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PARTIAL DIFFERENTIAL EQUATIONS

Classical Solutions of Hyperbolic Differential–Difference Equations in a Half-Space

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Abstract—Three-parameter families of solutions are constructed for two hyperbolic differential–difference equations with shift operators of the general-type acting with respect to all spatial variables. We prove theorems showing that the solutions obtained are classical provided that the real parts of the symbols of the corresponding differential–difference operators are positive. Classes of equations for which these conditions are satisfied are given.

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INTRODUCTION

In recent years, functional–differential equations, or, in other words, differential equations with a deviating argument, have become widespread in applications of mathematics. The systematic study of equations with a deviating argument began in the 1940s in connection with applications to automatic control theory and was associated with the research by Myshkis [1], Pinney [2], Bellman and Cooke [3], Kamenskii and Skubachevskii [4], El'sgol'ts and Norkin [5], and Hale [6].

The equations

$$u'(t) = f\big(t, u(t), u(t-h)\big)$$

and

$$u'(t) = f\big(t, u(t), u(t-h), u'(t-h)\big),$$

where $h > 0$ is a given constant, are simple examples of ordinary differential equations with a deviating argument.

Such equations arise where there is a delay element in the system to be modeled, with the effect that the system evolution rate is determined by the state of the system not only at the current time t but also at the previous time $t-h$.

The presence of the "delay" h sometimes leads not only to quantitative but also to qualitative changes in the statements of the problems and the properties of their solutions. For example, all the values of the unknown function $u(t)$ for $t_0 - h \le t \le t_0$ rather than $u(t_0)$ alone (as for classical equations) are specified as an initial condition for first-order equations.

The stability of systems described by such equations can be analyzed by studying the roots of the corresponding characteristic equations, which turn out to be transcendental rather than algebraic as for ordinary differential equations.

Real-life objects may be described by functional–differential equations of a more complex structure. In particular, the equations may include several discrete delays rather than one and also contain a "distributed delay;" this leads to the study of integro-differential equations that, in the linear case, can have, for example, the form

$$u'(t) = \int\limits_{t_0}^{t} K(t,s)u(s)ds + f(t), \quad t \ge t_0.$$

These equations describe systems with memory.

Essential results in the study of problems for functional–differential equations of various classes were obtained by Skubachevskii [7, 8], Vlasov and Medvedev [9, 10], Zarubin [11, 12], Muravnik [14–19], Razgulin [20], Rossovskii [21], and other authors.

Differential–difference equations form a special class of functional–differential equations for which the theory of boundary value problems is currently developed. Problems for elliptic differential–difference equations in bounded domains have been studied quite comprehensively by now; the theory for such equations was created and developed by Skubachevskii [7, 8].

Problems for elliptic differential–difference equations in unbounded domains have been studied to a much lesser extent. An extensive study of such problems is presented in Muravnik's papers [15–20]. In particular, boundary value problems for multidimensional elliptic differential–difference equations are considered in [18–20].

Parabolic equations with time deviations (or with variable delays) in higher derivatives were studied by Vlasov [9]. Boundary value problems in bounded domains for parabolic differential–difference equations with shifts in the spatial variables were studied by Shamin and Skubachevskii [23] and Selitskii and Skubachevskii [22]. In the case of an unbounded domain, problems for such equations were studied in Muravnik's monograph [13].

Zarubin [12] considered the Cauchy problem for a hyperbolic equation with time delay arising in the mathematical modeling of processes in media with fractal geometry. Vlasov and Medvedev [10] studied hyperbolic differential–difference equations for the case where the shift operators also act on the time variable.

As far as the present author is aware, at present, there are few papers dealing with hyperbolic differential–difference equations containing shifts with respect to the spatial variable. In [25–29], families of classical solutions are constructed for two-dimensional hyperbolic equations with shifts in the only space variable x ranging on the entire real line; the shifts occur either in the potentials or in the highest derivative.

In the present paper, we study the existence of smooth solutions of two hyperbolic differential–difference equations in the half-space $\{(x,t)|\, x\in\mathbb{R}^n,\ t>0\}$. The first of these equations contains superpositions of differential operators and shift operators with respect to each of the spatial variables,

$$u_{tt}(x,t) = L_1 u \overset{\text{def}}{=} a^2\sum_{j=1}^{n} u_{x_jx_j}(x,t) + \sum_{j=1}^{n} b_j u_{x_jx_j}(x_1,\dots,x_{j-1},x_j-h_j,x_{j+1},\dots,x_n,t), \tag{1}$$

where $a>0$, $b_1,\dots,b_n$, and $h_1,\dots,h_n$ are given real numbers.

The second equation contains a sum of differential operators and shift operators with respect to each of the spatial variables,

$$u_{tt}(x,t) = L_2 u \overset{\text{def}}{=} c^2\sum_{j=1}^{n} u_{x_jx_j}(x,t) - \sum_{j=1}^{n} d_j u(x_1,\dots,x_{j-1},x_j-l_j,x_{j+1},\dots,x_n,t), \tag{2}$$

where $c>0$, $d_1,\dots,d_n$, and $l_1,\dots,l_n$ are given real numbers.

Definition 1. A function $u(x,t)$ is called a *classical solution* of Eq. (1) (respectively, Eq. (2)) if the derivatives u_{tt} and $u_{x_jx_j}$ $(j=1,\dots,n)$ exist in the classical sense (i.e., as limits of finite-difference ratios) at each point of the half-space $(x,t)\in\mathbb{R}^n\times(0,+\infty)$ and if Eq. (1) (respectively, Eq. (2)) holds at each point of the half-space.

Definition 2. A differential–difference operator L is said to be *positive* if the real part of the symbol of this operator is positive, i.e., the condition $\mathrm{Re}L(\xi)>0$ is satisfied for all $\xi\in\mathbb{R}^n$.

The real part of the symbol of the differential–difference operator $-L_1$ in Eq. (1) is equal to

$$-\mathrm{Re}L_1(\xi) = a^2|\xi|^2 + \sum_{j=1}^{n} b_j\xi_j^2\cos(h_j\xi_j).$$

In what follows, we assume the operator $-L_1$ to be positive, i.e., the condition

$$a^2|\xi|^2 + \sum_{j=1}^{n} b_j\xi_j^2\cos(h_j\xi_j) > 0 \tag{3}$$

to be satisfied for all $\xi\in\mathbb{R}^n$.

The operator $-L_2$ of Eq. (2) will also be assumed to be positive in the subsequent exposition; i.e., we assume that the condition

$$c^2|\xi|^2 + \sum_{j=1}^{n} d_j \cos(l_j \xi_j) > 0 \tag{4}$$

is satisfied for all $\xi \in \mathbb{R}^n$.

1. CONSTRUCTION OF SOLUTIONS OF EQUATION (1)

To find solutions of the equation, we use the classical operational scheme [29, Sec.10], whereby one formally applies the Fourier transform with respect to the n-dimensional variable x to Eq. (1), $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{i\xi\cdot x} dx$, and passes to the dual variable ξ.

In view of the formulas $F_x[\partial_x^\alpha \partial_t^\beta f] = (-i\xi)^\alpha \partial_t^\beta F_x[f]$ and $F_x[f(x-x_0)] = e^{ix_0\cdot\xi} F_x[f]$ [31, Sec. 9], for the function $\widehat{u}(\xi,t) := F_x[u](\xi,t)$ we obtain the initial value problem

$$\frac{d^2\widehat{u}}{dt^2} = -\left(a^2|\xi|^2 + \sum_{j=1}^{n} b_j \xi_j^2 \cos(h_j\xi_j) + i\sum_{j=1}^{n} b_j \xi_j^2 \sin(h_j\xi_j)\right)\widehat{u}, \quad \xi \in \mathbb{R}^n, \tag{5}$$

$$\widehat{u}(0) = 0, \quad \widehat{u}_t(0) = 1. \tag{6}$$

For convenience, in the subsequent calculations we use the notation

$$\alpha(\xi) := \sum_{j=1}^{n} b_j \xi_j^2 \cos(h_j\xi_j),$$

$$\beta(\xi) := \sum_{j=1}^{n} b_j \xi_j^2 \sin(h_j\xi_j).$$

Then Eq. (5) becomes

$$\frac{d^2\widehat{u}}{dt^2} = -\left(a^2|\xi|^2 + \alpha(\xi) + i\,\beta(\xi)\right)\widehat{u}, \quad \xi \in \mathbb{R}^n,$$

and the roots of the corresponding characteristic equation are determined by the formula

$$k_{1,2} = \pm\sqrt{-\left(a^2|\xi|^2 + \alpha(\xi) + i\,\beta(\xi)\right)} = \pm\, i\sqrt{a^2|\xi|^2 + \alpha(\xi) + i\,\beta(\xi)} = \pm\, i\rho(\xi)e^{i\,\varphi(\xi)},$$

where

$$\rho(\xi) := \left[\left(a^2|\xi|^2 + \alpha(\xi)\right)^2 + \beta^2(\xi)\right]^{1/4}, \tag{7}$$

$$\varphi(\xi) := \frac{1}{2}\arctan\frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)}. \tag{8}$$

Note that the functions (7) and (8) are well defined for any $\xi \in \mathbb{R}^n$ under condition (3), because the radicand in formula (7) is always positive, and the denominator in the argument of the arctangent in (8) does not vanish.

Thus, the general solution of Eq. (5) has the form

$$\widehat{u}(\xi,t) = C_1(\xi)e^{i\,t\,\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)]} + C_2(\xi)e^{-i\,t\,\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)]},$$

where $C_1(\xi)$ and $C_2(\xi)$ are arbitrary constants depending on the parameter ξ; to determine these constants, we substitute the function $\widehat{u}(\xi,t)$ into the initial conditions (6). From the system

$$\begin{cases} C_1(\xi) + C_2(\xi) = 0 \\ C_1(\xi) - C_2(\xi) = \dfrac{1}{i\,\rho(\xi)\left[\cos\varphi(\xi) + i\sin\varphi(\xi)\right]}, \end{cases}$$

we find the values of these constants,

$$C_1(\xi) = \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)},$$
$$C_2(\xi) = -\frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}.$$

As a result, the solution of problem (5), (6) is given by the formula

$$\begin{aligned}\widehat{u}(\xi,t) &= \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}\left[e^{i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]} - e^{-i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]}\right]\\ &= \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}\left[e^{-t\,\rho(\xi)\sin\varphi(\xi)}e^{i\,t\,\rho(\xi)\cos\varphi(\xi)} - e^{t\,\rho(\xi)\sin\varphi(\xi)}e^{-i\,t\,\rho(\xi)\cos\varphi(\xi)}\right]\\ &= \frac{1}{2i\,\rho(\xi)}\left[e^{-t\,\rho(\xi)\sin\varphi(\xi)}e^{i(t\,\rho(\xi)\cos\varphi(\xi)-\varphi(\xi))} - e^{t\,\rho(\xi)\sin\varphi(\xi)}e^{-i(t\,\rho(\xi)\cos\varphi(\xi)+\varphi(\xi))}\right]\\ &= \frac{1}{2i\,\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi))} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi))}\right],\end{aligned} \tag{9}$$

where we use the notation

$$\begin{aligned}G_1(\xi) &:= \rho(\xi)\sin\varphi(\xi),\\ G_2(\xi) &:= \rho(\xi)\cos\varphi(\xi).\end{aligned} \tag{10}$$

Now we formally apply the inverse Fourier transform F_ξ^{-1} to relation (9) and obtain

$$\begin{aligned}u(x,t) &= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\frac{1}{2i\,\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi))} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi))}\right]e^{-ix\cdot\xi}d\xi\\ &= \frac{1}{2i(2\pi)^n}\int\limits_{\mathbb{R}^n}\frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi.\end{aligned}$$

Since the functions $\alpha(\xi)$, $\rho(\xi)$, and $G_2(\xi)$ are even and the functions $\beta(\xi)$, $\varphi(\xi)$, and $G_1(\xi)$ are odd in each of the variables ξ_j, we transform the last expression as follows:

$$\begin{aligned}&\frac{1}{2i(2\pi)^n}\int\limits_{\mathbb{R}^n}\frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi\\ &\quad= \frac{1}{2i(2\pi)^n}\left[\int\limits_{\mathbb{R}^n_-}\frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi\right.\\ &\qquad\left.+\int\limits_{\mathbb{R}^n_+}\frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi\right]\\ &\quad= \frac{1}{2i(2\pi)^n}\left[\int\limits_{\mathbb{R}^n_+}\frac{1}{\rho(\xi)}\left[e^{t\,G_1(\xi)}e^{i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)} - e^{-t\,G_1(\xi)}e^{-i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)}\right]d\xi\right.\\ &\qquad\left.+\int\limits_{\mathbb{R}^n_+}\frac{1}{\rho(\xi)}\left[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\right]d\xi\right]\end{aligned}$$

$$= \frac{1}{2i(2\pi)^n} \int\limits_{\mathbb{R}^n_+} \frac{1}{\rho(\xi)} \Big[2i\, e^{t\, G_1(\xi)} \sin\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big) + 2i\, e^{-t\, G_1(\xi)} \sin\big(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\big)\Big] d\xi$$

$$= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n_+} \frac{1}{\rho(\xi)} \Big[e^{t\, G_1(\xi)} \sin\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big) + e^{-t\, G_1(\xi)} \sin\big(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\big)\Big] d\xi.$$

We use the resulting representation to prove the following assertion.

Theorem 1. *Under condition* (3), *the functions*

$$F(x,t;\xi) := e^{t\, G_1(\xi)} \sin\,(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi), \tag{11}$$

$$H(x,t;\xi) := e^{-t\, G_1(\xi)} \sin\,(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi), \tag{12}$$

where $\varphi(\xi)$ *is determined by formula* (8) *and* $G_1(\xi)$ *and* $G_2(\xi)$ *are determined by relations* (10), *satisfy Eq.* (1) *in the classical sense.*

Proof. First, let us substitute the function (11) directly into Eq. (1). To this end, we find the derivatives

$$\begin{aligned} F_{x_j}(x,t;\xi) &= \xi_j e^{tG_1(\xi)} \cos\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big), \\ F_{x_jx_j}(x,t;\xi) &= -\xi_j^2 e^{tG_1(\xi)} \sin\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big), \\ F_t(x,t;\xi) &= G_1(\xi) e^{tG_1(\xi)} \sin\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big) \\ &\quad + G_2(\xi) e^{tG_1(\xi)} \cos\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big), \\ F_{tt}(x,t;\xi) &= \big[G_1^2(\xi) - G_2^2(\xi)\big] e^{tG_1(\xi)} \sin\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big) \\ &\quad + 2G_1(\xi)G_2(\xi) e^{tG_1(\xi)} \cos\big(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\big). \end{aligned} \tag{13}$$

Now let us evaluate the expressions $2G_1(\xi)G_2(\xi)$ and $G_1^2(\xi) - G_2^2(\xi)$.
Since $G_1(\xi)$ and $G_2(\xi)$ are defined in (10), we conclude that

$$2G_1(\xi)G_2(\xi) = \rho^2(\xi) \sin 2\varphi(\xi).$$

It follows from formula (8) that $|2\varphi(\xi)| < \pi/2$ and hence $\cos 2\varphi(\xi) > 0$. Then we have

$$\begin{aligned} \sin 2\varphi(\xi) &= \frac{\tan 2\varphi(\xi)}{\sqrt{1 + \tan^2 2\varphi(\xi)}} \\ &= \tan\left(\arctan \frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)}\right) \left[1 + \tan^2\left(\arctan \frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)}\right)\right]^{-1/2} \\ &= \frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)} \left[1 + \frac{\beta^2(\xi)}{\big(a^2|\xi|^2 + \alpha(\xi)\big)^2}\right]^{-1/2} \\ &= \frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)} \left[\frac{\big(a^2|\xi|^2 + \alpha(\xi)\big)^2}{\big(a^2|\xi|^2 + \alpha(\xi)\big)^2 + \beta^2(\xi)}\right]^{1/2} = \frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)} \frac{\big|a^2|\xi|^2 + \alpha(\xi)\big|}{\rho^2(\xi)}. \end{aligned}$$

By virtue of condition (3), from the last relation we obtain

$$\sin 2\varphi(\xi) = \frac{\beta(\xi)}{a^2|\xi|^2 + \alpha(\xi)} \frac{a^2|\xi|^2 + \alpha(\xi)}{\rho^2(\xi)} = \frac{\beta(\xi)}{\rho^2(\xi)},$$

and hence

$$2G_1(\xi)G_2(\xi) = \beta(\xi). \tag{14}$$

With the inequality $\cos 2\varphi(\xi) > 0$ established above and under condition (3), now we find

$$\begin{aligned} G_1^2(\xi) - G_2^2(\xi) &= \rho^2(\xi)\big[\sin^2\varphi(\xi) - \cos^2\varphi(\xi)\big] \\ &= -\rho^2(\xi)\cos 2\varphi(\xi) = -\frac{\rho^2(\xi)}{\sqrt{1+\tan^2 2\varphi(\xi)}} \\ &= -\rho^2(\xi)\left[\frac{\big(a^2|\xi|^2+\alpha(\xi)\big)^2}{\big(a^2|\xi|^2+\alpha(\xi)\big)^2+\beta^2(\xi)}\right]^{1/2} = -a^2|\xi|^2 - \alpha(\xi). \end{aligned} \tag{15}$$

In view of the expressions (14) and (15), the function (13) becomes

$$\begin{aligned} F_{tt}(x,t;\xi) = \Big[&-\big(a^2|\xi|^2+\alpha(\xi)\big)\sin\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big) \\ &+ \beta(\xi)\cos\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big)\Big]e^{tG_1(\xi)}. \end{aligned}$$

Now let us substitute the derivatives F_{tt} and $F_{x_jx_j}$ into Eq. (1),

$$\begin{aligned} F_{tt}(x,t;\xi) - a^2\sum_{j=1}^n F_{x_jx_j}(x,t;\xi) &= \Bigg[-\big(a^2|\xi|^2+\alpha(\xi)\big)\sin\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big) \\ &\quad + \beta(\xi)\cos\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big) + a^2\sum_{j=1}^n \xi_j^2\sin\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big)\Bigg]e^{tG_1(\xi)} \\ &= \Big[-\alpha(\xi)\sin\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big) + \beta(\xi)\cos\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big)\Big]e^{tG_1(\xi)} \\ &= \Bigg[-\sum_{j=1}^n b_j\xi_j^2\cos(h_j\xi_j)\sin\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big) \\ &\quad + \sum_{j=1}^n b_j\xi_j^2\sin(h_j\xi_j)\cos\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big)\Bigg]e^{tG_1(\xi)} \\ &= -\sum_{j=1}^n b_j\xi_j^2\sin\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi-h_j\xi_j\big)e^{tG_1(\xi)} \\ &= \sum_{j=1}^n b_jF_{x_jx_j}(x_1,\dots,x_{j-1},x_j-h_j,x_{j+1},\dots,x_n,t;\xi). \end{aligned}$$

A straightforward substitution into Eq. (1) shows in a similar way that the function $H(x,t;\xi)$ satisfies this equation in the classical sense as well. The proof of the theorem is complete.

Corollary 1. *Under condition* (3), *the family of functions*

$$G(x,t;A,B,\xi) := A\,e^{t\,G_1(\xi)}\sin\big(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\big) + B\,e^{-t\,G_1(\xi)}\sin\big(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi\big),$$

where $\varphi(\xi)$ *is given by* (8) *and* $G_1(\xi)$ *and* $G_2(\xi)$ *are given by* (10), *satisfies Eq.* (1) *in the classical sense for any real values of the parameters* A, B, *and* ξ.

We have not answered yet as to exactly what conditions should be imposed on the real coefficients $a > 0$ and $b_1,\dots,b_n$ and the shifts $h_1,\dots,h_n$ in Eq. (1) for condition (3) to be satisfied for any n-dimensional parameter ξ.

We have originally assumed the coefficient a to be strictly positive based on the characteristics of physical problems leading to equations of hyperbolic type. Since Eq. (1) becomes the classical equation $u_{tt} = a^2\Delta u$ if all $b_j = 0$ $(j = 1,\dots,n)$, we see that the role of the constant a is quite clear. For example, in the one-dimensional case (the equation of small transverse vibrations of a string), a is the square root of the ratio of the tension to the material string density, which are assumed to be constant at each point [30, p. 29].

Condition (3),

$$a^2(\xi_1^2+\xi_2^2+\ldots+\xi_n^2)+b_1\cos(h_1\xi_1)\xi_1^2+b_2\cos(h_2\xi_2)\xi_2^2+\ldots+b_n\cos(h_n\xi_n)\xi_n^2>0,$$

obviously holds for any shifts $h_1,\ldots,h_n$ and any values $\xi_1,\ldots,\xi_n$ if the coefficients of the equation satisfy the condition

$$\max_{j=1,\ldots,n}|b_j|<a^2.$$

2. CONSTRUCTION OF SOLUTIONS OF EQUATION (2)

To find solutions of Eq. (2), we also formally apply the Fourier transform with respect to the n-dimensional variable x to this equation and obtain the ordinary differential equation

$$\frac{d^2\widehat{u}}{dt^2}=-\left(c^2|\xi|^2+\sum_{j=1}^n d_j\cos(l_j\xi_j)+i\sum_{j=1}^n d_j\sin(l_j\xi_j)\right)\widehat{u},\quad \xi\in\mathbb{R}^n, \tag{16}$$

for the function $\widehat{u}(\xi,t):=F_x[u](\xi,t)$. Following [31, p. 198], we supplement this equation with the two initial conditions

$$\widehat{u}(0)=0,\quad \widehat{u}_t(0)=1. \tag{17}$$

For convenience, in the subsequent calculations we use the notation

$$\lambda(\xi):=\sum_{j=1}^n d_j\cos(l_j\xi_j),$$
$$\mu(\xi):=\sum_{j=1}^n d_j\sin(l_j\xi_j).$$

Then Eq. (16) becomes

$$\frac{d^2\widehat{u}}{dt^2}=-\big(c^2|\xi|^2+\lambda(\xi)+i\,\mu(\xi)\big)\,\widehat{u},\quad \xi\in\mathbb{R}^n,$$

and the roots of the corresponding characteristic equation are given by the formula

$$k_{1,2}=\pm\sqrt{-\big(c^2|\xi|^2+\lambda(\xi)+i\,\mu(\xi)\big)}=\pm\, i\sqrt{c^2|\xi|^2+\lambda(\xi)+i\,\mu(\xi)}=\pm\, i\delta(\xi)e^{i\,\psi(\xi)},$$

where

$$\delta(\xi):=\left[\big(c^2|\xi|^2+\lambda(\xi)\big)^2+\mu^2(\xi)\right]^{1/4}, \tag{18}$$

$$\psi(\xi):=\frac{1}{2}\arctan\frac{\mu(\xi)}{c^2|\xi|^2+\lambda(\xi)}. \tag{19}$$

Note also that the functions (18) and (19) are well defined for all $\xi\in\mathbb{R}^n$ under condition (4).

By analogy with the solution of problem (5), (6), we obtain the solution of problem (16), (17),

$$\widehat{u}(\xi,t)=\frac{1}{2i\,\delta(\xi)}\left[e^{-t\,\widetilde{G}_1(\xi)}e^{i(t\,\widetilde{G}_2(\xi)-\psi(\xi))}-e^{t\,\widetilde{G}_1(\xi)}e^{-i(t\,\widetilde{G}_2(\xi)+\psi(\xi))}\right], \tag{20}$$

where

$$\begin{aligned}\widetilde{G}_1(\xi)&:=\delta(\xi)\sin\psi(\xi),\\ \widetilde{G}_2(\xi)&:=\delta(\xi)\cos\psi(\xi).\end{aligned} \tag{21}$$

Formally applying the inverse Fourier transform F_ξ^{-1} to Eq. (20) and using the fact that the functions $\lambda(\xi)$, $\delta(\xi)$, and $\widetilde{G}_2(\xi)$ are even and the functions $\mu(\xi)$, $\psi(\xi)$, and $\widetilde{G}_1(\xi)$ are odd in each of the variables ξ_j in subsequent transformations, we eventually obtain

$$\begin{aligned}
u(x,t) &= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} \frac{1}{2i\,\delta(\xi)}\left[e^{-t\,\widetilde{G}_1(\xi)}e^{i(t\,\widetilde{G}_2(\xi)-\psi(\xi))} - e^{t\,\widetilde{G}_1(\xi)}e^{-i(t\,\widetilde{G}_2(\xi)+\psi(\xi))}\right]e^{-ix\cdot\xi}d\xi \\
&= \frac{1}{2i(2\pi)^n}\int\limits_{\mathbb{R}^n} \frac{1}{\delta(\xi)}\left[e^{-t\,\widetilde{G}_1(\xi)}e^{i(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi)} - e^{t\,\widetilde{G}_1(\xi)}e^{-i(t\,\widetilde{G}_2(\xi)+\psi(\xi)+x\cdot\xi)}\right]d\xi \\
&= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n_+} \frac{1}{\delta(\xi)}\left[e^{t\,\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi)+\psi(\xi)+x\cdot\xi\right) + e^{-t\,\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right)\right]d\xi.
\end{aligned}$$

The resulting integral representation proves the following assertion.

Theorem 2. *Under condition* (4), *the functions*

$$\widetilde{F}(x,t;\xi) := e^{t\,\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi)+\psi(\xi)+x\cdot\xi\right), \tag{22}$$

$$\widetilde{H}(x,t;\xi) := e^{-t\,\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right), \tag{23}$$

where $\psi(\xi)$ *is given by* (19) *and* $\widetilde{G}_1(\xi)$ *and* $\widetilde{G}_2(\xi)$ *are given by* (21), *satisfy Eq.* (2) *in the classical sense.*

Proof. First, let us verify that the function (23) satisfies Eq. (2). To this end, we calculate the derivatives

$$\begin{aligned}
\widetilde{H}_{x_j}(x,t;\xi) &= -\xi_j e^{-t\widetilde{G}_1(\xi)}\cos\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right), \\
\widetilde{H}_{x_jx_j}(x,t;\xi) &= -\xi_j^2 e^{-t\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right), \\
\widetilde{H}_t(x,t;\xi) &= -\widetilde{G}_1(\xi)e^{-t\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right) \\
&\quad + \widetilde{G}_2(\xi)e^{-t\widetilde{G}_1(\xi)}\cos\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right), \\
\widetilde{H}_{tt}(x,t;\xi) &= \left[\widetilde{G}_1^2(\xi)-\widetilde{G}_2^2(\xi)\right]e^{-t\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right) \\
&\quad - 2\widetilde{G}_1(\xi)\widetilde{G}_2(\xi)e^{-t\widetilde{G}_1(\xi)}\cos\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right).
\end{aligned}$$

By analogy with the argument in Theorem 1, we evaluate the expressions

$$\begin{aligned}
2\widetilde{G}_1(\xi)\widetilde{G}_2(\xi) &= \mu(\xi), \\
\widetilde{G}_1^2(\xi)-\widetilde{G}_2^2(\xi) &= -c^2|\xi|^2-\lambda(\xi),
\end{aligned}$$

so that the expression for $\widetilde{H}_{tt}$ acquires the form

$$\begin{aligned}
\widetilde{H}_{tt}(x,t;\xi) = \Big[&-\left(c^2|\xi|^2+\lambda(\xi)\right)\sin\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right)- \\
&-\mu(\xi)\cos\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right)\Big]e^{-t\widetilde{G}_1(\xi)}.
\end{aligned}$$

Now let us directly substitute the derivatives $\widetilde{H}_{tt}$ and $\widetilde{H}_{x_jx_j}$ into Eq. (2),

$$\widetilde{H}_{tt}(x,t;\xi) - c^2\sum_{j=1}^{n}\widetilde{H}_{x_jx_j}(x,t;\xi) = \Big[-\left(c^2|\xi|^2+\lambda(\xi)\right)\sin\left(t\,\widetilde{G}_2(\xi)-\psi(\xi)-x\cdot\xi\right)$$

$$
\begin{aligned}
&\quad - \mu(\xi)\cos\left(t\,\widetilde{G}_2(\xi) - \varphi(\xi) - x\cdot\xi\right) + c^2\sum_{j=1}^{n}\xi_j^2\sin\left(t\,\widetilde{G}_2(\xi) - \psi(\xi) - x\cdot\xi\right)\Bigg]\, e^{-t\widetilde{G}_1(\xi)} \\
&= -\left[\lambda(\xi)\sin\left(t\,\widetilde{G}_2(\xi) - \psi(\xi) - x\cdot\xi\right) + \mu(\xi)\cos\left(t\,\widetilde{G}_2(\xi) - \psi(\xi) - x\cdot\xi\right)\right] e^{-t\widetilde{G}_1(\xi)} \\
&= -\Bigg[\sum_{j=1}^{n} d_j\cos\left(l_j\xi_j\right)\sin\left(t\,\widetilde{G}_2(\xi) - \psi(\xi) - x\cdot\xi\right) \\
&\quad + \sum_{j=1}^{n} d_j\sin\left(l_j\xi_j\right)\cos\left(t\,\widetilde{G}_2(\xi) - \psi(\xi) - x\cdot\xi\right)\Bigg] e^{-t\widetilde{G}_1(\xi)} \\
&= -\sum_{j=1}^{n} d_j\sin\left(t\,\widetilde{G}_2(\xi) - \psi(\xi) - x\cdot\xi + l_j\xi_j\right) e^{-t\widetilde{G}_1(\xi)} \\
&= -\sum_{j=1}^{n} d_j\widetilde{H}(x_1,\ldots,x_{j-1},x_j - l_j,x_{j+1},\ldots,x_n,t;\xi).
\end{aligned}
$$

In a similar way, a straightforward substitution into Eq. (2) shows that the function $\widetilde{F}(x,t;\xi)$ satisfies Eq. (2) in the classical sense as well. The proof of the theorem is complete.

Corollary 2. *Under condition* (4), *the three-parameter family of functions*

$$
\begin{aligned}
\widetilde{G}(x,t;\widetilde{A},\widetilde{B},\xi) := \widetilde{A}\,e^{t\,\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi) + \psi(\xi) + x\cdot\xi\right) \\
+ \widetilde{B}\,e^{-t\,\widetilde{G}_1(\xi)}\sin\left(t\,\widetilde{G}_2(\xi) - \psi(\xi) - x\cdot\xi\right),
\end{aligned}
$$

where $\psi(\xi)$ *is given by* (19) *and* $\widetilde{G}_1(\xi)$ *and* $\widetilde{G}_2(\xi)$ *are given by* (21), *satisfies Eq.* (2) *in the classical sense for any real values of the parameters* $\widetilde{A}$, $\widetilde{B}$, *and* ξ.

Now let us find out whether there indeed exist equations of the form (2), whose classical solutions we have obtained, such that condition (4) is satisfied for any $\xi \in \mathbb{R}^n$. The answer is "yes." Let us give examples of such equations.

We represent condition (4) in the form

$$
\left(c^2\xi_1^2 + d_1\cos\left(l_1\xi_1\right)\right) + \left(c^2\xi_2^2 + d_2\cos\left(l_2\xi_2\right)\right) + \ldots + \left(c^2\xi_n^2 + d_n\cos\left(l_n\xi_n\right)\right) > 0.
$$

It was shown in the paper [27] that each of the n terms on the left-hand side in this inequality is positive under the condition

$$
0 < d_j l_j^2 \le 2c^2, \quad j = 1,\ldots,n.
$$

At present, finding other examples of Eqs. (2) for which (4) holds remains an open question.

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