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On the Spectrum of Two-Point Boundary Value Problems for the Dirac Operator

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Abstract— We consider the spectral problem for a Dirac operator with arbitrary two-point boundary conditions and an arbitrary complex-valued integrable potential. The existence of nontrivial boundary value problems of this type with an unbounded growth of the multiplicity of eigenvalues is established.

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INTRODUCTION

In the present paper, we study the Dirac system

$$B\mathbf{y}' + V\mathbf{y} = \lambda\mathbf{y}, \tag{1}$$

where $\mathbf{y} = \operatorname{col}(y_1(x), y_2(x))$, $\lambda \in \mathbb{C}$ is the spectral parameter,

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

and the functions $p, q \in L_1(0,\pi)$ are complex-valued, with the two-point boundary conditions

$$U(\mathbf{y}) \equiv C\mathbf{y}(0) + D\mathbf{y}(\pi) = 0, \tag{2}$$

where

$$C = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad D = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix},$$

the coefficients a_{ij} can be any complex numbers, and the rows of the matrix

$$A = (CD) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

are linearly independent.

We denote by $\|f\| = (|f_1|^2 + |f_2|^2)^{1/2}$ the norm of an arbitrary vector $f = \operatorname{col}(f_1, f_2) \in \mathbb{C}^2$ and set $\langle f, g\rangle = f_1 g_1 + f_2 g_2$. We denote the norm of an arbitrary 2×2 matrix W by $\|W\| = \sup_{\|f\|=1} \|Wf\|$. Let $L_{2,2}(a,b)$ be the space of two-dimensional vector functions $f(t) = \operatorname{col}(f_1(t), f_2(t))$ with the norm $\|f\|_{L_{2,2}(a,b)} = (\int_a^b \|f(t)\|\,dt)^{1/2}$, and let $L_{2,2}^{2,2}(a,b)$ be the space of 2×2 matrix functions $W(t)$ with the norm $\|W\|_{L_{2,2}^{2,2}(a,b)} = (\int_a^b \|W(t)\|\,dt)^{1/2}$. We treat the operator $\mathbb{L}\mathbf{y} = B\mathbf{y}' + V\mathbf{y}$ as a linear operator in the space $L_{2,2}(0,\pi)$ with domain $D(\mathbb{L}) = \{\mathbf{y} \in W_1^1[0,\pi] : \mathbb{L}\mathbf{y} \in L_{2,2}(0,\pi),\ U_j(\mathbf{y}) = 0\ (j = 1,2)\}$.

Let

$$E(x,\lambda) = \begin{pmatrix} c_1(x,\lambda) & -s_2(x,\lambda) \\ s_1(x,\lambda) & c_2(x,\lambda) \end{pmatrix}$$

be the fundamental matrix of Eq. (1) with the boundary condition $E(0,\lambda)=I$, where I is the identity matrix, and let $E_0(x,\lambda)$ be the fundamental matrix of the unperturbed equation $B\mathbf{y}'=\lambda\mathbf{y}$ with the boundary condition $E_0(0,\lambda)=I$. It is obvious that

$$E_0(x,\lambda)=\begin{pmatrix}\cos(\lambda x) & -\sin(\lambda x)\\ \sin(\lambda x) & \cos(\lambda x)\end{pmatrix}.$$

It is well known that the entries of the matrix $E(x,\lambda)$ are related by

$$c_1(x,\lambda)c_2(x,\lambda)+s_1(x,\lambda)s_2(x,\lambda)=1 \tag{3}$$

for any x and λ. Let J_{ij} be the determinant formed by the ith and jth columns of A. Set $J_0=J_{12}+J_{34}$, $J_1=J_{14}-J_{23}$, and $J_2=J_{13}+J_{24}$.

It was shown in [1] by the transformation operator method that the characteristic determinant $\Delta(\lambda)$ of problem (1), (2), which is equal to

$$\Delta(\lambda)=J_{12}+J_{34}+J_{14}c_2(\pi,\lambda)-J_{23}c_1(\pi,\lambda)-J_{13}s_2(\pi,\lambda)-J_{24}s_1(\pi,\lambda), \tag{4}$$

can be reduced to the form

$$\Delta(\lambda)=\Delta_0(\lambda)+\int\limits_0^\pi r_1(t)e^{-i\lambda t}\,dt+\int\limits_0^\pi r_2(t)e^{i\lambda t}\,dt=\Delta_0(\lambda)+R(\lambda), \tag{5}$$

where the function

$$\begin{aligned}\Delta_0(\lambda)&=J_0+J_1\cos(\pi\lambda)-J_2\sin(\pi\lambda)\\ &=J_{12}+J_{34}+\frac{1}{2}\big(e^{i\pi\lambda}(J_1+iJ_2)+e^{-i\pi\lambda}(J_1-iJ_2)\big)=J_0+C_1e^{i\pi\lambda}+C_2e^{-i\pi\lambda},\end{aligned} \tag{6}$$

$C_1=(J_1+iJ_2)/2$, $C_2=(J_1-iJ_2)/2$, is the characteristic determinant of the unperturbed problem

$$B\mathbf{y}'=\lambda\mathbf{y},\quad U(\mathbf{y})=0 \tag{7}$$

and the functions r_j belong to the space $L_1(0,\pi)$, $j=1,2$. If $p,q\in L_2(0,\pi)$ (for short, we write $V\in L_2(0,\pi)$), then $r_j\in L_2(0,\pi)$. It follows that the function $\Delta(\lambda)$ is an entire function of exponential type; therefore, we only have the following possibilities for the operator $\mathbb{L}$ of problem (1), (2):

1. The spectrum is empty.
2. The spectrum is a finite nonempty set.
3. The spectrum is a countable set without finite limit points.
4. The spectrum fills the entire complex plane.

Relations (5) and (6) imply that case 1 is realized for problem (7), for example, with the boundary conditions defined by the matrix

$$A=\begin{pmatrix}1 & i & -1 & i\\ 1 & -i & 1 & i\end{pmatrix},$$

and case 4, with the boundary conditions defined by the matrix

$$A=\begin{pmatrix}1 & -i & 0 & 0\\ 0 & 0 & i & 1\end{pmatrix}.$$

Let us prove that case 2 is impossible. Let the equation

$$\Delta(\lambda)=0$$

have finitely many roots λ_k, $k = 1, \ldots, n$. If $C_1C_2 \neq 0$, then conditions (2) are regular and problem (1), (2) has a countable set of eigenvalues; therefore, $C_1C_2 = 0$. Set $P(\lambda) = \prod_{k=1}^{n}(\lambda - \lambda_k)$. By [2],

$$\Delta(\lambda) = P(\lambda)e^{a\lambda+b},$$

where a and b are some constants. Assume, for example, that $C_2 = 0$. Setting $\lambda = -iy$ in relation (5), where $y > 0$, we obtain

$$J_0 + C_1e^{\pi y} + R(-iy) = P(-iy)e^{-iay+b},$$

which implies that

$$J_0e^{-\pi y} + C_1 + e^{-\pi y}R(-iy) = P(-iy)e^{b-i\,\mathrm{Re}\,ay}e^{(\mathrm{Im}\,a-\pi)y}. \tag{8}$$

According to [3, p. 36], the expression on the left-hand side in relation (8) tends to C_1 as $y \to \infty$. If $\mathrm{Im}\,a-\pi \geq 0$, then the expression on the right-hand side in relation (8) tends to infinity in absolute value, and if $\mathrm{Im}\,a - \pi < 0$, then it tends to zero. It follows that $C_1 = 0$. If $C_1 = C_2 = 0$, then

$$R(\lambda) = P(\lambda)e^{a\lambda+b}. \tag{9}$$

Obviously, the left-hand side of relation (9) is bounded on the real axis, while the right-hand side is not; that is, we arrive at a contradiction.

Definition. We say that problem (1), (2) has the *classical spectral asymptotics* if its spectrum is a countable set and the multiplicities of the eigenvalues are uniformly bounded.

The present paper is aimed at constructing problems (1), (2) for which case 3 is realized and the multiplicities of the eigenvalues grow unboundedly, i.e., problems with nonclassical spectral asymptotics.

MAIN RESULTS

Set $c_j(\lambda) = c_j(\pi, \lambda)$ and $s_j(\lambda) = s_j(\pi, \lambda)$, $j = 1, 2$. In addition, let PW_σ be the class of entire functions $f(z)$ of the exponential type $\leq \sigma$ such that $\|f\|_{L_2(R)} < \infty$. It is well known [4] that the functions $c_j(\lambda)$ and $s_j(\lambda)$ admit the representation

$$c_j(\lambda) = \cos(\pi\lambda) + g_j(\lambda), \quad s_j(\lambda) = \sin(\pi\lambda) + h_j(\lambda),$$

where $g_j, h_j \in PW_\pi$, $j = 1, 2$.

Lemma 1 [5]. *The functions $u(\lambda)$ and $v(\lambda)$ admit the representations*

$$u(\lambda) = \sin(\pi\lambda) + h(\lambda), \quad v(\lambda) = \cos(\pi\lambda) + g(\lambda),$$

where $h, g \in PW_\pi$, if and only if

$$u(\lambda) = -\pi(\lambda_0 - \lambda)\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{\lambda_n - \lambda}{n},$$

where $\lambda_n = n + \varepsilon_n$ and $\{\varepsilon_n\} \in l_2$, and

$$v(\lambda) = \prod_{n=-\infty}^{\infty}\frac{\lambda_n - \lambda}{n - 1/2},$$

where $\lambda_n = n - 1/2 + \kappa_n$ and $\{\kappa_n\} \in l_2$.

Consider the Dirac system with the boundary conditions defined by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{10}$$

We will assume that $V \in L_2(0,\pi)$. It follows from the representation (4) that the characteristic determinant $\Delta(\lambda)$ of problem (1), (2) with matrix A defined in (10) can be reduced to the form

$$\Delta(\lambda) = s_1(\lambda) - s_2(\lambda) = \int_{-\pi}^{\pi} r(t)e^{i\lambda t}\,dt = f(\lambda),$$

where $r \in L_2(0,\pi)$, and $f \in PW_\pi$. The converse statement holds true as well.

Theorem. *For each function $f \in PW_\pi$, there exists a potential $V \in L_2(0,\pi)$ such that the characteristic determinant $\Delta(\lambda)$ of problem (1), (2) with the matrix A defined by relation (10) and the potential $V(x)$ is identically equal to $f(\lambda)$.*

Proof. Let $f(\lambda)$ be an arbitrary function in the class PW_π. It follows from the Paley–Wiener theorem and [3, p. 36] that

$$\lim_{|\lambda|\to\infty} e^{-\pi|\operatorname{Im}\lambda|} f(\lambda) = 0; \tag{11}$$

consequently, there exists a positive integer N_0 so large that $|f(\lambda)| < 1/100$ if $\operatorname{Im}\lambda = 0$ and $|\operatorname{Re}\lambda| \ge N_0$.

Let $\{\lambda_n\}$, $n \in \mathbb{Z}$, be a strictly monotone increasing sequence of real numbers such that $N_0 < \lambda_n < N_0 + 1/100$ if $1 \le n \le N_0$, $\lambda_n = n - 1/2$ if $n > N_0$, and $\lambda_n = -\lambda_{-n+1}$ for any n. Set

$$c(\lambda) = \prod_{n=-\infty}^{\infty} \frac{\lambda_n - \lambda}{n - 1/2}.$$

Lemma 1 implies the relation

$$c(\lambda) = \cos(\pi\lambda) + g(\lambda), \tag{12}$$

where $g \in PW_\pi$. It follows from the Paley–Wiener theorem and [3, p. 36] that

$$\lim_{|\lambda|\to\infty} e^{-\pi|\operatorname{Im}\lambda|} g(\lambda) = 0;$$

therefore,

$$\bigl|c(\lambda)\bigr| \ge c_0 e^{\pi|\operatorname{Im}\lambda|} \tag{13}$$

($c_0 = \text{const} > 0$) for $|\operatorname{Im}\lambda| \ge M$, where M is a sufficiently large number.

Differentiating relation (12), we obtain

$$\dot{c}(\lambda) = -\pi\sin(\pi\lambda) + \dot{g}(\lambda). \tag{14}$$

Since the function $\dot{g}$ belongs to the class PW_π, we have, according to [6],

$$\dot{c}(\lambda_n) = -\pi\sin(\pi\lambda_n) + \tau_n,$$

where

$$\sum_{n=-\infty}^{\infty} |\tau_n|^2 < \infty.$$

Based on this, by the definition of the numbers λ_n, we obtain

$$\dot{c}(\lambda_n) = \pi(-1)^n + \rho_n, \tag{15}$$

where

$$\sum_{n=-\infty}^{\infty} |\rho_n|^2 < \infty.$$

Consequently, for all even n sufficiently large in modulus one has the inequality $\dot{c}(\lambda_n) > 0$. One can readily see that the inequality $\dot{c}(\lambda_n)\dot{c}(\lambda_{n+1}) < 0$ holds for all $n \in \mathbb{Z}$. It follows that

$$(-1)^n \dot{c}(\lambda_n) > 0 \tag{16}$$

for all $n \in \mathbb{Z}$. Note that (15) implies the relation

$$\frac{1}{\dot{c}(\lambda_n)} = \frac{(-1)^n}{\pi} + \sigma_n, \tag{17}$$

where

$$\sum_{n=-\infty}^{\infty} |\sigma_n|^2 < \infty.$$

Consider the quadratic equation

$$w^2 + f(\lambda_n)w - 1 = 0. \tag{18}$$

It has the roots

$$s_n^{\pm} = \frac{-f(\lambda_n) \pm \sqrt{f^2(\lambda_n) + 4}}{2}.$$

By $\Gamma(z, r)$ we denote the disk of radius r centered at point z. One can readily see that all numbers s_n^+ lie inside the disk $\Gamma(1, 1/10)$ and all numbers s_n^- lie inside the disk $\Gamma(-1, 1/10)$. Let $s_n = s_n^+$ if n is odd and $s_n = s_n^-$ if n is even. Since [6] $\{f(\lambda_n)\} \in l_2$, it follows from the definition of the numbers s_n that

$$s_n = (-1)^{n+1} + \vartheta_n, \tag{19}$$

where $\{\vartheta_n\} \in l_2$. It also follows from the definition of the numbers s_n and inequality (16) that all numbers $z_n = s_n/\dot{c}(\lambda_n)$ lie strictly to the left of the imaginary axis, while (17) and (19) imply the relation

$$z_n = -\frac{1}{\pi} + \rho_n,$$

where $\{\rho_n\} \in l_2$. Let $\beta_n = s_n - \sin(\pi\lambda_n)$; then $\{\beta_n\} \in l_2$ in view of (19). Set

$$h(\lambda) = c(\lambda) \sum_{n=-\infty}^{\infty} \frac{\beta_n}{\dot{c}(\lambda_n)(\lambda - \lambda_n)}.$$

According to [7, p. 120], the function h belongs to the class PW_π, and $h(\lambda_n) = \beta_n$. Set $s(\lambda) = \sin(\pi\lambda) + h(\lambda)$; then $s(\lambda_n) = s_n \neq 0$, and consequently, the functions $s(\lambda)$ and $c(\lambda)$ do not have common roots.

Set

$$Y_0(x, \lambda) = \begin{pmatrix} \cos(\lambda x) \\ \sin(\lambda x) \end{pmatrix}.$$

In the subsequent exposition, we need the following elementary assertion.

Lemma 2. *If function systems $\{\varphi_n\}$ and $\{\psi_n\}$ are complete in $L_2(a, b)$ $(n \in \mathbb{N})$, then the system of vectors*

$$\Psi_{n,n} = \begin{pmatrix} \{\varphi_n\} \\ \{\psi_n\} \end{pmatrix} \cup \begin{pmatrix} \{\varphi_n\} \\ \{-\psi_n\} \end{pmatrix}$$

is complete in $L_{2,2}(a, b)$.

Proof. Assume that there exists a vector $f(x) = \operatorname{col}(f_1(x), f_2(x)) \neq 0$ such that

$$\int_a^b \left(\varphi_n(x)\overline{f_1(x)} + \psi_n(x)\overline{f_2(x)}\right) dx = 0, \quad \int_a^b \left(\varphi_n(x)\overline{f_1(x)} - \psi_n(x)\overline{f_2(x)}\right) dx = 0$$

for all $n \in \mathbb{N}$. Then

$$\int_a^b \varphi_n(x)\overline{f_1(x)}\,dx = 0, \quad \int_a^b \psi_n(x)\overline{f_2(x)}\,dx = 0;$$

consequently, $f_1(x) \equiv f_2(x) \equiv 0$. The proof of the lemma is complete.

It follows from [8] that the function systems $\{\cos(\lambda_n x)\}$ and $\{\sin(\lambda_n x)\}$ $(n \in \mathbb{N})$ are complete in $L_2(0,\pi)$. Based on this, it follows from the definition of the numbers λ_n and Lemma 2 that the system of vectors

$$Y_0(x,\lambda_n) = \begin{pmatrix} \cos(\lambda_n x) \\ \sin(\lambda_n x) \end{pmatrix}$$

$(n \in \mathbb{Z})$ is complete in $L_{2,2}(0,\pi)$. Set

$$F(x,t) = -\sum_{n=-\infty}^{\infty} \left(\frac{s_n}{\dot{c}(\lambda_n)} (Y_0(x,\lambda_n) Y_0^{\mathrm{T}}(t,\lambda_n)) + \frac{1}{\pi} Y_0(x, n-1/2) Y_0^{\mathrm{T}}(t, n-1/2) \right). \tag{20}$$

It follows from [4] that

$$\|F(\,\cdot\,,x)\|_{L^{2,2}_{2,2}(0,\pi)} + \|F(x,\cdot\,)\|_{L^{2,2}_{2,2}(0,\pi)} < C,$$

where C is a constant independent of x. Let us prove that for each $x \in [0,\pi]$ the homogeneous equation

$$f^{\mathrm{T}}(t) + \int_0^x f^{\mathrm{T}}(s) F(s,t)\,ds = 0, \tag{21}$$

where $f(t) = \mathrm{col}\,(f_1(t), f_2(t))$, $f \in L_{2,2}(0,x)$, $f(t) = 0$ for $x < t \le \pi$, has only the trivial solution. Multiplying Eq. (21) by $\overline{f^{\mathrm{T}}(t)}$ and integrating the resulting equation over the segment $[0,x]$, we obtain

$$\|f\|^2_{L_{2,2}(0,x)} + \int_0^x \left\langle \int_0^x f^{\mathrm{T}}(s) F(s,t)\,ds, \overline{f^{\mathrm{T}}(t)} \right\rangle dt = 0.$$

Taking into account definition (20), by simple calculations we find that

$$\begin{aligned}
&f^{\mathrm{T}}(s)F(s,t) \\
&= -\Bigg\{ \sum_{n=-\infty}^{\infty} \Big\{ z_n \big[f_1(s)\cos(\lambda_n s)\cos(\lambda_n t) + f_2(s)\sin(\lambda_n s)\cos(\lambda_n t), \\
&\qquad f_1(s)\cos(\lambda_n s)\sin(\lambda_n t) + f_2(s)\sin(\lambda_n s)\sin(\lambda_n t) \big] \\
&\quad + \frac{1}{\pi}\Big[f_1(s)\cos\big((n-1/2)s\big)\cos\big((n-1/2)t\big) + f_2(s)\sin\big((n-1/2)s\big)\cos\big((n-1/2)t\big), \\
&\qquad f_1(s)\cos\big((n-1/2)s\big)\sin\big((n-1/2)t\big) + f_2(s)\sin\big((n-1/2)s\big)\sin\big((n-1/2)t\big) \Big] \Big\} \Bigg\} \\
&= -\Bigg\{ \sum_{n=-\infty}^{\infty} \Big\{ z_n \big[f_1(s)\cos(\lambda_n s)\cos(\lambda_n t) + f_2(s)\sin(\lambda_n s)\cos(\lambda_n t) \big] \\
&\quad + \frac{1}{\pi}\Big[f_1(s)\cos\big((n-1/2)s\big)\cos\big((n-1/2)t\big) + f_2(s)\sin\big((n-1/2)s\big)\cos\big((n-1/2)t\big) \Big], \\
&\qquad z_n \big[f_1(s)\cos(\lambda_n s)\sin(\lambda_n t) + f_2(s)\sin(\lambda_n s)\sin(\lambda_n t) \big] \\
&\quad + \frac{1}{\pi}\Big[f_1(s)\cos\big(((n-1/2)s)\big)\sin\big((n-1/2)t\big) f_2(s)\sin\big((n-1/2)s\big)\sin\big((n-1/2)t\big) \Big] \Big\} \Bigg\},
\end{aligned}$$

which implies that

$$\int\limits_0^x \left\langle \int\limits_0^x f^{\mathrm{T}}(s)F(s,t)\,ds, \overline{f^{\mathrm{T}}(t)} \right\rangle dt$$

$$= -\Biggl\{ \sum_{n=-\infty}^{\infty} \int\limits_0^x \Biggl(\int\limits_0^x \Bigl\{ z_n \bigl[f_1(s)\cos(\lambda_n s)\cos(\lambda_n t) + f_2(s)\sin(\lambda_n s)\cos(\lambda_n t) \bigr]$$

$$+ \frac{1}{\pi}\Bigl[f_1(s)\cos\bigl((n-1/2)s\bigr)\cos\bigl((n-1/2)t\bigr)$$

$$+ f_2(s)\sin\bigl((n-1/2)s\bigr)\cos\bigl((n-1/2)t\bigr)\Bigr]\Bigr\}\,ds\Biggr)\overline{f_1(t)}\,dt$$

$$+ \sum_{n=-\infty}^{\infty} \int\limits_0^x \Biggl(\int\limits_0^x \Bigl\{ z_n \bigl[f_1(s)\cos(\lambda_n s)\sin(\lambda_n t) + f_2(s)\sin(\lambda_n s)\sin(\lambda_n t) \bigr]$$

$$+ \frac{1}{\pi}\Bigl[f_1(s)\cos\bigl((n-1/2)s\bigr)\sin\bigl((n-1/2)t\bigr)$$

$$+ f_2(s)\sin\bigl((n-1/2)s\bigr)\sin\bigl((n-1/2)t\bigr)\Bigr]\Bigr\}\,ds\Biggr)\overline{f_2(t)}\,dt\Biggr\}$$

$$= -\Biggl\{ \sum_{n=-\infty}^{\infty} \Biggl(\int\limits_0^x z_n \bigl[f_1(s)\cos(\lambda_n s) + f_2(s)\sin(\lambda_n s) \bigr]\,ds \int\limits_0^x \cos(\lambda_n t)\overline{f_1(t)}\,dt$$

$$+ \frac{1}{\pi}\int\limits_0^x \Bigl[f_1(s)\cos\bigl((n-1/2)s\bigr) + f_2(s)\sin\bigl((n-1/2)s\bigr)\Bigr]\,ds \int\limits_0^x \cos\bigl((n-1/2)t\bigr)\overline{f_1(t)}\,dt\Biggr)$$

$$+ \sum_{n=-\infty}^{\infty} \Biggl(\int\limits_0^x z_n \bigl[f_1(s)\cos(\lambda_n s) + f_2(s)\sin(\lambda_n s) \bigr]\,ds \int\limits_0^x \sin(\lambda_n t)\overline{f_2(t)}\,dt$$

$$+ \frac{1}{\pi}\int\limits_0^x \Bigl[f_1(s)\cos\bigl((n-1/2)s\bigr) + f_2(s)\sin\bigl((n-1/2)s\bigr)\Bigr]\,ds \int\limits_0^x \sin\bigl((n-1/2)t\bigr)\overline{f_2(t)}\,dt\Biggr)\Biggr\}$$

$$= -\Biggl\{ \sum_{n=-\infty}^{\infty} \Biggl(\int\limits_0^x z_n \bigl[f_1(s)\cos(\lambda_n s) + f_2(s)\sin(\lambda_n s) \bigr]\,ds \int\limits_0^x \cos(\lambda_n t)\overline{f_1(t)}\,dt$$

$$+ \int\limits_0^x \bigl[f_1(s)\cos(\lambda_n s) + f_2(s)\sin(\lambda_n s) \bigr]\,ds \int\limits_0^x \sin(\lambda_n t)\overline{f_2(t)}\,dt\Biggr)$$

$$+ \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \Biggl(\int\limits_0^x \Bigl[f_1(s)\cos\bigl((n-1/2)s\bigr) + f_2(s)\sin\bigl((n-1/2)s\bigr)\Bigr]\,ds\Biggr) \int\limits_0^x \cos\bigl((n-1/2)t\bigr)\overline{f_1(t)}\,dt$$

$$+ \int\limits_0^x \Bigl[f_1(s)\cos\bigl((n-1/2)s\bigr) + f_2(s)\sin\bigl((n-1/2)s\bigr)\Bigr]\,ds \int\limits_0^x \sin\bigl((n-1/2)t\bigr)\overline{f_2(t)}\,dt\Biggr)\Biggr\}$$

$$= -\Bigg\{ \sum_{n=-\infty}^{\infty} \Bigg(\int_0^x z_n \big[f_1(t)\cos(\lambda_n t) + f_2(t)\sin(\lambda_n t)\big]\,dt \int_0^x \cos(\lambda_n t)\overline{f_1(t)}\,dt$$

$$+ \int_0^x \big[f_1(t)\cos(\lambda_n t) + f_2(t)\sin(\lambda_n t)\big]\,dt \int_0^x \sin(\lambda_n t)\overline{f_2(t)}\,dt \Bigg)$$

$$+ \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Bigg(\int_0^x \Big[f_1(t)\cos\big((n-1/2)t\big) + f_2(t)\sin\big((n-1/2)t\big)\Big]\,dt \int_0^x \cos\big((n-1/2)t\big)\overline{f_1(t)}\,dt$$

$$+ \int_0^x \Big[f_1(t)\cos(nt) + f_2(t)\sin\big((n-1/2)t\big)\Big]\,dt \int_0^x \sin\big((n-1/2)t\big)\overline{f_2(t)}\,dt \Bigg) \Bigg\}$$

$$= -\Bigg\{ \sum_{n=-\infty}^{\infty} z_n \int_0^x \big[f_1(t)\cos(\lambda_n t) + f_2(t)\sin(\lambda_n t)\big]\,dt \int_0^x \big[\overline{f_1(t)}\cos(\lambda_n t) + \overline{f_2(t)}\sin(\lambda_n t)\big]\,dt$$

$$+ \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \int_0^x \Big[f_1(t)\cos\big((n-1/2)t\big) + f_2(t)\sin\big((n-1/2)t\big)\Big]\,dt$$

$$\times \int_0^x \Big[\overline{f_1(t)}\cos\big((n-1/2)t\big) + \overline{f_2(t)}\sin\big((n-1/2)t\big)\Big]\,dt \Bigg\}$$

$$= -\sum_{n=-\infty}^{\infty} z_n \left| \int_0^x \big\langle f(t), Y_0(t,\lambda_n)\big\rangle\,dt \right|^2 - \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \left| \int_0^x \big\langle f(t), Y_0(t,(n-1/2))\big\rangle\,dt \right|^2.$$

In view of Parseval's identity, we obtain

$$\|f\|^2_{L_{2,2}(0,x)} = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \left| \int_0^x \big\langle f(t), Y_0(t,(n-1/2))\big\rangle\,dt \right|^2;$$

therefore,

$$\sum_{n=-\infty}^{\infty} z_n \left| \int_0^x \big\langle f(t), Y_0(t,\lambda_n)\big\rangle\,dt \right|^2 = 0. \tag{22}$$

Since $\operatorname{Re} z_n < 0$ for each n, Eq. (22) implies that $\int_0^x \langle f(t), Y_0(t,\lambda_n)\rangle\,dt = 0$. The latter and the completeness of the system of vectors $\{Y_0(t,\lambda_n)\}$ in $L_{2,2}(0,\pi)$ imply the identity $f(t) \equiv 0$. The unique solvability of Eq. (21) implies [4] that the functions $c(\lambda)$ and $-s(\lambda)$ are the entries of the first row of the monodromy matrix

$$\tilde{U}(\pi,\lambda) = \begin{pmatrix} \tilde{c}_1(\pi,\lambda) & -\tilde{s}_2(\pi,\lambda) \\ \tilde{s}_1(\pi,\lambda) & \tilde{c}_2(\pi,\lambda) \end{pmatrix}$$

of problem (1), (2) with the matrix A defined in (10) and some potential $\tilde{V} \in L_2(0,\pi)$; i.e.,

$$c(\lambda) = \tilde{c}_1(\pi,\lambda), \quad s(\lambda) = \tilde{s}_2(\pi,\lambda). \tag{23}$$

By virtue of (4), the characteristic determinant $\tilde{\Delta}(\lambda)$ of this problem has the form

$$\tilde{\Delta}(\lambda) = \tilde{s}_1(\pi,\lambda) - \tilde{s}_2(\pi,\lambda) = \tilde{f}(\lambda),$$

where $\tilde{f} \in PW_\pi$. Relations (3), (18), and (23) imply the equality

$$\tilde{\Delta}(\lambda_n) = \tilde{s}_1(\pi, \lambda_n) - \tilde{s}_2(\pi, \lambda_n) = \frac{1}{\tilde{s}_2(\pi, \lambda_n)} - \tilde{s}_2(\pi, \lambda_n) = \frac{1}{s(\lambda_n)} - s(\lambda_n) = f(\lambda_n).$$

It follows from the last equality that the function

$$\Phi(\lambda) = \frac{f(\lambda) - \tilde{\Delta}(\lambda)}{c(\lambda)} = \frac{f(\lambda) - \tilde{f}(\lambda)}{c(\lambda)}$$

is entire. Since

$$\left| f(\lambda) - \tilde{f}(\lambda) \right| < c_1 e^{\pi |\operatorname{Im} \lambda|}, \quad c_1 = \text{const}, \tag{24}$$

we conclude in view of inequality (13) that $|\Phi(\lambda)| \le c_2 = \text{const}$ if $|\operatorname{Im} \lambda| \ge M$.

Let H stand for the union of vertical segments $\{z : |\operatorname{Re} z| = n, |\operatorname{Im} z| \le M\}$, where $|n| = N_0 + 1, N_0 + 2, \ldots$ Since the function $c(\lambda)$ is a sine-type function [9], we have $|c(\lambda)| > \delta > 0$ for $\lambda \in H$. The last inequality, the estimate (24), and the maximum principle imply the inequality $|\Phi(\lambda)| < c_3 = \text{const}$ in the strip $|\operatorname{Im} \lambda| \le M$. Consequently, the function $\Phi(\lambda)$ is bounded in the entire complex plane and is constant by Liouville's theorem. Let $|\operatorname{Im} \lambda| = M$. Then, in view of relation (11), we have $\lim_{|\lambda| \to \infty} (f(\lambda) - \tilde{f}(\lambda)) = 0$; therefore, $\Phi(\lambda) \equiv 0$, and hence $f(\lambda) \equiv \tilde{\Delta}(\lambda)$. The proof of the theorem is complete.

Examples of functions in the class PW_π with roots of arbitrarily high multiplicity are known in the literature (see, e.g., [10, 11]). Note that the existence of one-dimensional boundary value problems with an unboundedly increasing multiplicity of eigenvalues was previously established for the Sturm–Liouville operator and an ordinary differential operator of any even order [10–12].

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