mathbf{e}_d)Z(\partial_z)$ for $Z(\zeta) = \operatorname{diag}\{\zeta^{t_1}, \ldots, \zeta^{t_K}\}$; i.e.,

$$0 = \mathcal{M}(\nabla)P(z) = \mathcal{M}(\mathbf{e}_d)a.$$

Condition (6) applied to the polynomial (31) gives a vector polynomial $\mathbf{q}$ such that

$$\xi_d^{\rho_{\mathcal{M}}}(\overline{a_1}\xi_d^{t_1}, \ldots, \overline{a_K}\xi_d^{t_K}) = \mathbf{q}(\xi)\mathcal{M}(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^d. \tag{32}$$

Set $\xi_1 = \ldots = \xi_{d-1} = 0$ in relation (32) and multiply the result by $Z(\xi_d)^{-1}a$ on the right. As a result, we arrive at a contradiction, which proves the lemma,

$$\xi_d^{\rho_{\mathcal{M}}}|a|^2 = \mathbf{q}(\mathbf{e}_d\xi_d)\mathcal{M}(\mathbf{e}_d\xi_d)Z(\xi_d)^{-1}a = \mathbf{q}(\mathbf{e}_d\xi_d)\mathcal{M}(\mathbf{e}_d)a = 0.$$

The following assertion is a more specific form of Proposition 1 in [25] (see also the survey [23, Proposition 3.2]).

Theorem 2. *There exists a $\gamma_0 > 0$ such that the pencil (24) with parameters $\theta = 0 \in \mathbb{R}^{d-1}$ and $\lambda = 0$ has only one eigenvalue $\mu = 0$ with total algebraic multiplicity $2T = 2(t_1 + \ldots + t_K)$ in the strip $\{\mu \in \mathbb{C} : |\operatorname{Re}\mu| < \gamma_0\}$. Associated with this eigenvalue is a canonical system of Jordan chains*

$$\{\mathbf{e}_k, 0, \ldots, 0, U^{k,t_k}, \ldots, U^{k,2t_k-1}\}, \quad k = 1, \ldots, K, \tag{33}$$

where the first t_k elements are specified explicitly and the remaining ones are the solutions of Eqs. (27) for $\theta = 0$, $\lambda = 0$, and $q = t_k, \ldots, 2t_k - 1$.

The first t_k elements of the Jordan chain (33) provide, according to formula (28), the elements of the basis (30) in the linear space (29). Since the Neumann boundary condition operator (3) taken from Green's formula (8) can be represented as $\mathcal{N}(x,\nabla) = \mathcal{N}^{\#}(x,\nabla)\mathcal{M}(\nabla)$ with an appropriate matrix differential operator $\mathcal{N}^{\#}(x,\nabla)$, according to relation (21) one has the equality

$$\mathcal{N}(x,\nabla)p^{k,q}(x) = 0, \quad x \in \varpi, \quad k = 1, \ldots, K, \quad q = 0, \ldots, t_k - 1.$$

Thus, the polynomials (30) satisfy the entire problem (2)–(4) for $\theta = 0$, and this explains the next assertion, proved in [23, item 3, Sec. 5] and [25, Sec. 5].

Theorem 3. *If for some $\gamma \in (0, \gamma_0)$ the functional*

$$H_0^{\mathbf{t}}(\Pi) \ni v \mapsto f^\gamma(v) = f(e^{\gamma z}v) \tag{34}$$

turnes out to be continuous, then problem (22) with $\theta = 0$ has a solution $u \in H_0^{\mathbf{t}}(\Pi)$ if and only if the T orthogonality conditions

$$f(p) = 0 \quad \text{for each} \quad p \in \mathcal{P}^0$$

are satisfied. Moreover, this solution is unique and satisfies the inclusion $e^{\gamma z}u \in H_0^{\mathbf{t}}(\Pi)$ and the estimate

$$\|e^{\gamma z}u; H_0^{\mathbf{t}}(\Pi)\| \le c_\gamma \|f^\gamma; H_0^{\mathbf{t}}(\Pi)^*\|.$$

Here the number $\gamma_0 > 0$ is taken from Theorem 2, and the factor c_γ is independent of the functional (34) but grows indefinitely as $\gamma \to +0$.

Let us restate the result for the inhomogeneous Neumann problem

$$\begin{aligned} \mathcal{L}(\nabla)u(x) &= f(x), \quad x \in \Pi, \\ \mathcal{N}(x,\nabla)u(x) &= g(x), \quad x \in \varpi, \end{aligned} \tag{35}$$

in the case of periodic ($\theta = 0$) smooth right-hand sides f, g and end ϖ. For an exponentially decaying vector function f, problem (35) with the periodicity conditions (4) and $\theta = 0$ has a periodic smooth solution $u \in H_0^{\mathbf{t}}(\Pi)$ if and only if the orthogonality conditions

$$(f, p)_\Pi + (g, \mathcal{D}p)_\varpi = 0 \quad \text{for each} \quad p \in \mathcal{P}^0,$$

following from the Green's formula with trial vector functions $v \in \mathcal{P}^0$, hold. At the same time, as the last explanation in the statement of Theorem 3 shows, the passage to the limit $\gamma \to +0$ is impossible, and indeed the operator of problem (35), (4), $\theta = 0$, in the space $H_0^{\mathbf{t}}(\Pi)$ loses its Fredholm property, because the linear space $\mathcal{P}^0$ contains at least all constant vectors from which it is easy to construct a singular Weyl sequence [31, Ch. 9, Sec. 1] for the operator $\mathcal{T}_0$ at the point $\tau = 1$, and hence $\Sigma_0^e = [0, 1]$ and $\sigma_0^c = [0, +\infty)$ according to relation (18) between the spectral parameters τ and λ.

Theorem 1 equips the operator $\mathcal{T}_\theta$ with other properties if $\theta \neq 0$—it becomes an isomorphism; this agrees with the following assertion following from Proposition 3.2 (1) in [23], since problem (22) with the quasiperiodicity conditions remains self-adjoint, but none of the polynomials in the linear space (21) satisfies the above conditions for $\theta \neq 0$.

Theorem 4. *For $\theta \in [-\pi, \pi)^{d-1} \setminus \{0\}$, there exists a positive number $\gamma(\theta)$ such that the strip $\{\mu \in \mathbb{C} : |\operatorname{Re} \mu| < \gamma(\theta)\}$ is free from the spectrum of the pencil $\mu \mapsto \mathfrak{A}_\theta^0(\mu)$. Here $\gamma(\theta) \to +0$ as $\theta \to 0 \in \mathbb{R}^{d-1}$.*

Corollary 2. *For $\theta \in [-\pi, \pi)^{d-1} \setminus \{0\}$ and $|\gamma| < \gamma(\theta)$, where the number $\gamma(\theta) > 0$ is taken from Theorem 4, the solution $u \in H_\theta^{\mathbf{t}}(\Pi)$ of problem (22) with right-hand side f obeying condition (34) satisfies the inclusion $e^{\gamma z} u \in H_\theta^{\mathbf{t}}(\Pi)$ and the estimate*

$$\left\| e^{\gamma z} u; H_\theta^{\mathbf{t}}(\Pi) \right\| \le c_\gamma \left\| f^\gamma; H_\theta^{\mathbf{t}}(\Pi)^* \right\|.$$

The **proof** can be derived from Theorem 4 using the general results in the paper [27] (see also [22, Chs. 3 and 5] and [23, Sec. 3]); however, for small γ it can be obtained by elementary means. For the trial vector function in the integral identity (22) we take the product $v^\gamma = e^{\gamma z} v \in C_c^\infty(\overline{\Pi})^K \cap H_\theta^{\mathbf{t}}(\Pi)$. Since its support is compact, we can make the change of the unknown $u \mapsto u^\gamma = e^{\gamma z} u$. As a result, the integral identity acquires the form

$$\left(\mathcal{A}\mathcal{M}(\nabla - \gamma \mathbf{e}_d) u^\gamma, \mathcal{M}(\nabla + \gamma \mathbf{e}_d) v^\gamma \right)_\Pi = f(v^\gamma). \tag{36}$$

By closure, formula (36) holds for $v^\gamma \in H_\theta^{\mathbf{t}}(\Pi)$, and its left-hand side generates a small (of norm $O(|\gamma|)$) perturbation of the operator of problem (22); hence for sufficiently small $|\gamma|$ the modified problem remains uniquely solvable, as desired. It remains to note that the simplified approach does not allow bringing the weight index γ closer to the critical value $\gamma(\theta)$.

Since the operator of the embedding $\mathcal{H}_\theta \subset L^2(\Pi)^K$ is noncompact, the same property holds for the operator $\mathcal{T}_\theta$ given by formula (18). Therefore, by Theorem 9.2.1 in [31], the essential spectrum Σ_θ^e cannot consist of the single point $\tau = 0$. Thus, there exist points $\tau \in (0, 1)$ and $\lambda = \tau^{-1} - 1 > 0$ for which the operator $\mathcal{T}_\theta - \tau$ and the operator $\mathcal{O}_\theta : H_\theta^{\mathbf{t}}(\Pi) \to H_\theta^{\mathbf{t}}(\Pi)^*$ of problem (23) lose their Fredholm property. Let $\lambda = \lambda_\theta^\dagger$ be the smallest of such points; it is positive for $\theta \neq 0$ by Theorem 1. According to the theory of boundary value problems in cylindrical domains [22, Chs. 3 and 5; 27], the pencil (24) with the threshold parameter $\lambda = \lambda_\theta^\dagger$ has an eigenvalue $\mu = i\zeta$ on the imaginary axis; to the eigenvalue there corresponds an exponential solution

$$w(y, z) = e^{i\zeta z} W(y) \tag{37}$$

of problem (2)–(4) in the cylinder $\omega \times \mathbb{R}$, where W is the corresponding eigenvector satisfying the quasiperiodicity conditions (25). There can be several such eigenvalues—they arise because the eigenvalues of the pencil $\mathfrak{A}_\theta^\lambda$ move from the half-planes $\{\mu \in \mathbb{C} : \pm \operatorname{Im} \mu > 0\}$ to the imaginary axis as $\lambda \to \lambda_\theta^\dagger - 0$. At least one of them remains on the imaginary axis for $\lambda > \lambda_\theta^\dagger$, since otherwise the continuous spectrum would lose connectedness (see [22, Ch. 1, Sec. 2] and the proof of Corollary 1 in [26]). According to [32], the latter is possible only if the eigenvector W has an associated vector W^1 found from Eq. (27) for $q = 1$, which takes the form

$$\begin{aligned}\big(\mathcal{AM}(\nabla_y, i\zeta)W^1, \mathcal{M}(\nabla_y, i\zeta)V\big)_\omega - \lambda_\theta^\dagger(\mathcal{B}W^1, V)_\omega \\ = -\big(\mathcal{AM}'(\nabla_y, i\zeta)W, \mathcal{M}(\nabla_y, i\zeta)V\big)_\omega - \big(\mathcal{AM}(\nabla_y, i\zeta)W, \mathcal{M}'(\nabla_y, i\zeta)V\big)_\omega\end{aligned} \tag{38}$$

for each vector $V \in H_\theta^{\mathfrak{t}}(\omega)$. Here $\mathcal{M}'(\nabla_y, \mu)$ is the derivative of the matrix $\mathcal{M}(\nabla_y, \mu)$ with respect to the last argument. One of the solvability conditions for Eq. (38) is obtained by substituting $V = W$ into its right-hand side,

$$\operatorname{Re}\big(\mathcal{AM}'(\nabla_y, i\zeta)W, \mathcal{M}(\nabla_y, i\zeta)W\big)_\omega = 0. \tag{39}$$

In what follows, we deal with the wave (37) for which relation (39) holds.

5. NONEMPTINESS OF THE DISCRETE SPECTRUM

By the maximin principle [31, Theorem 10.2.2], the lower bound $-\underline{\Sigma_\theta}$ of the spectrum of the operator $-\mathcal{T}_\theta$ (with the minus sign but lower semibounded) is calculated by the formula

$$-\underline{\Sigma_\theta} = \inf_{u \in H_\theta^{\mathfrak{t}}(\Pi)\setminus\{0\}} \frac{-\langle \mathcal{T}_\theta u, u\rangle_\theta}{\langle u, u\rangle_\theta}. \tag{40}$$

Taking into account definitions (14), (16), (17) and relation (18) between the spectral parameters, we conclude that

$$-\frac{1}{1+\underline{\sigma_\theta}} = \inf_{u \in H_\theta^{\mathfrak{t}}(\Pi)\setminus\{0\}} \frac{-(\mathcal{B}u, u)_\Pi}{a(u, u; \Pi) + (\mathcal{B}u, u)_\Pi} \quad \Leftrightarrow \quad \underline{\sigma_\theta} = \inf_{u \in H_\theta^{\mathfrak{t}}(\Pi)\setminus\{0\}} \frac{a(u, u; \Pi)}{(\mathcal{B}u, u)_\Pi}. \tag{41}$$

Here $\underline{\sigma_\theta}$ is the lower bound of the spectrum of problem (10), which (bound) falls into the interval $(0, \lambda_\theta^\dagger)$ and hence into the discrete spectrum σ_θ^d if and only if there exists a trial vector function $\varphi \in H_\theta^{\mathfrak{t}}(\Pi)$ satisfying the inequality

$$a(\varphi, \varphi; \Pi) - \lambda_\theta^\dagger(\mathcal{B}\varphi, \varphi)_\Pi < 0. \tag{42}$$

Let us use the trick in the paper [33]. For $\theta \neq 0$, we set

$$\varphi^\varepsilon(y, z) = e^{(i\zeta-\varepsilon)z}W(y) + \sqrt{\varepsilon}\psi(x), \tag{43}$$

where $\varepsilon > 0$ is a small parameter, $\{i\zeta, W\}$ is an eigenpair of the pencil (24) for $\lambda = \lambda_\theta^\dagger$ that has given rise to the wave (37) and satisfies condition (39), and ψ is a vector function in the space $C_c^\infty(\omega \times \mathbb{R})^K$ with small support around some point $x^0 \in \varpi$. We have

$$\begin{aligned}(\mathcal{B}\varphi^\varepsilon, \varphi^\varepsilon)_\Pi &= \big(\mathcal{B}(e^{(i\zeta-\varepsilon)z}W + \sqrt{\varepsilon}\psi), e^{(i\zeta-\varepsilon)z}W + \sqrt{\varepsilon}\psi\big)_{\Pi(R)} + \int_R^\infty (\mathcal{B}W, W)_\omega e^{-2\varepsilon z}\,dz \\ &= \frac{1}{2\varepsilon}e^{-2\varepsilon R}(\mathcal{B}W, W)_\omega + (\mathcal{B}w, w)_{\Pi(R)} + 2\sqrt{\varepsilon}\operatorname{Re}(\mathcal{B}w, \psi)_{\Pi(R)} + O(\varepsilon).\end{aligned} \tag{44}$$

We proceed in a similar way with the first term in (42),

$$\begin{aligned}a(\varphi^\varepsilon, \varphi^\varepsilon; \Pi) &= (\mathcal{AM}(\nabla)\big(e^{(i\zeta-\varepsilon)z}W + \sqrt{\varepsilon}\psi), \mathcal{M}(\nabla)(e^{(i\zeta-\varepsilon)z}W + \sqrt{\varepsilon}\psi)\big)_{\Pi(R)} \\ &\quad + \int_R^\infty (\mathcal{AM}\big(\nabla_y, i\zeta-\varepsilon)W, \mathcal{M}(\nabla_y, i\zeta-\varepsilon)W\big)_\omega e^{-2\varepsilon z}\,dz \\ &= \frac{1}{2\varepsilon}e^{-2\varepsilon R}\big(\mathcal{AM}(\nabla_y, i\zeta)W, \mathcal{M}(\nabla_y, i\zeta)W\big)_\omega \\ &\quad + e^{-2\varepsilon R}\operatorname{Re}\big(\mathcal{AM}'(\nabla_y, i\zeta)W, \mathcal{M}(\nabla_y, i\zeta)W\big)_\omega \\ &\quad + \big(\mathcal{AM}(\nabla)w, \mathcal{M}(\nabla)w\big)_{\Pi(R)} + 2\sqrt{\varepsilon}\operatorname{Re}\big(\mathcal{AM}(\nabla)w, \mathcal{M}(\nabla)\psi\big)_{\Pi(R)} + O(\varepsilon).\end{aligned} \tag{45}$$

Let us substitute the expressions (45) and (44) into the left-hand side of relation (42) with the trial vector function (43) and note that the terms of the order of ε^{-1} cancel out owing to the equality

$$\big(\mathcal{AM}(\nabla_y, i\zeta)W, \mathcal{M}(\nabla_y, i\zeta)W\big)_\omega = \lambda_\theta^\dagger(\mathcal{B}W, W)_\omega, \tag{46}$$

which can be derived by integration by parts from the equation $\mathfrak{A}_\theta^{\lambda_\theta^\dagger}(i\zeta)W = 0$ multiplied in the sense of the inner product on $L^2(\omega)^K$ by the eigenvector W. Yet another simplification comes from Eq. (39). As a result, we write relation (42) in the form

$$I_R^0(w) + 2\sqrt{\varepsilon}\,\mathrm{Re}\, I_R^1(w,\psi) < -C\varepsilon \tag{47}$$

with some factor $C > 0$ and the ingredients

$$\begin{aligned} I_R^0(w) &= a(w, w; \Pi(R)) - \lambda_\theta^\dagger(\mathcal{B}w, w)_{\Pi(R)}, \\ I_R^1(w,\psi) &= a(w, \psi; \Pi(R)) - \lambda_\theta^\dagger(\mathcal{B}w, \psi)_{\Pi(R)}. \end{aligned} \tag{48}$$

The size R has been chosen so that $\{x \in \Pi : z > R\} = \omega \times (R, +\infty)$, and the number (48) does not change as R grows by virtue of formula (46). Moreover, in view of the smallness of the support of the vector function ψ and the Green's formula (8), one has the equality

$$I_R^1(w,\psi) = (\mathcal{N}w, \mathcal{D}\psi)_\varpi. \tag{49}$$

Let us summarize the calculations made in what is the main assertion of this paper.

Theorem 5. *Let the wave* (37) *constructed based on an eigenpair* $\{i\zeta, W\}$ *of the pencil* (24) *with parameters* $\lambda = \lambda_\theta^\dagger$ *and* $\theta \in [-\pi,\pi)^{d-1}\setminus\{0\}$ *satisfy relation* (39)*. Then the discrete spectrum* σ_θ^d *of problem* (10) (*or* (2)–(4) *in differential setting*) *is necessarily nonempty in the following two cases*:

1. *The number* (49) *is negative.*
2. *The equality* $I_R^0(w) = 0$ *holds, and the vector function* $x \mapsto \mathcal{N}(x,\nabla)w(x)$ *does not degenerate at least at one point of the end* ϖ *of the half-strip* Π.

Proof. Assertion 1 is beyond doubt—it suffices to take ε small, thus satisfying inequality (47). In assertion 2, owing to the general properties of the Dirichlet system (see [2, Ch. 2, Sec. 2]), we select a point $x^0 \in \varpi$ for which $b := \mathcal{N}(x^0,\nabla)w(x^0) \in \mathbb{C}^T\setminus\{0\}$ and a trial vector function ψ such that $b^{\mathrm{T}}\overline{\mathcal{D}(x^0,\nabla)\psi(x^0)} < 0$. As a result, the real part of $I_R^1(w,\psi)$ becomes negative for small $\varepsilon > 0$, which means that inequality (47) holds; i.e., as in the first case, according to the minimum principle (40) and (41), we conclude that $\underline{\sigma_\theta} < \lambda_\theta^\dagger$ and $\underline{\sigma_\theta} \in \sigma_\theta^d$. This is exactly what needed to be verified.

Theorem 5 provides sufficient conditions for the nonemptiness of the discrete spectrum of problem (2)–(4) in the prism Π for $\theta \neq 0$ and the existence of the Rayleigh waves (12) in the half-space Ω with periodic boundary Γ (see formulas (1)). In what follows, we present specific problems in which these sufficient conditions prove useful.

6. EXAMPLES

1°. *Second-order scalar operator.* Let us reproduce the result in [33] in a slightly more general setting. Let $K = 1$, $N = d \geq 2$, $\mathcal{M}(\nabla) = \nabla$, and $\mathcal{B} = 1$. Using an affine transformation, we reduce the operator (11) to the form $-\nabla^{\mathrm{T}}\mathcal{A}^0\nabla$, where $\mathcal{A}^0$ is a diagonal matrix $\mathrm{diag}\{a_1^0,\ldots,a_d^0\}$ with $a_j^0 > 0$. The simple inequality

$$\int_{-1/2}^{1/2} \left|\frac{dV}{dt}(t)\right|^2 dt \geq \theta^2 \int_{-1/2}^{1/2} \big|V(t)\big|^2\,dt \tag{50}$$

for all $V \in H^1(-1/2,1/2)$, $V(1/2) = e^{i\theta}V(-1/2)$, and $|\theta| \le \pi$ and the formula

$$w(y,z) = e^{i\theta_1 y_1} \times \dots \times e^{i\theta_{d-1}y_{d-1}}$$

for the wave (37) show that, first, $\lambda = a_1^0\theta_1^2 + \ldots + a_{d-1}^0\theta_{d-1}^2$ and second, the number (48) is zero, because $\overline{\nabla w}^{\mathrm{T}}\mathcal{A}^0\nabla w - \lambda_\theta^\dagger|w|^2 = 0$. Finally, $\mathcal{N}(x,\nabla) = n(x)^{\mathrm{T}}\mathcal{A}^0\nabla$, where n is the unit outward normal vector on the end ϖ. Further, $\mathcal{N}(x,\nabla)w(x) = 0$ almost everywhere on ϖ if ϖ is a finite union[2] of the straight end $\omega \times \{h_0\}$ and (two-sided) segments of hyperplanes $\omega_q \times \{h_q\}$, where $0 \le h_0 \le h_1 \le \ldots \le h_Q$ and $\overline{\omega_q} \subsetneqq \overline{\omega}$. In addition, inequality (50) convinces us that $\sigma_\theta^d = \varnothing$ for all $\theta \in [-\pi,\pi)^{d-1}$ for the end geometry.

2°. *Spatial problem of elasticity theory.* Let $K = d = 3$, $N = 6$, $\mathcal{B} = \mathbb{I}_3$, and

$$\mathcal{M}(\nabla)^{\mathrm{T}} = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_3 & 0 & 0 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 2^{-1/2}\partial_3 & 0 \\ 0 & 0 & 0 & 2^{-1/2}\partial_1 & 2^{-1/2}\partial_2 & \partial_3 \end{pmatrix}. \tag{51}$$

Problem (13) describes the propagation of waves in a homogeneous anisotropic elastic space with periodic boundary and with a (real) symmetric and positive definite stiffness matrix $\mathcal{A}$. The corresponding quadratic form (13) is twice the elastic energy of the deformable body Ξ and degenerates on the space of rigid displacements,

$$\mathcal{P} = \left\{ d(x)b : b \in \mathbb{R}^6 \right\} \tag{52}$$

with the linear matrix function

$$d(x) = \begin{pmatrix} 1 & 0 & -2^{-1/2}x_2 & 2^{-1/2}x_3 & 0 & 0 \\ 0 & 1 & 2^{-1/2}x_1 & 0 & -2^{-1/2}x_3 & 0 \\ 0 & 0 & 0 & -2^{-1/2}x_1 & 2^{-1/2}x_2 & 1 \end{pmatrix}. \tag{53}$$

The factors $2^{-1/2}$ are convenient in the matrix notation of the constitutive relations in the theory of elasticity [34, Ch. 4; 35, Ch 3; 36, Ch. 2]; in particular, one has the equalities $\mathcal{M}(\nabla)\mathcal{M}(x)^{\mathrm{T}} = \mathbb{I}_6$ and $d(\nabla)^{\mathrm{T}}d(x)|_{x=0} = \mathbb{I}_6$, where $\mathbb{I}_m$ is the $m \times m$ identity matrix. The fact that the section ω is a unit square and not a rectangle is not restrictive, since in the theory of elasticity the matrix implementation of the operators of the system of differential equations (5) and the boundary conditions

$$\mathcal{N}(x,\nabla)w(x) := \mathcal{M}(n(x))^{\mathrm{T}}\mathcal{A}\mathcal{M}(\nabla)w(x) \tag{54}$$

can be preserved under an affine transformation of coordinates by introducing nonphysical columns of displacements and stresses (see, e.g., the paper [37]).

As was already mentioned in Sec. 1, similar 3D and 2D (see Example 3°) problems are in demand in practical engineering and therefore have been studied to a large extent (see [9–21] and many other publications). We present only a few corollaries of Theorem 5 that may be of interest.

Let us introduce the scalar function[3]

$$\Phi_\theta(w;y) = \overline{\mathcal{M}(\nabla_y, i\zeta)W(y)}^{\mathrm{T}}\mathcal{A}\mathcal{M}(\nabla_y, i\zeta)W(y) - \lambda_\theta^\dagger\left|W(y)\right|^2, \tag{55}$$

constructed based on wave (37) satisfying condition (39). If this function vanishes on the square $\omega = (-1/2,1/2) \ni y = (y_1,y_2)$, then the number (48) is zero and by Theorem 5 the surface Rayleigh waves (12) exist in the case where the normal stress vector (54) is distinct from zero at some point $x^0 \in \varpi$. Of course, the shape of the end ϖ can always be chosen so that condition (54)

[2] In this case, the boundary $\partial\Pi$ is not Lipschitz; however, the prism Π itself is representable as a union of Lipschitz domains, and this property is sufficient for all reasoning below.

[3] Our constructions are also suitable for the general elliptic systems considered. Complex conjugation is not needed in problems of elasticity theory.

for normal stresses is satisfied, because the six-dimensional stress vector $\mathcal{AM}(\nabla)w(x)$ cannot completely degenerate everywhere in Π.

Now let $\pm\Phi_\theta(w;y^\pm) > 0$ for some points $y^\pm \in \omega$; since the function (55) has zero mean over the square ω, it follows that both sets

$$\omega_\theta^\pm = \{y \in \omega : \pm\Phi_\theta(w;y) > 0\}$$

are nonempty. Now it is easy to construct a prism Π for which the condition $I_R^0(\omega) < 0$ in Theorem 5 is satisfied. This occurs, for example, if

$$\begin{gathered}\Pi = \left(\Pi_0 \cup \Upsilon_\theta^-\right) \setminus \Upsilon_\theta^+, \quad \Pi_0 = \omega \times \mathbb{R}_+,\\ \Upsilon_\theta^\pm = \omega_\theta^\pm \cap \{x : y \in \omega, \pm z \geq 0\},\end{gathered} \tag{56}$$

because the difference of integrals

$$I_R^0(w) = \int\limits_{\Upsilon_\theta^-} \Phi_\theta(w;y)\,dx - \int\limits_{\Upsilon_\theta^+} \Phi_\theta(w;y)\,dx$$

is negative if at least one of the sets (56) is nonempty. If, however, $\Upsilon_\theta^\pm = \varnothing$ and $\Pi = \Pi_0$ is a half-cylinder with straight end, then $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}_+$ is a half-space and the corresponding displacement field (12) is a classical Rayleigh wave (see [38]).

3°. *Isotropic half-strip.* Let $K = d = 2$, $N = 3$, $\mathcal{B} = \mathbb{I}_2$, and

$$\mathcal{M}(\nabla) = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ 2^{-1/2}\partial_2 & 2^{-1/2}\partial_1 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \boldsymbol{\lambda} + 2\boldsymbol{\mu} & \boldsymbol{\lambda} & 0 \\ \boldsymbol{\lambda} & \boldsymbol{\lambda} + 2\boldsymbol{\mu} & 0 \\ 0 & 0 & \boldsymbol{\mu} \end{pmatrix},$$

where $\boldsymbol{\lambda} \geq 0$ and $\boldsymbol{\mu} > 0$ are the Lamé constants of a homogeneous isotropic elastic body $\Pi \subset (-1/2, 1/2) \times \mathbb{R}$. (Scaling reduces the half-strip width to unity.) In this case, $\theta \in [-\pi, \pi)$ is a scalar, and straightforward calculations (see, e.g., [39]) show that $\lambda_\theta^\dagger = \theta^2\boldsymbol{\mu}$, $\zeta = 0$, and $W(y) = (0, e^{i\theta y})^{\mathrm{T}}$ in the wave (37). Consequently,

$$\begin{gathered}\Phi_\theta(w;y) = 0 \quad \text{and} \quad \mathcal{AM}(\nabla)w(y) = (0,0,\boldsymbol{\mu})^{\mathrm{T}} i\theta e^{i\theta y},\\ \mathcal{M}\big(n(x)\big)^{\mathrm{T}} \mathcal{AM}(\nabla)w(y) = n(x)\boldsymbol{\mu} i\theta e^{i\theta y}.\end{gathered}$$

Thus, conditions 2 in Theorem 5 are completely satisfied; i.e., there exists a Rayleigh wave for any Floquet parameter $\theta \in [-\pi, \pi) \setminus \{0\}$ and any profile of the periodic boundary Γ of the isotropic deformable half-plane Ω.

7. PROBLEMS WITH SINGULAR MATRIX $\mathcal{B}$ OR ALTERNATING MATRIX $\mathcal{A}$

4°. *Kirchhoff plate.* Let $d = 2$, $K = 3$, $N = 6$, and

$$\mathcal{M}(\nabla)^{\mathrm{T}} = \begin{pmatrix} \mathcal{M}^{\mathsf{M}}(\nabla)^{\mathrm{T}} & & \mathbb{O}_{2\times 3} & \\ \mathbb{O}_{1\times 3} & 2^{-1/2}\partial_1^2 & \partial_1\partial_2 & 2^{-1/2}\partial_2^2 \end{pmatrix}, \tag{57}$$

where $\mathcal{M}^{\mathsf{M}}(\nabla)$ is the 3×2 matrix of differential operators in the list (37) and $\mathbb{O}_{p\times q}$ is the zero $p \times q$ matrix. The two-dimensional problem (13) serves as an asymptotic model of vibrations of a thin 3D plate (see [36, Ch. 7], [39–43], and many other papers). Moreover, $u'(x) = (u_1(x), u_1(x))^{\mathrm{T}}$ and $u_3(x)$ are the thickness-averaged longitudinal displacement vector and plate deflection, respectively. At the considered low frequencies, the kinetic energy of longitudinal vibrations in the adopted model is negligibly small, and hence $\mathcal{B}$ is the singular diagonal matrix $\mathrm{diag}\{0, 0, 1\}$. Note that in the mid-frequency range, on the contrary, transverse vibrations are damped, and the plane problem of the theory of elasticity in Example 3° (and Example 2°) in Sec. 6 acts as a two-dimensional model of longitudinal plate vibrations. The form (9) with the operator matrix (57) degenerates on the space of rigid displacements (52) with the following linear matrix function:

$$d(x) = \begin{pmatrix} 1 & 0 & -2^{-1/2}x_2 & 0 & 0 & 0 \\ 0 & 1 & 2^{-1/2}x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & x_2 & 1 \end{pmatrix}. \tag{58}$$

The differences between the matrices (58) and (53) are due to the fact that in the Kirchhoff theory the total displacement vector in a thin 3D plate is reconstructed using the formula

$$\big(u_1(x_1,x_2) - x_3\partial_1 u_3(x_1,x_2), u_2(x_1,x_2) - x_3\partial_2 u_3(x_1,x_2), u_3(x_1,x_2)\big)^{\mathrm{T}}.$$

Substituting the columns of the matrix (58) into the last expression, we obtain the columns of the matrix (53).

It is well known (see, e.g., [36, Ch. 4, Sec. 2]) that in the case of homogeneous and even layered plates, their midplanes can be fixed in such a way that the matrix $\mathcal{A}$ becomes block diagonal,

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^{\leftrightarrows} & \mathbb{O}_{3\times 3} \\ \mathbb{O}_{3\times 3} & \mathcal{A}^{\updownarrow} \end{pmatrix},$$

and problem (13) splits into a static ($\lambda = 0$) plane problem of the theory of elasticity and a spectral equation of the fourth order, in particular, the Sophie Germain biharmonic equation [44, Sec. 30] for an isotropic plate; then, of course, Theorem 5 applies. At the same time, for a plate made of a composite material, the matrix $\mathcal{A}$ can be completely dense; i.e. all equations in system (2) are intertwined.

For a singular matrix $\mathcal{B}$, formulas (14) and (16) do not define a norm on the space $H_0^{\mathbf{t}}(\Pi)$, but for a nonzero Floquet parameter, the Hilbert space (11) can still be equipped with the inner product (14).

Lemma 2. *If $\theta \in [-\pi, 0) \cup (0, \pi)$, then for the norm on $H_\theta^{\mathbf{t}}(\Pi)$ one can take the expression $a(u,u;\Pi)^{1/2}$ or $(a(u,u;\Pi) + \|u_3; L^2(\Pi)\|^2)^{1/2}$, where a is the quadratic form (9) with the matrix (57) of differential operators of the first and second orders.*

Proof. Only the trivial rigid displacement in the linear space (52) with matrix (58) satisfies the quasiperiodicity conditions in formula (11) on the sides of the half-strip Π, and hence the Lebesgue norms $\|u; L^2(\Pi(R))\|$ and $\|u; L^2(Q_{R+m})\|$, respectively, can be removed from the right-hand sides of Korn's inequalities on the sets $\Pi(R)$ and Q_{R+m}, $m \in \mathbb{N}_0$ (see the remark on relation (15)). Thus, we have the estimate

$$\big\|u; H_\theta^{\mathbf{t}}(\Pi)\big\| \le c_{\Pi,\mathcal{A},\mathcal{M}}\, a(u,u;\Pi)^{1/2}.$$

The proof of the lemma is complete.

5°. *Piezoelectric problem.* Let $d = 3$, $K = 4$, $N = 9$, $\mathcal{B} = \operatorname{diag}\{1,1,1,0\}$, and

$$\mathcal{M}(\nabla)^{\mathrm{T}} = \begin{pmatrix} \mathcal{M}^{\mathsf{M}}(\nabla)^{\mathrm{T}} & \mathbb{O}_{3\times 3} \\ \mathbb{O}_{1\times 6} & \nabla^{\mathrm{T}} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{A}^{\mathsf{MM}} & \mathcal{A}^{\mathsf{ME}} \\ \mathcal{A}^{\mathsf{EM}} & -\mathcal{A}^{\mathsf{EE}} \end{pmatrix}. \tag{59}$$

Here $\mathcal{M}^{\mathsf{M}}(\nabla)^{\mathrm{T}}$ is the matrix (51), $\mathcal{A}^{\mathsf{MM}}$ and $\mathcal{A}^{\mathsf{EE}}$ are (real) symmetric and positive definite matrices, and $\mathcal{A}^{\mathsf{ME}} = (\mathcal{A}^{\mathsf{EM}})^{\mathrm{T}}$ is a (6×3) matrix without any special properties but necessarily nonzero. In addition, $u^{\mathsf{M}} = (u_1,u_2,u_3)^{\mathrm{T}}$ is the displacement vector, and u_4 is the electric potential. Problem (13) describes time-harmonic oscillations of a piezoelectric medium in which free transformation of elastic energy into electromagnetic energy and vice versa is possible, which explains the minus sign of the lower right 3×3 block in the matrix $\mathcal{A}$. In the mid- and low-frequency ranges of the spectrum, where the mechanical vibrations are realized, electromagnetic oscillations should be neglected. Therefore, the spectral parameter λ disappears from the bottom line of the system of differential equations (2); i.e., $\mathcal{B}_{44} = 0$. A detailed explanation of the physical statement of the piezoelectric problem can be found in [45, 46] and other monographs. In particular, the form (2) is associated not with the total energy of the medium but with its electric enthalpy [47].

The matrix $\mathcal{A}$ in (59) is not positive, and since the case of $\mathcal{A}^{\mathsf{ME}} = \mathbb{O}_{6\times 3}$ is uninteresting owing to the disappearance of the discussed piezoelectric effect, it is impossible to achieve the property of formal positivity of the operator (5) by any substitutions. Therefore, the techniques used in this paper are not suitable for the formally self-adjoint problem (2)–(4) with the matrices (59). However, following [48], we make the change of the unknown $u_4 \mapsto u^{\mathsf{E}} = iu_4$ and thus give the matrix in the differential operator (5) the form

$$\begin{pmatrix} \mathcal{A}^{\mathsf{MM}} & \mathbb{O}_{6\times 3} \\ \mathbb{O}_{3\times 6} & \mathcal{A}^{\mathsf{EE}} \end{pmatrix} - i \begin{pmatrix} \mathbb{O}_{6\times 6} & \mathcal{A}^{\mathsf{ME}} \\ \mathcal{A}^{\mathsf{EM}} & \mathbb{O}_{3\times 3} \end{pmatrix} \tag{60}$$

with two symmetric 9×9 matrices. According to the definitions and conclusions in the paper [25], the form (9) with the numerical matrix (60) and the differential operator $\mathcal{M}(\nabla)$ from the list (59) retains the polynomial property (20) in which the linear space of polynomials has the form

$$\mathcal{P} = \Big\{ \big(d(x) b^{\mathsf{M}}, b_4 \big)^{\mathrm{T}} : b^{\mathsf{M}} \in \mathbb{C}^6, b_4 \in \mathbb{C} \Big\},$$

where the matrix (53) of rigid mechanical displacements occurs together with a constant electric potential. Thus, the matrix (5) of second-order differential operators is elliptic, and hence Theorem 1 and Corollary 1, which come from the analysis of the corresponding formally self-adjoint Neumann boundary value problem (2)–(4) in a semi-infinite prism, remain valid. At the same time, we are still unable to define the self-adjoint operator $\mathcal{T}_\theta$ by formula (17) and have to complicate its construction by means of a trick [48]. In what follows, we deal with the slightly simpler case of $\theta \in [-\pi, \pi)^2 \setminus \{0\}$, in which the auxiliary problems in Π become uniquely solvable. Namely, let $\mathcal{J} u^{\mathsf{M}} := \mathbf{u}_4 \in H^1_\theta(\Pi)$ be a solution of the problem

$$(\mathcal{A}^{\mathsf{EE}} \nabla \mathbf{u}_4, \nabla v_4)_\Pi = \big(\mathcal{A}^{\mathsf{EM}} \mathcal{M}^{\mathsf{M}}(\nabla) u^{\mathsf{M}}, \nabla v_4 \big)_\Pi \quad \text{for all potentials} \quad v_4 \in H^1_\theta(\Pi), \tag{61}$$

found based on the displacement vector $u^{\mathsf{M}} \in H^1_\theta(\Pi)^3$. Here $H^1_\theta(\Pi)$ is a scalar Sobolev space with one quasi-periodicity condition on opposite faces of the prism Π. As has been verified in the papers [48, 49] and is easy to check by straightforward calculations, the variational statement of the piezoelectric problem (2)–(4) with the original matrices (59) is equivalent to integral identity

$$\big(\mathcal{A}^{\mathsf{MM}} \mathcal{M}^{\mathsf{M}}(\nabla) u^{\mathsf{M}}, \mathcal{M}^{\mathsf{M}}(\nabla) v^{\mathsf{M}} \big)_\Pi + \big(\mathcal{A}^{\mathsf{ME}} \mathcal{J} \nabla u^{\mathsf{M}}, \mathcal{M}^{\mathsf{M}}(\nabla) v^{\mathsf{M}} \big)_\Pi = \lambda (u^{\mathsf{M}}, v^{\mathsf{M}})_\Pi \quad \text{for all} \quad v^{\mathsf{M}} \in H^1_\theta(\Pi)^3, \tag{62}$$

and the sesquilinear Hermitian form

$$\langle u^{\mathsf{M}}, v^{\mathsf{M}} \rangle = \big(\mathcal{A}^{\mathsf{MM}} \mathcal{M}^{\mathsf{M}}(\nabla) u^{\mathsf{M}}, \mathcal{M}^{\mathsf{M}}(\nabla) v^{\mathsf{M}} \big)_\Pi + \big(\mathcal{A}^{\mathsf{ME}} \mathcal{J} \nabla u^{\mathsf{M}}, \mathcal{M}^{\mathsf{M}}(\nabla) v^{\mathsf{M}})_\Pi + (u^{\mathsf{M}}, v^{\mathsf{M}} \big)_\Pi,$$

incorporating the left-hand side of identity (62), turns out to be positive definite and can be taken for the inner product on the Hilbert space $\mathcal{H}^{\mathsf{M}}_\theta = H^1_\theta(\Pi)^3$. Furthermore, now it is possible to introduce an operator $\mathcal{T}^{\mathsf{M}}_\theta$ with the desired properties,

$$\langle \mathcal{T}^{\mathsf{M}}_\theta u^{\mathsf{M}}, v^{\mathsf{M}} \rangle = (u^{\mathsf{M}}, v^{\mathsf{M}})_\Pi \quad \text{for all displacement vectors} \quad u^{\mathsf{M}}, v^{\mathsf{M}} \in H^1_\theta(\Pi)^3,$$

and the new spectral parameter (18).

Everything is ready to apply the minimum principle, as in Sec. 5, to derive sufficient conditions for the discrete spectrum of the operator $\mathcal{T}^{\mathsf{M}}_\theta$ to be nonempty and hence also for the existence of isolated eigenvalues of the original piezoelectric problem; however, the integro-differential operator of problem (62) has ceased to be local, which significantly affects the result of the subsequent calculations.

Since the pencil (24) and its spectrum retain the properties indicated in Sec. 4, there exists a wave (37) with nonzero mechanical component $w^{\mathsf{M}} = (w_1, w_2, w_3)^{\mathrm{T}}$ and some electrical component w_4. The latter satisfies system (2) and the quasiperiodicity conditions (4), but leaves a discrepancy in the Neumann boundary condition (3) at the end ϖ,

$$g(x) = -\mathcal{M}\big(n(x)\big)^{\mathrm{T}}\mathcal{A}\mathcal{M}(\nabla)w(x)$$

with the fourth (lower, electrical) component

$$g_4(x) = -n(x)^{\mathrm{T}}\mathcal{A}^{\mathsf{ME}}\mathcal{M}^{\mathsf{M}}(\nabla)w^{\mathsf{M}}(x) + n(x)^{\mathrm{T}}\mathcal{A}^{\mathsf{EE}}\nabla w_4(x), \tag{63}$$

which we counterbalance by using the solution $\mathbf{w}_4 \in H^1_\theta(\Pi)$ of the following static (without a spectral parameter) problem similar to (61):

$$(\mathcal{A}^{\mathsf{EE}}\nabla \mathbf{w}_4, \nabla v_4)_\Pi = (g_4, v_4)_\varpi \quad \text{for all} \quad v_4 \in H^1_\theta(\Pi). \tag{64}$$

The existence of a solution exponentially decaying at infinity is ensured by Theorem 1 for a nonzero Floquet parameter. (If $\theta = 0$, then one has to use Theorem 3; this complicates the subsequent analysis; cf. the papers [48, 49].)

We apply the minimum principle (40) to the operator $\mathcal{T}^{\mathsf{M}}_\theta$ and, after transformations similar to (41), conclude that

$$-\underline{\Sigma^{\mathsf{M}}_\theta} = \inf_{u^{\mathsf{M}}\in H^1_\theta(\Pi)^3\setminus\{0\}} \frac{-\langle \mathcal{T}^{\mathsf{M}}_\theta u^{\mathsf{M}}, u^{\mathsf{M}}\rangle_\theta}{\langle u^{\mathsf{M}}, u^{\mathsf{M}}\rangle_\theta} \quad \Leftrightarrow \quad \underline{\sigma^{\mathsf{M}}_\theta} = \inf_{u^{\mathsf{M}}\in H^1_\theta(\Pi)^3\setminus\{0\}} \frac{a\big((u^{\mathsf{M}}, \mathcal{J}u^{\mathsf{M}}), (u^{\mathsf{M}}, \mathcal{J}u^{\mathsf{M}}); \Pi\big)}{\big\|u^{\mathsf{M}}; L^2(\Pi)\big\|^2}.$$

Here, just as in Sec. 5, it is required to find a trial vector function $\varphi^{\mathsf{M}} \in H^1_\theta(\Pi)^3$ for which one has the following inequality similar to (42):

$$a\big((\varphi^{\mathsf{M}}, \mathcal{J}\varphi^{\mathsf{M}}), (\varphi^{\mathsf{M}}, \mathcal{J}\varphi^{\mathsf{M}}); \Pi\big) - \lambda^\dagger_\theta \big\|\varphi^{\mathsf{M}}; L^2(\Pi)\big\|^2 < 0. \tag{65}$$

Since the electrical component $\mathcal{J}u^{\mathsf{M}} \in H^1_\theta(\Pi)$ is determined from the mechanical component $u^{\mathsf{M}} \in H^1_\theta(\Pi)^3$ as a solution of problem (61), the construction of (43) needs to be modified. Let us show how to derive a sufficient condition comparable with the first assertion in Theorem 5. Set

$$\varphi^{\mathsf{M}\varepsilon}(y,z) = \mathcal{E}^\varepsilon_R(z)w^{\mathsf{M}}(y,z), \tag{66}$$

where

$$\mathcal{E}^\varepsilon_R(z) = 1 \quad \text{for} \quad z < R, \quad \mathcal{E}^\varepsilon_R(z) = e^{-\varepsilon(z-R)} \quad \text{for} \quad z \ge R, \quad \partial_z\mathcal{E}^\varepsilon_R(z) = -\varepsilon e^{-\varepsilon(z-R)}X_R(z), \tag{67}$$

and X_R is the Heaviside function with a jump at the point $z = R$. We point out that the use of a continuous piecewise smooth decaying factor $\mathcal{E}^\varepsilon_R$ is possible because the derivative $\partial_z\mathcal{E}^\varepsilon_R$ is a bounded piecewise smooth function and $t_1 = t_2 = t_3 = 1$; i.e., the inclusion $\varphi^{\mathsf{M}\varepsilon} \in H^1_\theta(\Pi)^3$ holds. We represent the electrical component $\varphi^\varepsilon_4 = \mathcal{J}\varphi^{\mathsf{M}\varepsilon}$ as

$$\varphi^\varepsilon_4(y,z) = \mathcal{E}^\varepsilon_R(z)w_4(y,z) - \mathbf{w}_4(y,z) + \varepsilon\mathcal{E}^\varepsilon_R(z)w'_4(y,z), \tag{68}$$

where $w = (w^{\mathsf{M}}, w_4)$ is the threshold wave (37) and $\mathbf{w}_4 \in H^1_\theta(\Pi)$ and w'_4 are solutions of problems (64) and

$$\begin{aligned}(\mathcal{A}^{\mathsf{EE}}\nabla w'_4, \nabla v_4)_\Pi &- \varepsilon(\mathcal{A}^{\mathsf{EE}}X\mathbf{e}_3X_Rw'_4, \nabla v_4)_\Pi + \varepsilon(\mathcal{A}^{\mathsf{EE}}\nabla w'_4, \mathbf{e}_3X_Rv_4)_\Pi \\ &= (\mathcal{A}^{\mathsf{EE}}\mathbf{e}_3X_Rw_4, \nabla v_4)_\Pi - (\mathcal{A}^{\mathsf{EE}}\nabla w_4, \mathbf{e}_3X_Rv_4)_\Pi - \big(\mathcal{A}^{\mathsf{EM}}\mathcal{M}(\mathbf{e}_3)X_Rw^{\mathsf{M}}, \nabla v_4\big)_\Pi \\ &\quad + \big(\mathcal{A}^{\mathsf{EM}}\mathcal{M}(\nabla)w^{\mathsf{M}}, \mathbf{e}_3X_Rv_4\big)_\Pi \quad \text{for all} \quad v_4 \in H^1_\theta(\Pi)\cap C^\infty_c(\overline{\Pi})^K.\end{aligned} \tag{69}$$

Lemma 3. *Problem* (69) *has a solution*

$$w_4'(y,z) = e^{i\zeta z} W_4'(y) + \widetilde{w}_4'(y,z), \tag{70}$$

where $W_4' \in H_\theta^l(\omega)$ *and* $e^{\gamma z}\widetilde{w}_4' \in H_\theta^1(\Pi)$ *for any* $l \in \mathbb{N}$ *and* $\gamma \in (0, \gamma(\theta))$.

Proof. Since the functions occurring first in the inner products on the right-hand side in the integral identity (69) are essentially the products $e^{i\zeta z}F(y)$, one must take into account the asymptotic constructions in [27] (see also [22, Ch. 3, Sec. 3]) and solve the scalar equation

$$\begin{aligned} -(\nabla_y, i\zeta - \varepsilon)^{\mathrm T} \mathcal{A}^{\mathsf{EE}} (\nabla_y, i\zeta - \varepsilon) W_4' = \frac{1}{\varepsilon}\Big(&(\nabla_y, i\zeta - \varepsilon)^{\mathrm T} \mathcal{A}^{\mathsf{EE}} (\nabla_y, i\zeta - \varepsilon) W_4(y) \\ &- (\nabla_y, i\zeta - \varepsilon)^{\mathrm T} \mathcal{A}^{\mathsf{EM}} \mathcal{M}^{\mathsf M} (\nabla_y, i\zeta - \varepsilon) W^{\mathsf M}(y)\Big), \quad y \in \omega, \end{aligned}$$

with the quasiperiodicity conditions (25), $m = 0, 1$. We point out that the amplitude part $W = ((W^{\mathsf M})^{\mathrm T}, W_4)^{\mathrm T}$ of the wave (37) satisfies the equality (the transformed bottom line of system (2))

$$(\nabla_y, i\zeta)^{\mathrm T} \mathcal{A}^{\mathsf{EE}} (\nabla_y, i\zeta) W_4(y) = (\nabla_y, i\zeta)^{\mathrm T} \mathcal{A}^{\mathsf{EM}} \mathcal{M}^{\mathsf M} (\nabla_y, i\zeta) W^{\mathsf M}(y), \quad y \in \omega,$$

and therefore, the right-hand side of Eq. (70) is uniformly bounded as $\varepsilon \to +0$. By Theorem 4, problem (70), (25) is uniquely solvable. In addition, the functional left unaccounted for on the right-hand side of a problem of the form (69) for the remainder $\widetilde{w}_4'$ has acquired a compact support, which means that Corollary 2 completes the verification of the lemma, and the ingredients of the representation (69) remain bounded as $\varepsilon \to +0$.

The verification of the fact that the expression (68) indeed solves problem (61) with right-hand side constructed based on the product (66) is carried out on the basis of the integral identities (64) and (69) with suitable trial functions.

Let us repeat calculations (44) and (45) to arrive at the relations

$$\begin{aligned} \left\|\varphi^{\mathsf M\varepsilon}; L^2(\Pi)\right\|^2 &= \int_R^\infty \int_\omega e^{-2\varepsilon(z-R)} \left|W^{\mathsf M}(y)\right|^2 dy\, dz + \left\|w^{\mathsf M}; L^2(\Pi(R))\right\|^2 \\ &= \frac{1}{2\varepsilon}\left\|W^{\mathsf M}; L^2(\omega)\right\|^2 + \left\|w^{\mathsf M}; L^2(\Pi(R))\right\|^2 \end{aligned} \tag{71}$$

and

$$\begin{aligned} &a\big((\varphi^{\mathsf M\varepsilon}, \varphi_4^\varepsilon), (\varphi^{\mathsf M\varepsilon}, \varphi_4^\varepsilon); \Pi\big) \\ &\quad= \int_R^\infty e^{-2\varepsilon(z-R)} \Big(\big(\mathcal{A}\mathcal{M}(\nabla_y, i\zeta)W, \mathcal{M}(\nabla_y, i\zeta)W\big)_\omega + \varepsilon\, 2\mathrm{Re}\,\big(\mathcal{A}\mathcal{M}(e_3)W, \mathcal{M}(\nabla_y, i\zeta)W\big)_\omega \Big)\, dz \\ &\qquad\qquad + \big(\mathcal{A}\mathcal{M}(\nabla)w, \mathcal{M}(\nabla)w\big)_{\Pi(R)} - (\mathcal{A}^{\mathsf{EE}}\nabla \mathbf{w}_4, \nabla \mathbf{w}_4)_\Pi \\ &\qquad\qquad + 2\,\mathrm{Re}\,\Big((\mathcal{A}^{\mathsf{EE}}\nabla w_4, \nabla \mathbf{w}_4)_\Pi - \big(\mathcal{A}^{\mathsf{EM}}\mathcal{M}^{\mathsf M}(\nabla)w^{\mathsf M}, \nabla \mathbf{w}_4\big)_\Pi \Big) + O(\varepsilon) \\ &\quad= \frac{1}{2\varepsilon}\big(\mathcal{A}\mathcal{M}(\nabla_y, i\zeta)W, \mathcal{M}(\nabla_y, i\zeta)W\big)_\omega + \big(\mathcal{A}\mathcal{M}(\nabla)w, \mathcal{M}(\nabla)w\big)_{\Pi(R)} \\ &\qquad\qquad - (\mathcal{A}^{\mathsf{EE}}\nabla \mathbf{w}_4, \nabla \mathbf{w}_4)_\Pi + 2\,\mathrm{Re}\,(g_4, \mathbf{w}_4)_\varpi + O(\varepsilon). \end{aligned} \tag{72}$$

While the transformation (71) is quite simple (it has led to an equality owing to the choice (67) of the exponential weight function $\mathcal{E}_R^\varepsilon(z)$), the transformation (72) is rather confusing owing to additional terms in the definition of the transformation (68), so let us give an explanation. The factors ε and $\mathcal{E}_R^\varepsilon(z)$ in the last term in (68), as well as the representation (70), which essentially means that the solution w_4' of problem (69) is bounded, show that the contribution of the expression $\varepsilon\mathcal{E}_R^\varepsilon w_4'$ is $O(\varepsilon)$ and can be neglected. The integral containing the matrix $-\mathcal{M}(\mathbf{e}_3) = \partial_\varepsilon \mathcal{M}(\nabla_y, i\zeta - \varepsilon)$ has disappeared owing to the customary convention (39). According to Corollary 2, the solution $\mathbf{w}_4$ of problem (64) with a compactly supported functional on the right-hand side decays at infinity at a fixed (independent of ε) rate $O(e^{-\gamma z})$, $\gamma > 0$, and hence in the inner products $(\mathcal{A}^{\mathsf{EM}}\mathcal{M}(\nabla)\mathcal{E}^{\varepsilon}_R w^{\mathsf{M}}, \nabla \mathbf{w}_4)_\Pi$ and alike, the substitution $\mathcal{E}^{\varepsilon}_R(z) \mapsto 1$ also generates an admissible error $O(\varepsilon)$. In addition, the last transition in the calculation (72) uses definition (63), and with $v_4 = \mathbf{w}_4$ equality (64) shows that the expression under Re equals $(\mathcal{A}^{\mathsf{EE}}\nabla\mathbf{w}_4, \nabla\mathbf{w}_4)_\Pi$. Finally, the substitution of the expressions (72) and (71) into inequality (65) with the trial vector function (66), (68) taking into account equalities (46) and (64) leads to the relation

$$\mathbf{I}^0_R(w) < C\varepsilon,$$

where $C > 0$ is some constant and

$$\mathbf{I}^0_R(w) = \big(\mathcal{A}\mathcal{M}(\nabla)w, \mathcal{M}(\nabla)w\big)_{\Pi(R)} - \lambda^{\dagger}_{\theta}\Big\|w^{\mathsf{M}}; L^2\big(\Pi(R)\big)\Big\|^2 + (\mathcal{A}^{\mathsf{EE}}\nabla\mathbf{w}_4, \nabla\mathbf{w}_4)_\Pi. \tag{73}$$

Now the arguments accompanying the verification of Theorem 5 lead to the following assertion.

Theorem 6. *If, for $\theta = (\theta_1, \theta_2)$ and $|\theta_j| \in (0, \pi]$, the expression (73) calculated for the piezoelectric wave (37), which satisfies problem (2)–(4) with the threshold parameter $\lambda = \lambda^{\dagger}_{\theta} > 0$, is negative and satisfies relation (39), then the discrete spectrum σ^d_θ of problem (2)–(4) with the matrices (59) is nonempty.*

Compared to the number (48) found for the problem of elasticity theory (Sec. 6, 2°), the number (73) contains the additional positive term

$$(\mathcal{A}^{\mathsf{EE}}\nabla\mathbf{w}_4, \nabla\mathbf{w}_4)_\Pi \tag{74}$$

that has appeared as a result of compensation for the inhomogeneity (63) in the boundary condition (3) when forming the operator $\mathcal{J}w^{\mathsf{M}}$. This observation is consistent with the physical nature of the piezoelectric problem: apart from the elastic energy $(\mathcal{A}^{\mathsf{MM}}\mathcal{M}^{\mathsf{M}}(\nabla)w^{\mathsf{M}}, \mathcal{M}^{\mathsf{M}}(\nabla)w^{\mathsf{M}})_{\Pi(R)}$, the body $\Pi(R)$ stores the electromagnetic energy $(\mathcal{A}^{\mathsf{EE}}\nabla w, \nabla w_4)_{\Pi(R)}$. At first glance, it seems that the inequality $\mathbf{I}^0_R(w) < 0$ is a more difficult goal to achieve than the inequality $I^0_R(w) < 0$ in a "purely elastic" situation; in particular, manipulations with sets (56) are useless precisely because of the term (74). At the same time, it is impossible to compare $\mathbf{I}^0_R(w)$ and $I^0_R(w)$, if only because the cutoff points of the continuous spectrum in the piezoelectric and elastic problems are in no way related. Finally, again owing to the nonlocal operator $\mathcal{J}$, it was impossible to apply the trick in [33] and obtain an analog of item 2 in Theorem 5 for problem (2)–(4) with the matrices (59).

REFERENCES

1. Nečas, J., *Les méthodes en théorie des équations elliptiques,* Paris: Masson, Prague: Academia, 1967.
2. Lions, J.-L. and Magenes, E., *Problèmes aux limites inhomogènes et applications*, Paris: Dunod, 1968.
3. Ladyzhenskaya, O.A., *Kraevye zadachi matematicheskoi fiziki* (Boundary Value Problems of Mathematical Physics), Moscow: Nauka, 1973.
4. Mittra, R. and Lee, S.W., *Analytical Techniques in the Theory of Guided Waves*, New York: Macmillan, 1971.
5. Wilcox, C.H., Scattering theory for diffraction gratings, in *Appl. Math. Sci. Ser.,* Singapore: Springer Sci. Bus. Media, vol. 46, 1997.
6. Rayleigh, J.W.S., On waves propagated along the plane surface of an elastic solid, *Proc. London Math. Soc.*, 1885, vol. 17, no. 253, pp. 4–11.
7. Lamb, H., On waves in an elastic plate, *Proc. R. Soc.*, 1917, vol. A93, pp. 114–128.
8. Stoneley, R., Elastic waves at the surface of separation of two solids, *Proc. R. Soc. London A*, 1924, vol. 106, pp. 416–428.
9. Viktorov, I.A., *Zvukovye poverkhnostnye volny v tverdykh telakh* (Sonic Surface Waves in Solids), Moscow: Nauka, 1981.
10. Kaplunov, J.D., Kossovich, L.Y., and Nolde, E.V., *Dynamics of Thin Walled Elastic Bodies*, San Diego, CA: Academic Press, 1997.
11. Mikhasev, G.I. and Tovstik, P.E., *Lokalizovannye kolebaniya i volny v tonkikh obolochkakh. Asimptoticheskie metody* (Localized Vibrations and Waves in Thin Shells. Asymptotic Methods), Moscow: Nauka, 2009.

12. Konenkov, Yu.K., On the “Rayleigh”-type flexural wave, *Akust. Zh.*, 1960, vol. 6, pp. 124–126.
13. Grinchenko, V.T. and Meleshko, V.V., Specific features of energy distribution in a thin rectangular plate under edge resonance, *Dokl. Akad. Nauk USSR. Ser. A*, 1976, no. 7, pp. 612–616.
14. Kim, J.-Y. and Rokhlin, S.I., Surface acoustic wave measurements of small fatigue cracks initiated from a surface cavity, *Int. J. Solids Struct.*, 2002, vol. 39, pp. 1487–1504.
15. Zakharov, D.D. and Becker, W., Rayleigh type bending waves in anisotropic media, *J. Sound Vib.*, 2003, vol. 261, pp. 805–818.
16. Kamotskii, I.V., Surface wave running along the edge of an elastic wedge, *St. Petersburg Math. J.*, 2009, vol. 20, no. 1, pp. 59–63.
17. Kamotskii, I.V. and Kiselev, A.P., An energy approach to the proof of the existence of Rayleigh waves in an anisotropic elastic half-space, *Prikl. Mat. Mekh.*, 2009, vol. 73, no. 4, pp. 645–654.
18. Zavorokhin, G.L. and Nazarov, A.I., On elastic waves in a wedge, *J. Math. Sci.*, 2011, vol. 175, no. 6, pp. 646–650.
19. Krushynska, A.A., Flexural edge waves in semi-infinite elastic plates, *J. Sound Vib.*, 2011, vol. 330, pp. 1964–1976.
20. Nazarov, A., Nazarov, S., and Zavorokhin, G., On symmetric wedge mode of an elastic solid, *Eur. J. Appl. Math.*, 2021, vol. 33, no. 2, pp. 201–223.
21. Lawrie, J. and Kaplunov, J., Edge waves and resonance on elastic structures: an overview, *Math. Mech. Solids*, 2012, vol. 17, no. 1, pp. 4–16.
22. Nazarov, S.A. and Plamenevsky, B.A., *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, Berlin–New York: De Gruyter, 1994.
23. Nazarov, S.A., The polynomial property of self-adjoint elliptic boundary-value problems and an algebraic description of their attributes, *Russ. Math. Surv.*, 1999, vol. 54, no. 5, pp. 947–1014.
24. Nazarov, S.A., Self-adjoint elliptic boundary-value problems. The polynomial property and formally positive operators, *J. Math. Sci.,* 1998, vol. 92, no. 6, pp. 4338-4353.
25. Nazarov, S.A., Non-self-adjoint elliptic problems with a polynomial property in domains possessing cylindrical outlets to infinity, *J. Math. Sci.*, 2000, vol. 101, no. 5, pp. 3512–3522.
26. Agranovich, M.S. and Vishik, M.I., Elliptic problems with a parameter and parabolic problems of general type, *Russ. Math. Surv.*, 1964, vol. 19, no. 3, pp. 53–157.
27. Kondrat’ev, V.A., Boundary problems for elliptic equations in domains with conical or angular points, *Trans. Moscow Math. Soc.*, 1967, vol. 16, pp. 227–313.
28. Agmon, S., Douglis, A., and Nirenberg, L., Estimates near the boundary for solutions of elliptic differential equations satisfying general boundary conditions. 2, *Commun. Pure Appl. Math.*, 1964, vol. 17, pp. 35–92.
29. Solonnikov, V.A., General boundary value problems for Douglis–Nirenberg elliptic systems. II, *Proc. Steklov Inst. Mat.*, 1968, vol. 92, pp. 269–339.
30. Gohberg, I.C. and Krein, M.G., *Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monogr.,* Providence, R.I.: Am. Math. Soc., 1969, vol. 18.
31. Birman, M.Sh. and Solomyak, M.Z., *Spectral Theory of Self-Adjoint Operators in Hilbert Space, Math. Appl. (Sov. Ser.),* Dordrecht: Reidel, 1987, vol. 5.
32. Nazarov, S.A., Threshold resonances and virtual levels in the spectrum of cylindrical and periodic waveguides, *Izv. Math.*, 2020, vol. 84, no. 6, pp. 1105–1160.
33. Kamotskii, I.V. and Nazarov, S.A., Exponentially decreasing solutions of diffraction problems on a rigid periodic boundary, *Math. Notes*, 2003, vol. 73, no. 1, pp. 129–131.
34. Lekhnitskii, S.G., *Teoriya uprugosti anizotropnogo tela* (Theory of Elasticity of an Anisotropic Body), Moscow: Nauka, 1977.
35. Bertram, A., *Elasticity and Plasticity of Large Deformations*, Berlin–Heidelberg: Springer, 2005.
36. Nazarov, S.A., *Asimptoticheskaya teoriya tonkikh plastin i sterzhnei. Ponizhenie razmernosti i integral’nye otsenki* (Asymptotic Theory of Thin Plates and Rods. Dimension Reduction and Integral Estimates), Novosibirsk: Nauchn. Kniga, 2002.
37. Langer, S., Nazarov, S.A., and Specovius-Neugebauer, M., Affine transformations of three-dimensional anisotropic media and explicit formulas for fundamental matrices, *J. Appl. Mech. Techn. Phys.*, 2006, vol. 47, no. 2, pp. 229–235.
38. Kamotskii, I.V. and Nazarov, S.A., Elastic waves localized near periodic families of defects, *Dokl. Ross. Akad. Nauk*, 1999, vol. 368, no. 6, pp. 771–773.
39. Shoikhet, B.A., On asymptotically exact equations for thin plates of complex structure, *Prikl. Mat. Mekh.*, 1973, vol. 37, no. 5, pp. 914–924.

40. Ciarlet, P.G., *Mathematical Elasticity, II: Theory of Plates*, *Studies in Mathematics and Its Applications,* Amsterdam, vol. 27, 1997.
41. Dauge, M., Djurdjevic, I., Faou, E., and Rössle, A., Eigenmode asymptotics in thin elastic plates, *J. Math. Pures Appl.,* 1999, vol. 78, no. 9, pp. 925–964.
42. Nazarov, S.A., On the asymptotics of the spectrum of a thin plate problem of elasticity, *Sib. Math. J.*, 2000, vol. 41, no. 4, pp. 744–759.
43. Dauge, M. and Yosibash, Z., Eigen-frequencies in thin elastic 3-D domains and Reissner–Mindlin plate models, *Math. Meth. Appl. Sci.*, 2002, vol. 25, no. 1, pp. 21–48.
44. Mikhlin, S.G., *Variatsionnye metody v matematicheskoi fizike* (Variational Methods in Mathematical Physics), Moscow: Nauka, 1970.
45. Parton, V.Z. and Kudryavtsev, B.A., *Elektromagnitouprugost' p'ezoelektricheskikh i elektroprovodyashchikh sred* (Electromagnetoelasticity of Piezoelectric and Electrically Conductive Media), Moscow: Nauka, 1988.
46. Tiersten, H.F., *Linear Piezoelectric Plate Vibrations*, New York: Springer, 1964.
47. Suo, Z., Kuo, C.-M., Barnett, D.M., and Willis, J.R., Fracture mechanics for piezoelectric ceramics, *J. Mech. Phys. Solids*, 1992, vol. 40, no. 4, pp. 739–765.
48. Nazarov, S.A., Uniform estimates of remainders in asymptotic expansions of solutions to the problem on eigen-oscillations of a piezoelectric plate, *J. Math. Sci.*, 2003, vol. 114, no. 5, pp. 1657–1726.
49. Nazarov, S.A., Ruotsalainen, K.R., and Silvola, M., Trapped modes in piezoelectric and elastic waveguides, *J. Elasticity*, 2016, vol. 124, no. 2, pp. 193–223.