ISSN 0012-2661, Differential Equations, 2022, Vol. 58, No. 3, pp. 392–404. © Pleiades Publishing, Ltd., 2022.
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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Algorithm of the Regularization Method for a Nonlinear Singularly Perturbed Integro-Differential Equation with Rapidly Oscillating Inhomogeneities

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Received October 12, 2021; revised December 23, 2021; accepted March 9, 2022

Abstract— Lomov's regularization method is generalized to nonlinear singularly perturbed integro-differential equations with rapidly oscillating right-hand side. The influence of the kernel of the integral operator, the nonlinearity, and the rapidly oscillating part on the asymptotics of the solution of the initial value problem for these equations is established. Previously, singularly perturbed linear systems of this type and nonlinear systems without oscillating inhomogeneity were studied.

DOI: 10.1134/S0012266122030090

This work is dedicated to the blessed memory
of our dear teacher Sergei Aleksandrovich Lomov
(December 10, 1922–December 6, 1993)
in connection with the 100th anniversary of his birth

INTRODUCTION

The study of various applied problems related to the properties of media with a periodic structure leads to the study of differential equations with rapidly oscillating inhomogeneities. Equations of this type are often encountered, for example, in electrical systems under the influence of high-frequency external forces. The presence of these forces creates serious problems in the numerical integration of the corresponding differential equations. Therefore, asymptotic methods are usually applied to such equations, of which the most famous are the Feshchenko–Shkil'–Nikolenko splitting method [1–3] and the Lomov regularization method [4–14]. However, both of these methods were developed mainly for singularly perturbed differential equations that do not contain an integral operator. The transition from differential equations to integro-differential ones requires significantly restructuring the algorithm of the regularization method. The integral term generates new types of singularities in solutions that differ from those already known, which complicates the development of the algorithm of the regularization method. Previously, mainly linear problems of this type were studied.

In the present paper, we consider a nonlinear problem of the form

$$\begin{aligned} L_\varepsilon y(t,\varepsilon) \equiv \varepsilon\frac{dy}{dt} - A(t)y - \int\limits_0^t K(t,s)y(s,\varepsilon)\,ds - \varepsilon f(y,t) \\ = h_1(t) + h_2(t)e^{i\beta(t)/\varepsilon}, \quad y(0,\varepsilon) = y^0, \quad t\in[0,T], \end{aligned} \tag{1}$$

where $y = y(t,\varepsilon)$ is the unknown function, $A(t)$, $K(t,s)$, $h_j(t)$, $j = 1,2$, $f(y,t)$, and $\beta(t)$ are given scalar functions ($\beta'(t)$ is the frequency of the rapidly oscillating inhomogeneities), y^0 is a constant, $\varepsilon > 0$ is a small parameter, and T is a given number.

Problem (1) will be considered under the following conditions:

1. One has the inclusions $A(t) \in C^\infty([0,T],\mathbb{R})$, $h_1(t), h_2(t), \beta(t) \in C^\infty([0,T],\mathbb{R})$, and $K(t,s) \in C^\infty(\{0 \le s \le t \le T\},\mathbb{R})$; the function $f(y,t) = \sum_{k=0}^N f_k(t)y^k$ is a polynomial in the variable y with coefficients $f_k(t) \in C^\infty([0,T],\mathbb{R})$, $k = 0,\dots,N$, $N \in \mathbb{N}$.
2. For all $t \in [0,T]$, one has the inequalities $A(t) < 0$ and $\beta'(t) > 0$.

The aim of this paper is to generalize the algorithm of the Lomov regularization method to problems of the form (1) and analyze the singularities in the solution $y(t,\varepsilon)$ introduced by the nonlinearity $f(y,t)$ and the rapidly oscillating inhomogeneity $h_2(t)e^{i\beta(t)/\varepsilon}$. For simplicity, we study the scalar version of this problem. It is assumed that its study in the multidimensional case will be the subject of our subsequent papers.

1. SOLUTION SPACE AND REGULARIZATION OF PROBLEM (1)

For convenience, we write $\lambda_1(t) \equiv A(t)$, $\lambda_2(t) \equiv \beta'(t)$, and $\sigma = \sigma(\varepsilon) = e^{i\beta(0)/\varepsilon}$, introduce the regularizing variables (see [4])

$$\tau_j = \frac{1}{\varepsilon}\int\limits_0^t \lambda_j(\theta)\,d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, \quad j = 1, 2, \tag{2}$$

and consider the extended problem

$$\begin{aligned} L_\varepsilon \tilde{y}(t,\tau,\sigma,\varepsilon) &\equiv \varepsilon\frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{2}\lambda_j(t)\frac{\partial \tilde{y}}{\partial \tau_j} - \lambda_1(t)\tilde{y} - \int\limits_0^t K(t,s)\tilde{y}(s,\psi(s)/\varepsilon,\varepsilon)\,ds - \varepsilon f(y,t) \\ &= h_1(t) + h_2(t)e^{\tau_2}\sigma, \\ \tilde{y}(t,\tau,\sigma,\varepsilon)\Big|_{\substack{t=0\\ \tau=0\\ \sigma=e^{i\beta(0)/\varepsilon}}} &= y^0 \end{aligned} \tag{3}$$

for the function $\tilde{y} = \tilde{y}(t,\tau,\sigma,\varepsilon)$, where, according to (2), $\tau = (\tau_1,\tau_2)$ and $\psi = (\psi_1,\psi_2)$. Obviously, if $\tilde{y} = \tilde{y}(t,\tau,\sigma,\varepsilon)$ is a solution of problem (3), then the function $\tilde{y} = \tilde{y}(t,\psi(t)/\varepsilon,\sigma,\varepsilon)$ is an exact solution of problem (1); therefore, problem (3) is extended with respect to problem (1). However, it cannot be considered completely regularized, since regularization with respect to the integral term

$$J\tilde{y} \equiv J\left(\tilde{y}(t,\tau,\sigma,\varepsilon)\Big|_{\substack{t=s\\ \tau=\psi(s)/\varepsilon}}\right) = \int\limits_0^t K(t,s)\tilde{y}\big(s,\psi(s)/\varepsilon,\sigma,\varepsilon\big)\,ds$$

has not been performed in it. To regularize this problem, we introduce the class M_ε asymptotically invariant with respect to the action of the operator J (see [4, p. 62]).

Consider the space U of functions $y(t,\tau,\sigma)$ representable by the sums[1]

$$\begin{gathered} y(t,\tau,\sigma) = y_0(t,\sigma) + \sum_{j=1}^{2} y_j(t,\sigma)e^{\tau_j} + \sum_{|m|=2}^{N_y} y^{(m)}(t)e^{(m,\tau)}, \\ y_j(t,\sigma), y^{(m)}(t) \in C^\infty\big([0,T],\mathbb{C}\big), \\ j = 0,\ldots,2, \quad m = (m_1,m_2), \quad 2 \le |m| \equiv m_1 + m_2 \le N_y. \end{gathered} \tag{4}$$

Note that the elements of the space U in (4) depend on the constant $\sigma = \sigma(\varepsilon)$ bounded in $\varepsilon > 0$, which does not affect the development of the algorithm set forth below; therefore, in what follows we omit the dependence on σ for brevity in the notation of the element (4) of this space. In addition, we point out that the degree of the polynomial in the exponentials in (4) depends on the element $y(t,\tau) \in U$.

[1] Here and in the following, the superscript m in parentheses in $y^{(m)}$ means $y^{(m_1,m_2)}$ and is the number of the coefficient $y^{(m)}$. It should not be confused with the number of derivative.

Let us show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the action of the operator J. The image of the operator J on the element (4) has the form

$$Jy(t,\tau) = \int_0^t K(t,s)y_0(s)\,ds + \sum_{j=1}^{2}\int_0^t K(t,s)y_j(s)e^{\varepsilon^{-1}\int_0^s \lambda_j(\theta)\,d\theta}\,ds$$
$$+ \sum_{|m|=2}^{N_y}\int_0^t K(t,s)y^{(m)}(s)e^{\varepsilon^{-1}\int_0^s (m,\lambda(\theta))\,d\theta}\,ds,$$
$$\big(m,\lambda(t)\big) \equiv m_1\lambda_1(t) + m_2\lambda_2(t).$$

Integrating by parts, we have

$$\begin{aligned}
&\int_0^t K(t,s)y_j(s)e^{\varepsilon^{-1}\int_0^s \lambda_j(\theta)\,d\theta}\,ds \\
&\quad= \varepsilon\int_0^t \frac{K(t,s)y_j(s)}{\lambda_j(s)}\,de^{\varepsilon^{-1}\int_0^s \lambda_j(\theta)\,d\theta} \\
&\quad= \varepsilon\left[\frac{K(t,t)y_j(t)}{\lambda_j(t)}e^{\varepsilon^{-1}\int_0^t \lambda_j(\theta)\,d\theta} - \frac{K(t,0)y_j(0)}{\lambda_j(0)}\right] - \varepsilon\int_0^t \left(\frac{\partial}{\partial s}\frac{K(t,s)y_j(s)}{\lambda_j(s)}\right)e^{\varepsilon^{-1}\int_0^s \lambda_j(\theta)\,d\theta}\,ds \\
&\quad= \sum_{\nu=0}^{\infty}(-1)^{\nu+1}\left[\Big(I_j^\nu\big(K(t,s)y_j(s)\big)\Big)\Big|_{s=t}e^{\tau_j} - \Big(I_j^\nu\big(K(t,s)y_j(s)\big)\Big)\Big|_{s=0}\right]\Bigg|_{\tau=\psi_j(t)/\varepsilon},
\end{aligned} \tag{5}$$

where

$$I_j^0 = \frac{1}{\lambda_j(s)}, \quad I_j^\nu = \frac{1}{\lambda_j(s)}\frac{\partial}{\partial s}I_j^{\nu-1}, \quad j=1,2, \quad \nu\ge 1.$$

Likewise, taking into account the fact that $(m,\lambda(t)) \neq 0$ for each $t\in[0,T]$, $|m|\ge 2$, we obtain

$$\begin{aligned}
&\int_0^t K(t,s)y^{(m)}(s)e^{\varepsilon^{-1}\int_0^s (m,\lambda(\theta))\,d\theta}\,ds \\
&\quad= \int_0^t \frac{K(t,s)y^{(m)}(s)}{\big(m,\lambda(s)\big)}\,de^{\varepsilon^{-1}\int_0^s (m,\lambda(\theta))\,d\theta} \\
&\quad= \varepsilon\left[\frac{K(t,t)y^{(m)}(t)}{\big(m,\lambda(t)\big)}e^{\varepsilon^{-1}\int_0^t (m,\lambda(\theta))\,d\theta} - \frac{K(t,0)y^{(m)}(0)}{\big(m,\lambda(0)\big)}\right] \\
&\qquad - \varepsilon\int_0^t \left(\frac{\partial}{\partial s}\frac{K(t,s)y^{(m)}(s)}{\big(m,\lambda(s)\big)}\right)e^{\varepsilon^{-1}\int_0^s (m,\lambda(\theta))\,d\theta}\,ds \\
&\quad= \sum_{\nu=0}^{\infty}(-1)^{\nu+1}\left[\Big(I_m^\nu\big(K(t,s)y^{(m)}(s)\big)\Big)\Big|_{s=t}e^{(m,\tau)} - \Big(I_m^\nu\big(K(t,s)y^{(m)}(s)\big)\Big)_{s=0}\right]\Bigg|_{\tau=\psi_j(t)/\varepsilon},
\end{aligned} \tag{6}$$

where we have introduced the operators

$$I_m^0 = \frac{1}{\big(m,\lambda(s)\big)}, \quad I_m^\nu = \frac{1}{\big(m,\lambda(s)\big)}\frac{\partial}{\partial s}I_m^{\nu-1}, \quad |m|\ge 2, \quad \nu\ge 1.$$

One can readily show (see, e.g., [15, pp. 291–294]) that the series (5) and (6) asymptotically converge uniformly with respect to $t \in [0,T]$ as $\varepsilon \to +0$. It follows that the class M_ε is asymptotically invariant (as $\varepsilon \to +0$) with respect to the action of the operator J.

We introduce operators $R_\nu : U \to U$ acting on each element $y(t,\tau) \in U$ of the form (4) by the rule

$$R_0 y(t,\tau) = \int\limits_0^t K(t,s) y_0(s)\, ds, \tag{7_0}$$

$$R_1 y(t,\tau) = \sum_{j=1}^{2} \left[\frac{K(t,t)y_j(t)}{\lambda_j(t)} e^{\tau_j} - \frac{K(t,0)y_j(0)}{\lambda_j(0)} \right] + \sum_{|m|}^{N_y} \left[\frac{K(t,t)y^{(m)}(t)}{(m,\lambda(t))} e^{(m,\tau)} - \frac{K(t,0)y^{(m)}(0)}{(m,\lambda(0))} \right], \tag{7_1}$$

$$\begin{aligned} R_\nu y(t,\tau) = (-1)^{\nu+1} &\left[\Big(I_j^\nu \big(K(t,s) y_j(s) \big) \Big)\Big|_{s=t} e^{\tau_j} - \Big(I_j^\nu \big(K(t,s) y_j(s) \big) \Big)\Big|_{s=0} \right] \\ &+ (-1)^{\nu+1} \sum_{|m|=2}^{N_y} \left[\Big(I_m^\nu \big(K(t,s) y^{(m)}(s) \big) \Big)\Big|_{s=t} e^{(m,\tau)} - \Big(I_m^\nu \big(K(t,s) y^{(m)}(s) \big) \Big)\Big|_{s=0} \right]. \end{aligned} \tag{7_ν}$$

Let $\tilde{y}(t,\tau,\varepsilon)$ be an arbitrary continuous function of $(t,\tau) \in [0,T] \times \{\tau : \operatorname{Re}\tau_j \le 0,\ j = 1,2\}$ with the asymptotic expansion

$$\tilde{y}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t,\tau), \quad y_k(t,\tau) \in U, \tag{8}$$

converging as $\varepsilon \to +0$ uniformly with respect to $(t,\tau) \in [0,T] \times \{\tau : \operatorname{Re}\tau_j \le 0,\ j = 1,2\}$. Then the image $J\tilde{y}(t,\tau,\varepsilon)$ of this function expands into the asymptotic series

$$J\tilde{y}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k J y_k(t,\tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} y_s(t,\tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing the extension $\tilde{J}$ of the operator J to series of the form (8); namely, we set

$$\tilde{J}\tilde{y}(t,\tau,\varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k y_k(t,\tau) \right) \overset{\text{def}}{=} \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} y_s(t,\tau). \tag{9}$$

Even though the operator $\tilde{J}$ has been introduced formally, its usefulness is obvious, because in practice one usually constructs the Nth approximation to the asymptotic solution of problem (2), which only involves the Nth partial sums of the series (8), having not a formal but a substantive meaning.

Now we can write a problem completely regularized with respect to the original problem (2),

$$\begin{gathered} L_\varepsilon \tilde{y}(t,\tau,\varepsilon) \equiv \varepsilon \frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{2} \lambda_j(t) \frac{\partial \tilde{y}}{\partial \tau_j} - A(t)\tilde{y} - \tilde{J}\tilde{y} = h_1(t) + h_2(t) e^{\tau_2}\sigma, \\ \tilde{y}(t,\tau,\varepsilon)\Big|_{\substack{t=0 \\ \tau=0 \\ \sigma=e^{i\beta(0)/\varepsilon}}} = y^0, \quad t \in [0,T], \end{gathered} \tag{10}$$

where the operator $\tilde{J}$ is given by relation (9).

2. ITERATION PROBLEMS AND THEIR SOLVABILITY IN U

Substituting the series (8) into problem (10) and matching the coefficients of like powers of ε, we obtain the following iteration problems:

$$Ly_0(t,\tau) \equiv \sum_{j=1}^{2} \lambda_j(t) \frac{\partial y_0}{\partial \tau_j} - \lambda_1(t) y_0 - R_0 y_0 = h_1(t) + h_2(t) e^{\tau_2} \sigma, \quad y_0(0,0) = y^0; \tag{11$_0$}$$

$$Ly_1(t,\tau) = -\frac{\partial y_0}{\partial t} + f(y_0,t) + R_1 y_0, \quad y_1(0,0) = 0; \tag{11$_1$}$$

$$Ly_2(t,\tau) = -\frac{\partial y_1}{\partial t} + \frac{\partial f(y_0,t)}{\partial y} y_1 + R_1 y_1 + R_2 y_0, \quad y_2(0,0) = 0; \tag{11$_2$}$$

$$\cdots$$

$$Ly_k(t,\tau) = -\frac{\partial y_{k-1}}{\partial t} + P_k(y_0,\cdots,y_{k-1},t) + R_k y_0 + \ldots + R_1 y_{k-1}, \quad y_k(0,0) = 0, \quad k > 2, \tag{11$_k$}$$

where $P_k(y_0,\ldots,y_{k-1},t)$ is some polynomial of $y_0,\ldots,y_{k-1}$ linear in y_{k-1}. We write each of the iteration problems (11_k) in the form

$$Ly(t,\tau) \equiv \sum_{j=1}^{2} \lambda_j(t) \frac{\partial y}{\partial \tau_j} - \lambda_1(t) y - R_0 y = H(t,\tau), \quad y(0,0) = y_*, \tag{12}$$

where $H(t,\tau) = H_0(t) + \sum_{j=1}^{2} H_j(t) e^{\tau_j} + \sum_{|m|=2}^{N_H} H^{(m)}(t) e^{(m,\tau)}$ is a known function in the space U, $y_* \in \mathbb{C}$ is a constant, and the operator R_0 is given by relation (7_0).

We introduce the inner (for each $t \in [0,T]$) product in the space U,

$$\langle z, w \rangle \equiv \left\langle z_0(t) + \sum_{j=1}^{2} z_j(t) e^{\tau_j} + \sum_{|m|=2}^{N_z} \big(z^{(m)}(t) e^{(m,\tau)}, w_0(t)\big) + \sum_{j=1}^{2} w_j(t) e^{\tau_j} + \sum_{|m|=2}^{N_w} w^{(m)}(t) e^{(m,\tau)} \right\rangle$$
$$\stackrel{\text{def}}{=} \big(z_0(t), w_0(t)\big) + \sum_{j=1}^{2} \big(z_j(t), w_j(t)\big) + \sum_{|m|=2}^{\min(N_z,N_w)} \big(z^{(m)}(t), w^{(m)}(t)\big),$$

where by $(\cdot,\cdot)$ we have denoted the usual inner product on $\mathbb{C}$; i.e., $(u(t), v(t)) = u(t) \cdot \bar{v}(t)$. Let us prove the following assertion.

Theorem 1. *Let conditions* 1 *and* 2 *be satisfied, and let the right-hand side* $H(t,\tau)$ *of Eq.* (12) *belong to the space* U. *Then for this equation to be solvable in the space* U *it is necessary and sufficient that the following identity holds*:

$$\big\langle H(t,\tau), e^{\tau_1} \big\rangle \equiv 0 \quad \textit{for all} \quad t \in [0,T]. \tag{13}$$

Proof. We seek a solution of Eq. (10) in the form of the element (4) of the space U (we omit the dependence of the element (4) on σ for the reason explained above),

$$y(t,\tau) = y_0(t) + \sum_{j=1}^{2} y_j(t) e^{\tau_j} + \sum_{|m|=2}^{N_H} y^{(m)}(t) e^{(m,\tau)}. \tag{14}$$

Substituting the representation (14) into Eq. (12), we obtain

$$\sum_{j=1}^{2} \big[\lambda_j(t) I - \lambda_1(t)\big] y_j(t) e^{\tau_j} + \sum_{|m|=2}^{N_H} \Big[\big(m, \lambda(t)\big) - \lambda_1(t)\Big] e^{(m,\tau)} - \lambda_1(t) y_0(t) - \int_0^t K(t,s) y_0(s)\, ds$$

$$= H_0(t) + \sum_{j=1}^{2} H_j(t)e^{\tau_j} + \sum_{|m|=2}^{N_H} H^{(m)}(t)e^{(m,\tau)}.$$

Separately matching the free terms and the coefficients of like exponentials in the last relation, we obtain the following equations:

$$-\lambda_1(t)y_0(t) - \int_0^t K(t,s)y_0(s)\,ds = H_0(t), \tag{14_0}$$

$$\left[\lambda_j(t)I - \lambda_1(t)\right]y_j(t) = H_j(t), \quad j = 1, 2, \tag{14_j}$$

$$\Big[\big(m, \lambda(t)\big) - \lambda_1(t)\Big]y^{(m)}(t) = H^{(m)}(t), \quad 2 \le |m| \le N_H. \tag{14_m}$$

By virtue of the fact that $\lambda_1(t)\neq 0$ for all $t\in[0,T]$ and the inclusion $K(t,s)\in C^\infty(\{0\le s\le t\le T\},\mathbb{R})$, Eq. ($14_0$) has a unique solution $y_0(t) \in C^\infty([0,T],\mathbb{C})$. Equation ($14_1$) has the form $0 \cdot y_1(t) = H_1(t)$. It is solvable in the space $C^\infty([0,T],\mathbb{C})$ if and only if $H_1(t) \equiv 0$, i.e., if $\langle H(t,\tau), e^{\tau_1}\rangle \equiv 0$ for all $t \in [0,T]$. Equation (14_2) has a unique solution $y_2(t) = [\lambda_2(t) - \lambda_1(t)]^{-1}H_2(t)$ in the class $C^\infty([0,T],\mathbb{C})$.

Consider Eq. (14_m) in greater detail,

$$\left[(m_1-1)\lambda_1(t) + im_2\beta'(t)\right]y^{(m)}(t) = H^{(m)}(t), \quad 2 \le m_1 + m_2 \le N_H.$$

Let us show that the coefficient $(m_1 - 1)\lambda_1(t) + im_2\beta'(t)$ is nonzero for any $t \in [0,T]$ and $2 \le m_1 + m_2 \le N_H$. Indeed, assume the contrary: $(m_1 - 1)\lambda_1(t) + im_2\beta'(t) = 0$ for some t. Separating the imaginary and real parts in this equality, we obtain

$$(m_1 - 1)\lambda_1(t) = 0, \quad m_2\beta'(t) = 0.$$

The second equality in this system is possible only if $m_2 = 0$. However, then $m_1 \ge 2$ and the first equality is not satisfied, because $\lambda_1(t) < 0$. Hence $(m_1 - 1)\lambda_1(t) + im_2\beta'(t) \neq 0$ for all $t \in [0,T]$, $2 \le m_1 + m_2 \le N_H$. It follows that each equation (14_m) has a unique solution, and this solution can be written in the form

$$y^{(m)}(t) = \Big[\big(m, \lambda(t)\big) - \lambda_1(t)\Big]^{-1} H^{(m)}(t), \quad 2 \le |m| \le N_H.$$

Thus, for Eq. (12) to be solvable, it is necessary and sufficient that identity (13) holds. The proof of the theorem is complete.

Remark. Under the assumptions of Theorem 1 and condition (13), Eq. (12) has the following solution in the space U:

$$\begin{aligned} y(t,\tau) = y_0(t) + \alpha_1(t)e^{\tau_1} + \left[\lambda_2(t) - \lambda_1(t)\right]^{-1}H_2(t)e^{\tau_2} \\ + \sum_{|m|=2}^{N_H}\Big[\big(m,\lambda(t)\big) - \lambda_1(t)\Big]^{-1}H^{(m)}(t)e^{(m,\tau)}, \end{aligned} \tag{15}$$

where $\alpha_1(t) \in C^\infty([0,T],\mathbb{C})$ is an arbitrary function.

3. UNIQUE SOLVABILITY OF THE GENERAL ITERATION PROBLEM IN SPACE U. REMAINDER THEOREM

As can be seen from (15), the solution of Eq. (12) is determined nonuniquely. However, if we subject it to the additional conditions

$$y(0,0) = y_*, \quad \left\langle -\frac{\partial y}{\partial t} + R_1 y + Q(t,\tau), e^{\tau_1}\right\rangle \equiv 0 \quad \text{for all} \quad t \in [0,T], \tag{16}$$

where $Q(t,\tau) = Q_0(t) + \sum_{j=1}^{2} Q_i(t)e^{\tau_j} + \sum_{|m|=2}^{N_Q} Q^{(m)}(t)e^{(m,\tau)}$ is a known function in the space U and y_* is a constant in $\mathbb{C}$, then problem (12) will be uniquely solvable in the space U. Note that conditions (16) are natural for the entire series of iteration problems (11_k) and arise as the solvability conditions (13) when switching from problem (11_k) to problem (11_{k+1}). The following assertion holds true.

Theorem 2. *Let conditions* 1 *and* 2 *be satisfied, and let the right-hand side* $H(t,\tau)$ *of Eq.* (12) *belong to the space* U *and satisfy the orthogonality condition* (13). *Then Eq.* (12) *is uniquely solvable in* U *under the additional conditions* (16).

Proof. Under condition (13), Eq. (12) has a solution (15) in the space U, where $\alpha_1(t) \in C^\infty([0,T],\mathbb{C})$ is yet an arbitrary function. Let us subject solution (15) to the first condition in (16), i.e., require that $y(0,0) = y_*$. We conclude that $\alpha_1(0) = y^*$, where we have denoted

$$y^* = y_* + \lambda_1^{-1}(0)H_0(0) - \big[\lambda_2(0) - \lambda_1(0)\big]^{-1} H_2(0) - \sum_{|m|=2}^{N_H} \Big[\big(m,\lambda(0)\big) - \lambda_1(0)\Big]^{-1} H^{(m)}(0). \quad (17)$$

Now we subject solution (15) to the second condition in (16). Let us first calculate[2] the expression

$$\begin{aligned} -\frac{\partial y_0}{\partial t} &+ R_1 y_0 + Q(t,\tau) \\ &= -\dot{y}_0(t) - \dot{\alpha}_1(t)e^{\tau_1} - \left(\frac{H_2(t)}{\lambda_2(t)-\lambda_1(t)}\right)^{\bullet} e^{\tau_2} - \sum_{|m|=2}^{N_H}\left(\frac{H^{(m)}(t)}{\big(m,\lambda(t)\big)-\lambda_1(t)}\right)^{\bullet} e^{(m,\tau)} \\ &\quad + \frac{K(t,t)\alpha_1(t)}{\lambda_1(t)}e^{\tau_1} - \frac{K(t,0)\alpha_1(0)}{\lambda_1(0)} + \frac{K(t,t)H_2(t)}{\lambda_1(t)\big[\lambda_2(t)-\lambda_1(t)\big]}e^{\tau_2} \\ &\quad - \frac{K(t,0)H_2(0)}{\lambda_1(0)\big[\lambda_2(0)-\lambda_1(0)\big]} + Q_0(t) + \sum_{j=1}^{2} Q_i(t)e^{\tau_j} + \sum_{|m|=2}^{N_Q} Q^{(m)}(t)e^{(m,\tau)}. \end{aligned}$$

Therefore, the second condition in (16) leads to the ordinary differential equation

$$-\dot{\alpha}_1(t) + \frac{K(t,t)\alpha_1(t)}{\lambda_1(t)} = Q_1(t).$$

Supplementing this equation with the initial condition (17), we uniquely determine the function

$$\alpha_1(t) = \exp\left(\int_0^t \frac{K(s,s)}{\lambda_1(s)}\,ds\right)\left(y^* + \int_0^t \exp\left(-\int_0^x \frac{K(\theta,\theta)}{\lambda_1(\theta)}\,d\theta\right) Q_1(x)dx\right)$$

and hence construct the solution (15) of problem (12) in the space U in a unique way. The proof of the theorem is complete.

Applying Theorems 1 and 2 to the iteration problems (11_k), we uniquely find their solutions in the space U and construct the series (8). Let us compose the partial sum $S_n(t,\tau,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k y_k(t,\tau)$ of this series and denote its restriction to $\tau = \psi(t)/\varepsilon$ by $y_{\varepsilon n}(t)$. The following assertion holds.

Lemma. *Let conditions* 1 *and* 2 *be satisfied. Then the function* $y_{\varepsilon n}(t)$ *satisfies problem* (1) *modulo terms containing* ε^{n+1}; *i.e.*,

$$\varepsilon\frac{dy_{\varepsilon n}(t)}{dt} - A(t)y_{\varepsilon n}(t) - \int_0^t K(t,s)y_{\varepsilon n}(s,\varepsilon)\,ds - \varepsilon f(y_{\varepsilon n},t)$$

[2] Here and in what follows, a bold dot as a superscript on a parenthesis means differentiation with respect to t of the expression in the parentheses.

$$= h_1(t) + h_2(t)e^{i\beta(t)/\varepsilon} + \varepsilon^{n+1}F(t,\varepsilon), \quad y_{\varepsilon n}(0,\varepsilon) = y^0, \quad t \in [0,T],$$

where $\|F(t,\varepsilon)\|_{C[0,T]} \le \overline{F}$, $\overline{F} > 0$ *is a constant independent of* $\varepsilon \in (0,\varepsilon_0]$, *and* $\varepsilon_0 > 0$ *is sufficiently small.*

Proof. By L_0 we denote the operator

$$L_0 \equiv \sum_{j=1}^{2} \lambda_j(t)\frac{\partial}{\partial \tau_j} - A(t).$$

Substituting $y_0(t,\tau)$, ..., $y_n(t,\tau)$ into the first n equations in system (11_k), we obtain identities. Let us multiply them by 1, ε, ..., and ε^n, respectively, and add them up. As a result, we obtain

$$\begin{aligned}
&L_0(y_0 + \varepsilon y_1 + \dots + \varepsilon^n y_n) - R_0(y_0 + \varepsilon y_1 + \dots + \varepsilon^n y_n)\\
&\quad \equiv h_1(t) + h_2(t)e^{\tau_2}\sigma - \varepsilon\left(\frac{\partial y_0}{\partial t} + \varepsilon\frac{\partial y_1}{\partial t} + \dots + \varepsilon^{n-1}\frac{\partial y_{n-1}}{\partial t}\right) + \varepsilon R_1 y_0 + \varepsilon^2(R_1y_1 + R_2y_0)\\
&\qquad + \varepsilon^n(R_1y_{n-1} + \dots + R_ny_0) + \left[\varepsilon f(y_0,t) + \varepsilon^2\left(\frac{\partial f(y_0,t)}{\partial y}y_1\right) + \dots + \varepsilon^N P_n(y_0,\dots,y_{n-1},t)\right].
\end{aligned}$$

Let us restrict this to $\tau = \psi(t)/\varepsilon$ taking into account the identities

$$\begin{aligned}
&e^{\tau_2}\sigma|_{\tau=\psi(t)/\varepsilon} \equiv e^{i\beta(t)/\varepsilon}, \qquad \sum_{0\le|m|\le k} z^{(m)}(t)e^{(m,\tau)}|_{\tau=\psi(t)/\varepsilon} \equiv \sum_{0\le|m|\le k} z^{(m)}(t)e^{(m,\psi(t)/\varepsilon)},\\
&L_0 v(t,\tau,\varepsilon)|_{\tau=\psi(t)/\varepsilon} \equiv \varepsilon\frac{dv\big(t,\psi(t)/\varepsilon,\varepsilon\big)}{dt} - A(t)v\big(t,\psi(t)/\varepsilon,\varepsilon\big) - \varepsilon\frac{\partial v\big(t,\psi(t)/\varepsilon,\varepsilon\big)}{\partial t}.
\end{aligned}$$

We have

$$\begin{aligned}
\varepsilon\frac{dy_{\varepsilon n}}{dt} - A(t)y_{\varepsilon n}(t) &= \varepsilon^{n+1}\frac{\partial y_n(t,\psi)}{\partial t} + h_1(t) + h_2(t)e^{i\beta(t)/\varepsilon} + \sum_{r=0}^{N}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s(t,\psi)\\
&\quad + \varepsilon\left[f(y_0,t) + \varepsilon\frac{\partial f(y_0,t)}{\partial y}y_1 + \dots + \varepsilon^{n-1}P_n(y_0,\dots,y_{n-1},t)\right],
\end{aligned}$$

or

$$\begin{aligned}
\varepsilon\frac{dy_{\varepsilon n}}{dt} &= A(t)y_{\varepsilon n}(t) + \int_0^t K(t,s)y_{\varepsilon n}(s)\,ds + h_1(t) + h_2(t)e^{i\beta(t)/\varepsilon} + \varepsilon^{n+1}\frac{\partial y_n(t,\psi)}{\partial t}\\
&\quad + \varepsilon f\big(y_{\varepsilon n}(t),t\big) - \left[\int_0^t K(t,s)y_{\varepsilon n}(s)\,ds - \sum_{r=0}^{n}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s(t,\psi)\right]\\
&\quad - \varepsilon\left[f\big(y_{\varepsilon n}(t),t\big) - f(y_0,t) - \frac{\partial f(y_0,t)}{\partial y}y_1 - \dots - \varepsilon^{n-1}P_n(y_0,\dots,y_{n-1},t)\right].
\end{aligned}$$

According to the definition of the operators R_k, the first bracketed expression in the last identity can be represented in the form $\varepsilon^{n+1}M_1(t,\varepsilon)$, and by the construction of problems (11_0), ..., (11_n), the second bracketed expression can be represented in the form $\varepsilon^{n+1}M_2(t,\varepsilon)$, where $\|M_i\|_{C[0,T]} \le \text{const}$ (for all $\varepsilon \in (0,\varepsilon_0]$), $i = 1,2$. Denoting $F(t,\varepsilon) \equiv -M_1(t,\varepsilon) - M_2(t,\varepsilon) + \partial y_N(t,\psi)/\partial t$, we obtain the assertion of the Lemma.

When justifying the asymptotic convergence of the formal solution $y_{\varepsilon n}(t)$ to the exact solution $y(t,\varepsilon)$, we use the following assertion about the solvability of the operator equation $P_\varepsilon(u) = 0$ (see, e.g., [16, pp. 187–188]).

Theorem (Srubshchik–Yudovich). *Let an operator P_ε act from a Banach space B_1 into a Banach space B_2, and assume that the first two derivatives of P_ε are continuous in some ball $\{\|u-u_0\|\le r\}\subset B_1$. Assume also that there exists an operator $\Gamma_\varepsilon\equiv[P'_\varepsilon(u_0)]^{-1}$ and the conditions*

$$\begin{aligned}&(1\text{a})\quad \|\Gamma_\varepsilon\|\le c_1\varepsilon^{-k};\\&(2\text{a})\quad \|P_\varepsilon(u_0)\|\le c_2\varepsilon^{m}\quad (m>2k);\\&(3\text{a})\quad \|P''_\varepsilon(u)\|\le c_3\end{aligned}$$

are satisfied. Then for sufficiently small $\varepsilon>0$ ($\varepsilon\in(0,\varepsilon_0]$) the equation $P_\varepsilon(u)=0$ has a solution $u_*\in B_1$ satisfying the inequality $\|u_*-u_0\|_{B_1}\le c\varepsilon^{m-k}$. Here c, c_1, c_2, and c_3 are some positive constants independent of $\varepsilon\in(0,\varepsilon_0]$.

Applying this theorem to the equation

$$\begin{aligned}P_\varepsilon(u)\equiv\varepsilon\frac{du}{dt}-A(t)u-\int\limits_0^t K(t,s)u(s,\varepsilon)\,ds\\ -\varepsilon f(u+y^0,\varepsilon)-A(t)y^0-\int\limits_0^t K(t,s)y^0\,ds-h_1(t)-h_2(t)e^{i\beta(t)/\varepsilon}=0,\end{aligned}$$

we arrive at the following result (see [16, pp. 190–192]).

Theorem 3. *Let conditions* 1 *and* 2 *be satisfied for Eq.* (1). *Then for $\varepsilon\in(0,\varepsilon_0]$ ($\varepsilon_0>0$ is sufficiently small) problem* (1) *has a unique solution $y(t,\varepsilon)\in C^1([0,T],\mathbb{C})$, which satisfies the estimate*

$$\big\|y(t,\varepsilon)-y_{\varepsilon n}(t)\big\|_{C[0,T]}\le c_n\varepsilon^{n+1},\quad n=0,1,2,\dots,$$

where $y_{\varepsilon n}(t)$ is the restriction (for $\tau=\psi(t)/\varepsilon$) of the nth partial sum of the series (8) (*with coefficients $y_k(t,\tau)\in U$ satisfying the iteration problems* (11_k)) *and the constant $c_n>0$ is independent of ε for $\varepsilon\in(0,\varepsilon_0]$.*

4. CONSTRUCTING A SOLUTION OF THE FIRST ITERATION PROBLEM

Using Theorem 1, we find a solution of the first iteration problem (11_0). Since the right-hand side $h_1(t)+h_2(t)e^{\tau_2}\sigma$ of Eq. (14_0) satisfies condition (13), it follows (according to (15)) that this equation has a solution in the space U of the form

$$y_0(t,\tau)=y_0^{(0)}(t)+\alpha_1^{(0)}(t)e^{\tau_1}+y_2(t)e^{\tau_2},\tag{18}$$

where $\alpha_1^{(0)}(t)\in C^\infty([0,T],\mathbb{C})$ is for now an arbitrary function and $y_0^{(0)}(t)$ is a solution of the integral equation $-\lambda_1(t)y_0^{(0)}(t)-\int_0^t K(t,s)y_0^{(0)}(s)\,ds=h_1(t)$, $y_2(t)=[\lambda_2(t)-\lambda_1(t)]^{-1}h_2(t)\sigma$. Subjecting family (18) to the initial condition $y_0(0,0)=y^0$, we obtain $y_0^{(0)}(0)+\alpha_1(0)+y_2^{(0)}(0)=y^0$; i.e.,

$$\alpha_1(0)=y^0+\lambda_1^{-1}(0)h_1(0)-\big[\lambda_2(0)-\lambda_1(0)\big]^{-1}h_2(0)\sigma.\tag{19}$$

To calculate the function $\alpha_1^{(0)}(t)$ completely, we proceed to the next iteration problem (11_1). Substituting the solution (18) of Eq. (14_0) into this problem, we obtain the equation

$$\begin{aligned}Ly_1(t,\tau)=-\frac{d}{dt}y_0^{(0)}(t)-\dot\alpha_1^{(0)}(t)e^{\tau_1}-\dot y_2(t)e^{\tau_2}\\+f(y_0^{(0)}(t)+\alpha_1^{(0)}(t)e^{\tau_1}+y_2(t)e^{\tau_2},t)+R_1(y_0^{(0)}(t)+\alpha_1^{(0)}(t)e^{\tau_1}+y_2(t)e^{\tau_2}).\end{aligned}$$

Isolating terms with the exponential e^{τ_1} on the right-hand side in this equation and subjecting them to the orthogonality condition (13), we arrive at the equation

$$-\dot{\alpha}_1^{(0)}(t)+\left(\frac{\partial f\big(y_0^{(0)}(t),t\big)}{\partial y}+\frac{K(t,t)}{\lambda_1(t)}\right)\alpha_1^{(0)}(t)=0;$$

supplementing this equation with the initial condition (19), we find $\alpha_1^{(0)}(t)$,

$$\alpha_1^{(0)}(t)=\Big(y^0+\lambda_1^{-1}(0)h_1(0)-\big[\lambda_2(0)-\lambda_1(0)\big]^{-1}h_2(0)\sigma\Big)\exp\left(\int\limits_0^t\frac{\partial f\big(y_0^{(0)}(\theta),\theta\big)}{\partial y}+\frac{K(\theta,\theta)}{\lambda_1(\theta)}\right)d\theta,$$

and hence the solution (18) of problem (11_0) in the space U is uniquely determined. In this case, the leading term of the asymptotics has the form

$$\begin{aligned}y_{\varepsilon 0}(t)=y_0^{(0)}(t)&+\left(y^0+\lambda_1^{-1}(0)h_1(0)-\big[\lambda_2(0)-\lambda_1(0)\big]^{-1}h_2(0)\exp\left(\frac{i}{\varepsilon}\beta(0)\right)\right)\\&\times\exp\left(\int\limits_0^t\frac{\partial f(y_0^{(0)}(\theta),\theta)}{\partial y}+\frac{K(\theta,\theta)}{\lambda_1(\theta)}\right)d\theta\exp\left(\frac{1}{\varepsilon}\int\limits_0^t\lambda_1(\theta)\,d\theta\right)\\&+\big[\lambda_2(t)-\lambda_1(t)\big]^{-1}h_2(t)\exp\left(\frac{i}{\varepsilon}\beta(t)\right).\end{aligned}\tag{20}$$

Let us analyze it.

It can be seen from the expression (20) for $y_{\varepsilon 0}(t)$ that the construction of the leading term of the asymptotics of the solution of problem (1) is essentially affected by the rapidly oscillating inhomogeneity $h_2(t)e^{i/\varepsilon\beta(t)}$, the kernel $K(t,s)$ of the integral operator, and the nonlinearity $f(y,t)$. Leaving the point $y=y^0$ at the time $t=0$, the exact solution $y(t,\varepsilon)$ to problem (1) performs fast oscillations around the solution $y_0^{(0)}(t)$ of the following integral equation for $t>0$:

$$-\lambda_1(t)y_0^{(0)}(t)-\int\limits_0^t K(t,s)y_0^{(0)}(s)\,ds=h_1(t),$$

without tending to any limit as $\varepsilon\to+0$. One can readily see that this equation has been obtained from the degenerate equation ($\varepsilon=0$) for (1) after discarding the rapidly oscillating inhomogeneity in (1). If $h_2(t)\equiv 0$, i.e., there is no rapidly oscillating inhomogeneity, then, leaving the point $y=y^0$ at time $t=0$, the solution $y(t,\varepsilon)$ of problem (1) rapidly (at an exponential rate) tends as $\varepsilon\to+0$ to the solution of the degenerate equation $-A(t)\bar{\bar{y}}-\int_0^t K(t,s)\bar{\bar{y}}(s)\,ds=h_1(t)$.

5. SUPPLEMENT: A BRIEF OUTLINE OF THE DEVELOPMENT OF THE LOMOV REGULARIZATION METHOD FOR SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL EQUATIONS

In the late 1950s–early 1960s, S.A. Lomov, studying the Lighthill model equation, came to the idea of regularizing singular perturbations by passing to a higher-dimensional space. This idea was deeply developed by him in subsequent works and leads to the creation of a *method of regularization of singular perturbations*, most fully described in his monograph [4]. The regularization method allows constructing asymptotic solutions of singularly perturbed problems in the form of series in powers of a small parameter, the sum of which is *pseudoanalytic* under some additional restrictions on the initial data of the problem. The latter means that the regularized series converge not only asymptotically, but also in the usual sense in some annular neighborhood $0<|\varepsilon|<\varepsilon_0$ of the point $\varepsilon=0$. A new direction has been formed in the theory of differential equations—the analytical theory of singular perturbations. The results on pseudoanalyticity by Lomov were generalized to nonlinear ordinary differential equations, partial differential equations, and equations in Banach space by his students V.F. Safonov, V.I. Prokhorenko, A.A. Bobodzhanov, and V.I. Kachalov. At present, the analytical theory of singular perturbations, thanks to the research by V.I. Kachalov, is in a very satisfactory state.

However, the problem of regularizing singularly perturbed integro-differential equations remained practically unexplored. The first application of the regularization method to such equations was given by S.A. Lomov (1970) and described in detail in the monograph [4, Ch. 4]. In this paper, we consider a singularly perturbed Volterra-type system

$$\varepsilon\frac{dy}{dt} = A(t)y + \int\limits_0^t K(t,s)y(s,\varepsilon)\,ds + h(t), \quad y(0,\varepsilon) = y^0, \quad t\in[0,T], \tag{S.1}$$

under the conditions of *stability of the spectrum* $\{\lambda_j(t)\}$ of the operator $A(t)$,

$$\lambda_i(t)\neq 0, \quad \lambda_j(t)\neq\lambda_i(t), \quad i\neq j, \quad i,j=1,\dots,n, \quad \text{for all} \quad t\in[0,T]. \tag{S.2}$$

(Unlike the papers by the Vasil'eva–Butuzov–Imanaliev school, it is assumed here that the inequalities $\operatorname{Re}\lambda_j(t)\le 0$ hold, i.e., in particular, pure imaginary spectral points are allowed.) The main difficulty to be overcome in equations of the type (S.1) is the regularization of the integral operator

$$Jy = \int\limits_0^t K(t,s)y(s,\varepsilon)\,ds.$$

While the differential part of problem (S.1) admits a fairly self-apparent extension

$$\varepsilon\frac{\partial\tilde y}{\partial t} + \sum_{j=1}^n \lambda_j(t)\frac{\partial\tilde y}{\partial\tau_j} - A(t)\tilde y$$

after introducing the regularizing variables

$$\tau_j = \frac{1}{\varepsilon}\int\limits_0^t \lambda_j(\theta)\,d\theta \equiv \frac{\psi(t)}{\varepsilon}, \quad j=1,\dots,n, \tag{S.3}$$

in the indicated variables the integral operator takes the form

$$J\tilde y = \int\limits_0^t K(t,s)\tilde y\left(s,\frac{\psi(s)}{\varepsilon},\varepsilon\right)ds,$$

and its extension with respect to the independent variables τ_j becomes problematic.

The solution of this problem was given by S.A. Lomov himself. For the simplest case of integro-differential equations of the type (S.1), he proposed introducing a space that is invariant under the action of the integral operator J obtained in a natural way from the space of resonant solutions by integrating by parts its elements (see [4]). This idea of fundamental importance made it possible to move the process of generalization of the regularization method to integro-differential systems from the "dead point." Nevertheless, for ten years (1979–1989) not a single work devoted to this topic was published. Only in the 1990s, interest in integro-differential equations was renewed owing to the study of the relationship between the Lomov regularization method and the Larionov equivalent differential correspondence method [17]. Larionov's method considers integro-differential equations

$$\varepsilon\frac{dy}{dt} = Ay + \int\limits_0^t K\left(\frac{t}{\varepsilon}-\frac{s}{\varepsilon}\right)y(s,\varepsilon)\,ds + h(t), \quad y(0,\varepsilon) = y^0, \quad t\in[0,T],$$

with a constant matrix A and with rapidly varying kernels of the type $K(t/\varepsilon - s/\varepsilon) \equiv e^{-\nu(t/\varepsilon - s/\varepsilon)}$ ($\nu = \text{const}$). Such equations often occur in applications (see, e.g., [17]), but a regularization method for them has not been developed.

When generalizing Lomov's idea to integro-differential equations with rapidly varying kernels, Safonov proposed to consider systems of singularly perturbed integro-differential equations of the form

$$\varepsilon \frac{dy}{dt} = A(t)y + \int\limits_0^t e^{\varepsilon^{-1} \int_s^t \mu(\theta)\, d\theta} K(t,s) y(s,\varepsilon)\, ds + h(t),$$
$$y(0,\varepsilon) = y^0, \quad t \in [0,T], \quad \varepsilon > 0, \tag{S.4}$$

with a variable matrix $A(t)$ and with a scalar function $\mu(t)$ called the spectral value of the kernel of the integral operator. The paper [18] considered the case of instability of the spectral value ($\mu(t) = t^r l(t)$, $l(t) < 0$) under conditions (S.2) on the spectrum of the matrix $A(t)$, and it was shown that the regularization of problem (S.4) involves not only the regularizing functions (S.3) but also the special functions

$$\sigma_k = e^{\varepsilon^{-1} \int_0^t \mu(\theta)\, d\theta} \int\limits_0^t e^{\varepsilon^{-1} \int_0^s \mu(\theta)\, d\theta} \frac{s^k}{k!}\, ds \quad (k = 0, \ldots, r-1) \tag{S.5}$$

induced by the point $t = 0$ of instability of the spectral value $\mu(t)$ of the kernel of the integral operator as well as the function $\mu(t)$ itself. It follows that if $\mu(t) < 0$ (for all $t \in [0,T]$), then, to regularize problem (S.4), in addition to the regularizing functions (S.3), one should introduce the regularizing variables (S.5) and one more additional variable $\tau_{n+1} = \varepsilon^{-1} \int_0^t \mu(\theta)\, d\theta$.

The prospect of further research is related to the consideration of integro-differential partial differential equations, nonlinear integro-differential equations of the Volterra and Fredholm type, as well as nonlinear integro-differential equations with rapidly oscillating coefficients and inhomogeneities. The last type of equations includes the equation in problem (1) considered in the present paper, as well as the equations of problems in the papers [10–12, 14].

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