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NUMERICAL METHODS

Criterion for the Stability of Difference Schemes for Nonlinear Differential Equations

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Abstract—For abstract nonlinear difference schemes with operators acting in finite-dimensional Banach spaces, a stability criterion is stated and proved; namely, for a consistent finite-difference approximation to a well-posed differential problem, the solution of the difference scheme converges if and only if the scheme is unconditionally stable. In a sense, this criterion generalizes Lax's equivalence theorem to nonlinear differential problems. The results obtained are used to study the stability of difference schemes that approximate quasilinear parabolic equations with nonlinearities of unbounded growth.

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INTRODUCTION

The basic concepts of the theory of difference schemes are consistency, stability, and convergence. The connection between these concepts is given by the Filippov–Ryaben'kii theorem [1, p. 16; 2, p. 764], which is known abroad as the Lax equivalence theorem [3, p. 54]; namely, for a consistent finite-difference method for a well-posed linear initial–boundary value problem for partial differential equations, the difference method converges if and only if it is stable. By *consistency* we mean the requirement to approximate a well-posed differential problem. In the nonlinear case, convergence, generally speaking, does not imply stability [4].

Many authors have attempted to transfer the above-formulated statement to nonlinear difference problems [5–7]. A survey of some results in this direction is presented in [4] and is mainly related to other definitions of stability such as weak stability or weak generalized stability. Noteworthy is the series of papers [8–12] dealing with the study of the stability of difference schemes approximating quasilinear parabolic and hyperbolic equations of a special form. All studies in these papers are carried out only under assumptions related solely to the properties of the input data of the differential problem. Stability in the general case can be proved only up to a certain finite time instant $t \le t_0$, whose value is accounted for by the application of a grid analog of the Bihari lemma. In [13, 14], similar results were obtained for computational methods for the equations of a polytropic gas with subsonic flows.

In the present paper, the Lax equivalence theorem is generalized to abstract nonlinear difference problems with operators acting in finite-dimensional Banach spaces. In the nonlinear case, such a criterion can be established only for unconditionally stable computational methods, when the corresponding a priori estimates take place for a sufficiently small $|h| \le h_0$. In this case, the value of h_0 depends both on the consistency of discrete and continuous norms in Banach spaces and on the magnitude of the perturbation in the input data of the problem. The studies carried out here allow us to conclude that there is a close and inextricable connection between the concepts of stability in discrete and continuous cases.

1. STATEMENT OF THE PROBLEM

Let H_k be a Banach space with the norm $\|\cdot\|_k$, $k=1,2$, let $L: H_1 \to H_2$ be a nonlinear unbounded differential operator, and assume that we are given an element $f \in H_2$. Consider the operator equation

$$Lu = f. \tag{1}$$

In the sequel, we assume that problem (1) is Hadamard well posed; i.e., the following conditions are satisfied:

1. There exists a unique solution for all input data $f \in H_2$.
2. The solution continuously depends on the input data; i.e., there exists a positive constant $c_0 > 0$ for which the following inequality is satisfied:

$$\|\tilde{u} - u\|_1 \le c_0 \|\tilde{f} - f\|_2, \tag{2}$$

where $\tilde{u} \in H_1$ is the solution of problem (1) with the perturbed input data $\tilde{f} \in H_2$.

The property of the solution of the differential problem expressed by inequality (2) is called the *stability* of the solution with respect to a small perturbation in the input data.

For an approximate solution of problem (1), we use the difference scheme (abstract notation)

$$L_h y = \varphi_h. \tag{3}$$

Here $L_h : H_{1h} \to H_{2h}$ and $\varphi \in H_{2h}$ approximate L and f, respectively, and the H_{kh}, $k = 1, 2$, are finite-dimensional Banach spaces depending on a positive parameter h that is a vector of some normed space with norm $|h|$.

In the present paper, we stick to the main definitions of the theory of difference schemes given in [7, 15].

By the *approximation* of the difference scheme (3) on the solution of the differential problem (1) we mean the error

$$\psi_h = L_h u_h - \varphi_h = L_h u_h - (Lu)_h + f_h - \varphi_h, \tag{4}$$

for which

$$\|\psi_h\|_{2h} \le M h^{k_3}, \quad k_3 = \text{const} > 0. \tag{5}$$

We say that a difference scheme is *consistent* with the differential problem if

$$\left\|L_h u_h - (Lu)_h\right\|_{2h} \to 0 \quad \text{and} \quad \|f_h - \varphi_h\|_{2h} \to 0 \quad \text{as} \quad |h| \to 0. \tag{6}$$

For all elements in H_m and H_{mh}, we assume that $\Pi_{mh} g = g_h$, where Π_{mh} is the projection. In the case of continuous functions, the operator Π_{mh} is the unity (identity) one, i.e.,

$$g_{mh}(x) = \Pi_{mh} g_m(x) = g_m(x), \quad m = 1, 2; \quad x \in \overline{\omega}_h.$$

We will also assume that the mesh norms $\|\cdot\|_{kh}$ introduced in H_{kh} are consistent with the corresponding norms $\|\cdot\|_k$ in the spaces H_k, $k = 1, 2$; i.e.,

$$|\|g_h\|_{mh} - \|g\|_m| \le c_m |h|^{k_m}, \quad m = 1, 2,$$

for all $g_h \in H_{mh}$ and $g \in H_m$, where $k_m > 0$.

Moreover, we assume that the difference scheme (3) is consistent with the well-posed problem (1) in the sense of satisfying relations (5) and (6).

Recall also that the solution of the difference scheme *converges* to the solution of the differential problem at the rate $O(|h|^{k_3})$ if the following inequality holds:

$$\|y - u_h\|_{1h} \le c_3 |h|^{k_3}.$$

2. CONVERGENCE CRITERION

Let us state and prove the main result of the present paper.

Theorem. *If the well-posed problem* (1) *and its finite-difference approximation satisfy the consistency condition, then the unconditional stability is necessary and sufficient for the convergence of the difference scheme.*

Proof. The **necessity** was proved earlier (see, e.g., [7, p. 107]). For the completeness of the presentation, we reproduce this proof here. Thus, let the difference scheme (3) be unconditionally stable. It follows that there exists a constant c_4 independent of $h, y, \tilde{y}$ such that for all sufficiently small $|h| \le h_0$ one has the a priori estimate

$$\|\tilde{y} - y\|_{1h} \le c_4 \|\tilde{\varphi}_h - \varphi_h\|_{2h}, \tag{7}$$

where $\tilde{y}$ is the solution of problem (3) with the input data $\tilde{\varphi}_h \in H_{2h}$. Recall also that if inequality (7) is satisfied for arbitrary $|h|$, then such a scheme is said to be *absolutely stable* [7, p. 286].

From relation (4), which determines the error ψ_h, we express

$$L_h u_h = \psi_h + \varphi_h = \tilde{\varphi}_h;$$

based on this, by virtue of the definition of stability and inequalities (5) and (7), we obtain

$$\|y - u_h\|_{1h} \le c_4 \|\tilde{\varphi}_h - \varphi_h\|_{2h} = c_4 \|\psi_h\|_{2h} \to 0 \quad \text{as} \quad |h| \to 0.$$

The necessity is proved.

Sufficiency. Let us prove that the convergence implies the unconditional stability of the scheme; i.e., there exists a positive constant c_4 such that the estimate (7) holds for a sufficiently small

$$|h| \le h_0, \quad h_0 = c_5 \|\tilde{\varphi}_h - \varphi_h\|_{2h}^{1/k_4}, \quad k_4 = \min\{k_1, k_2, k_3\}. \tag{8}$$

In view of the above assumptions, using the triangle inequality for norms, we obtain

$$\begin{aligned}\|\tilde{y} - y\|_{1h} &= \|\tilde{y} - \tilde{u}_h - (y - u_h) + (\tilde{u}_h - u_h)\|_{1h} \le \|\tilde{y} - \tilde{u}_h\|_{1h} + \|y - u_h\|_{1h} + \|\tilde{u} - u_h\|_{1h} \\ &\le c_5 |h|^{k_3} + c_1 |h|^{k_1} + \|\tilde{u} - u\|_1 \le c_5 |h|^{k_3} + c_1 |h|^{k_1} + c_0 \|\tilde{f} - f\|_2\end{aligned}$$

and hence

$$\|\tilde{y} - y\|_{1h} \le c_5 |h|^{k_3} + c_1 |h|^{k_1} + c_0 c_2 |h|^{k_2} + c_0 \|\tilde{f}_h - f_h\|_{2h}.$$

Now, since

$$\|\tilde{f}_h - f_h\|_{2h} = \|\tilde{f}_h - \tilde{\varphi}_h - (f_h - \varphi_h) + \tilde{\varphi}_h - \varphi_h\|_{2h} \le \|\psi_h\|_{2h} + \|\tilde{\varphi}_h - \varphi_h\|_{2h} \le M|h|^{k_1} + \|\tilde{\varphi}_h - \varphi_h\|_{2h},$$

we arrive at the estimate (7), which means the unconditional stability of the difference scheme (3) under assumption (8).

The proof of the theorem is complete.

3. STABILITY OF DIFFERENCE SCHEMES APPROXIMATING A QUASILINEAR PARABOLIC EQUATION

3.1. The Case of Existence of Classical Solution

The theory of difference schemes for nonlinear equations of mathematical physics with nonlinearities of unbounded growth is one of the most difficult and topical areas of computational mathematics. The issues of convergence and well-posedness of difference schemes for this class of problems have been studied by many authors [16–19].

Despite the obtained estimates of the accuracy of solutions of difference schemes that approximate nonlinear equations of mathematical physics, the question of their stability remained open for a long time. In our opinion, the main reason for the lack of scientific results in this direction is associated with the need to obtain preliminary a priori estimates not only for the difference solution in the problem for the perturbation $\delta y = \tilde{y} - y$ but also for its derivatives in the strong uniform metric.

The criterion for the convergence of nonlinear difference schemes proved in this paper allows one to prove the unconditional stability of difference methods for which convergence has already been proved.

In the rectangle $\overline{Q}_T = \overline{\Omega} \times [0 \le t \le T]$, where $\overline{\Omega} = \{x : 0 \le x \le l\}$, we consider the Dirichlet boundary value problem for the quasilinear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right), \tag{9}$$

$$u(x, 0) = u_0(x), \quad x \in \overline{\Omega}; \quad u(0, t) = \mu_1(t), \quad u(l, t) = \mu_2(t), \quad t \in [0, T]. \tag{10}$$

We introduce the range of the exact solution

$$\overline{D} = \big\{u : m \le u \le M,\ \ \overline{u}_0 > 0,\ \ (x,t) \in \overline{Q}_T\big\},\quad m = \inf_{(x,t)\in\overline{Q}_T} u(x,t),\quad M = \sup_{(x,t)\in\overline{Q}_T} u(x,t),$$

and define its neighborhood

$$\overline{D}_1 = \big\{\tilde{u} : |\tilde{u} - u| < r,\ \ u \in \overline{D},\ \ r > 0\big\}.$$

In problems with an unbounded nonlinearity, it is assumed that there exists a constant $r_0 > 0$ such that

$$k(u) \ge r_0 \quad \text{for all} \quad u \in \overline{D}$$

and in addition, the function $k(\tilde{u})$ has all derivatives bounded in $\overline{D}_1$.

We assume that problem (9), (10) is well posed in the following sense:

(a) There exists its unique solution $u(x,t) \in C^{2+\lambda,1+\beta}(\overline{Q}_T)$, $0.5 < \lambda,\ \beta < 1$, with the function $\partial^2 u/\partial x^2$ being Lipschitz continuous in the variable t. Here $C^{m_1+\lambda,m_2+\beta}(\overline{Q}_T)$ is the class of functions whose x-derivatives of order $\le m_1$ and t-derivatives of order $\le m_2$ are continuous in $\overline{Q}_T$ and satisfy the Hölder condition with exponents λ and β, respectively.

(b) The solution is stable in the uniform norm for all $u, \tilde{u} \in C^{2+\lambda,1+\beta}(\overline{Q}_T)$ with respect to small disturbances in the initial data,

$$\|\tilde{u} - u\|_{C(\overline{Q}_T)} \le c_0 \|\tilde{u}_0 - u_0\|_{C(\overline{\Omega})},$$

where $\|\cdot\|_{C(\overline{Q}_T)} = \max\limits_{(x,t)\in\overline{Q}_T} |\cdot|$, $\|\cdot\|_{C(\overline{\Omega})} = \max\limits_{x\in\overline{\Omega}} |\cdot|$, and $\tilde{u}$ is the solution of problem (9), (10) with the perturbed initial condition $\tilde{u}_0$.

On the uniform mesh $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$, where $\overline{\omega}_h = \{x_i = ih, i = 0,\dots,N, hN = l\}$ and $\overline{\omega}_\tau = \{t_n = n\tau$, $N = 0,\dots,N_0,\ \tau N_0 = T\}$, we approximate the original differential problem by the conservative purely implicit difference scheme

$$y_t = \big(k(\hat{y}_{(0.5)})\hat{y}_{\overline{x}}\big)_x,\quad y_{(0.5)} = (y_{i-1} + y_i)/2, \tag{11}$$

$$y_i^0 = u_{0i},\quad i = 0,\dots,N;\quad y_0^{n+1} = \mu_1^{n+1},\quad y_N^{n+1} = \mu_N^{n+1}. \tag{12}$$

Here we have used the standard notation of the theory of difference schemes [7, p. 12],

$$y = y_i^n = y(x_i,t_n),\quad y_t = (\hat{y} - y)/\tau,\quad \hat{y} = y_i^{n+1},\quad y_{\overline{x}} = (y_i - y_{i-1})/h,$$

$$y_x = (y_{i+1} - y_i)/h,\quad (ay_{\overline{x}})_x = (a_{i+1}y_{\overline{x},i+1} - a_i y_{\overline{x},i})/h.$$

The accuracy of the difference scheme (11), (12) was studied in detail in the paper [18]. In particular, the estimate $\|\psi\|_{C(\overline{\omega}_{h\tau})} \le M(h^\lambda + \tau^\beta)$, $M = \text{const} > 0$, was obtained for the approximation error $\psi = -u_t + (k(\hat{u}_{(0.5)})\hat{u}_{\overline{x}})_x$ on the solution of the differential problem and the following accuracy estimate was proved:

$$\|y - u\|_{C(\overline{\omega}_{h\tau})} \le c_3(h^{\lambda-0.5} + \tau^{\beta-0.5}),$$

where, as usual, $\|\cdot\|_{C(\overline{\omega}_h)} = \max\limits_{x\in\overline{\omega}_h} |\cdot|$ and $\|\cdot\|_{C(\overline{\omega}_{h\tau})} = \max\limits_{(x,t)\in\overline{\omega}_{h\tau}} |\cdot|$.

Obviously, a similar estimate holds for the perturbed difference scheme,

$$\|\tilde{y} - \tilde{u}\|_{C(\overline{\omega}_{h\tau})} \le c_3(h^{\lambda-0.5} + \tau^{\beta-0.5});\quad 0.5 < \lambda,\quad \beta < 1.$$

Based on the above, we conclude that

$$\begin{aligned}\|\tilde{y} - y\|_{C(\overline{\omega}_{h\tau})} &\le \|y - u\|_{C(\overline{\omega}_{h\tau})} + \|\tilde{y} - \tilde{u}\|_{C(\overline{\omega}_{h\tau})} + \|\tilde{u} - u\|_{C(\overline{\omega}_{h\tau})} \\ &\le 2c_3(h^{\lambda-0.5} + \tau^{\beta-0.5}) + \|\tilde{u}_0 - u_0\|_{C(\overline{\Omega})}.\end{aligned}$$

Obviously, for sufficiently small $h \le h_0$ and $\tau \le \tau_0$ satisfying the inequality

$$2c_3(h^{\lambda-0.5} + \tau^{\beta-0.5}) \le \|\tilde{u}_0 - u_0\|_{C(\overline{\Omega})},$$

the difference scheme (11), (12) is unconditionally stable in the C-norm with respect to the initial data, and one has the inequality

$$\|\tilde{y} - y\|_{C(\overline{\omega}_{h\tau})} \le 2\|\tilde{u}_0 - u_0\|_{C(\overline{\Omega})}.$$

3.2. Stability of Difference Schemes for Problems with Generalized Solutions

In the rectangle $\overline{Q}_T$, consider problem (10) for the somewhat more general equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x,t,u)\frac{\partial u}{\partial x}\right), \quad (x,t) \in Q_T, \tag{13}$$

whose coefficient $k(x,t,u)$ has the properties

$$k(x,t,u) \in C(\overline{Q}_T \times \overline{D}), \quad \frac{\partial k(x,t,u)}{\partial u} \in C(\overline{Q}_T \times \overline{D}),$$
$$k(x,t,u) \ge k_1 > 0 \quad \text{for all } (x,t) \in \overline{Q}_T, u \in \overline{D}.$$

In accordance with [20], we call the function $u(x,t)$ a *generalized solution* of problem (13), (10) if for each infinitely differentiable function $\varphi(x,t)$ with compact support one has the equality

$$\iint\limits_{Q_T} \left(-u\frac{\partial \varphi}{\partial t} + k(x,t,u)\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\right) dx\, dt = 0. \tag{14}$$

If $u(x,t) \in C(\overline{Q}_T)$ and $\partial u/\partial x$ is a piecewise continuous function, then condition (14) is equivalent to the equality

$$\oint\limits_C u\, dx + k(x,t,u)\frac{\partial u}{\partial x}\, dt = 0 \tag{15}$$

along any contour bounding the subdomain $Q' \in \overline{Q}_T$. This is a common statement of conservative laws [20]. Using identity (14), we can define a generalized solution of problem (13), (10) in the space $L_2(0,T;H_0^1(\Omega))$. Such a solution—when the existence of the derivative $\partial u/\partial t$ is not required in any sense—is often referred to as a *weak generalized solution*. However, relation (14) implicitly contains information about the derivative [21]; namely,

$$\frac{\partial u}{\partial t} \in L_2(0,T;H^{-1}(\Omega)).$$

We cannot use the definition in the form (14), because first, in the case under consideration, the function $u(x,t)$ is not zero on the boundary, and second, we construct a theory that would be true in the case of nonself-adjoint operators and for problems of arbitrary dimension.

According to (15), on the line of discontinuity $\partial x/\partial t = D(t)$ one has the equality

$$\left[uD - k(x,t,u)\frac{\partial u}{\partial x}\right] = 0; \tag{16}$$

here and below by $[\cdot]$ we denote the difference between the values of the function on the left and on the right of the line of discontinuity.

Physical laws asserting the continuity of the solution and the flux are a special case of relation (16),

$$[u] = 0, \quad \left[k(x,t,u)\frac{\partial u}{\partial x}\right] = 0. \tag{17}$$

Since we assume that $u(x,t) \in C(\overline{Q}_T)$ in what follows, we restrict ourselves to the case in which $\partial u/\partial x$ has discontinuities of only the first kind on the lines of discontinuity $x_k = v_k(t)$, $k = 1, \dots, m$.

Note that in the case of linear problems, the second condition in (17) has the form

$$\left[k(x,t)\frac{\partial u}{\partial x}\right] = 0.$$

Therefore, if $\partial u/\partial x$ is a discontinuous function, then this implies a discontinuity in the coefficient $k = k(x,t)$. This is not necessary in the nonlinear case. The continuity of the flux can be ensured by the degeneracy of the coefficient $k = k(x,t,u)$ on weak lines of discontinuity, as we can observe in the case of running temperature waves along the zero background for the power-law nonlinearities $k = u^\sigma$, $\sigma > 0$ (see [7, p. 450]).

In the case of discontinuous coefficients along the straight lines $x = \xi$, Samarskii [7, p. 417] proved that the best conservative scheme with the stencil functional of the form

$$a_i = \left(\frac{1}{h}\int\limits_{x_{i-1}}^{x_i} \frac{dx}{k(x,t)}\right)^{-1}$$

converges in the L_2-norm with the second order in the spatial variable. A more complicated case where the line of discontinuity is not parallel to the coordinate axes was not considered. In the general case, to obtain the corresponding estimates of the approximation error, along with a negative space norm one also needs to use norms negative with respect to the time variable.

On the introduced uniform mesh $\overline{\omega}_{h\tau}$, we approximate the differential problem (13), (10) by the linearized difference scheme

$$y_t = (a\hat{y}_{\overline{x}})_x, \tag{18}$$

$$y(x,0) = u_0(x), \quad x \in \overline{\omega}_h; \quad y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}), \quad t_n \in \omega_\tau. \tag{19}$$

The stencil functional

$$a = a(y) = 0.5\big[k(x_{(0.5)}, t_n, y_{i-1}^n) + k(x_{(0.5)}, t_n, y_i^n)\big], \quad x_{(0.5)} = \frac{1}{2}(x_{i-1} + x_i),$$

is chosen, as usual, based on the condition of second-order consistency for the elliptic operator [7, p. 409],

$$(a\hat{u}_{\overline{x}})_x - \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) = O(h^2 + \tau).$$

Let us indicate some properties of the solution of the difference scheme (18), (19). Let us define the range of the generalized solution of problem (13), (10),

$$\overline{D} = \big\{m_1 \le u(x,t) \le m_2, \quad (x,t) \in \overline{Q}_T\big\},$$
$$m_1 = \min_{(x,t)\in\overline{Q}_T}\big\{\mu_1(t), \mu_2(t), u_0(x)\big\}, \quad m_2 = \max_{(x,t)\in\overline{Q}_T}\big\{\mu_1(t), \mu_2(t), u_0(x)\big\}.$$

Further, for the difference solution we use the two-sided estimate

$$m_1 \le y_i^n \le m_2, \quad i = 0, \dots, N, \quad n = 0, \dots, N_0, \tag{20}$$

proved in [22]; i.e., $y(x,t) \in \overline{D}_u$ for all $(x,t) \in \overline{\omega}_{h\tau}$. The proof is based on the maximum principle established in [22, 23] for difference schemes with alternating-sign input data.

The a priori estimate [22]

$$\|y^n\|_{C(\overline{\omega}_{h\tau})} \le \max\left\{\max_{t\in\overline{\omega}_\tau}\big\{|\mu_1(t)|, |\mu_2(t)|\big\}, \quad \|u_0\|_{C(\overline{\omega}_h)}\right\}$$

is a corollary of the two-sided estimate (20).

Now let us produce a problem for the error $z = y - u$ of the method. Owing to the nonlinearity of the scheme being explored, this problem will certainly be nontrivial. In fact, subtracting the equation $u_t = (a\hat{u}_{\overline{x}})_x$ for the approximation error from the difference equation (18), we obtain two equivalent forms for the equation for the error z of the method,

$$z_t = \big(a(y)\hat{z}_{\overline{x}}\big)_x + \Big(\big(a(y) - a(u)\big)\hat{u}_{\overline{x}}\Big)_x + \psi, \tag{21}$$

$$z_t = \big(a(y)\hat{\tilde{z}}_{\overline{x}}\big)_x + \Big(\big(a(y) - a(u)\big)\hat{y}_{\overline{x}}\Big)_x + \psi. \tag{22}$$

This equations should be equipped with the appropriate initial and boundary conditions,

$$z(x,0) = 0, \quad x \in \overline{\omega}_h, \quad z(0,\hat{t}) = z(l,\hat{t}) = 0, \quad t \in \omega_\tau.$$

Although these problems are equivalent, in the statement (22) we need to have preliminary information about local behavior of the difference derivative of the approximate solution y_x. Producing such an a priori estimate for the derivative y_x is not an easy problem. Similar problems arise in the direct study of stability. In this case, the problem for the disturbance $\delta y = \tilde{y} - y$ has the form

$$\delta y_t = \big(a(y)\delta\hat{y}_{\overline{x}}\big)_x + \Big(\big(a(\tilde{y}) - a(y)\big)\hat{\tilde{y}}_{\overline{x}}\Big)_x,$$
$$\delta y(x,0) = \tilde{u}_0 - u_0, \quad x \in \overline{\omega}_h; \quad \delta y(0,\hat{t}) = \hat{\tilde{\mu}}_1 - \hat{\mu}_1, \quad \delta y(l,\hat{t}) = \hat{\tilde{\mu}}_2 - \hat{\mu}_2, \quad t \in \omega_\tau.$$

Since $y, u \in \overline{D}_u$ and $k(u) \in C^1(\overline{D}_u)$, we have

$$\max_{(x,t)\in\overline{\omega}_{h\tau}} \big|a(y) - a(u)\big| \le L z_{(0.5)}, \quad L = \text{const} > 0.$$

Further, we will use the inner products and norms on the space of mesh functions $L_2(\omega_h)$, $W_2^1(\omega_h)$

$$(v,g) = \sum_{i=1}^{N-1} h v_i g_i, \quad \|v\|_h = \sqrt{(v,v)}, \quad \|v_{\overline{x}}]|^2 = \sum_{i=1}^{N} h v_{\overline{x},i}^2.$$

Taking the inner product of the difference equation (21) by $2\tau z$ in $L_2(\omega_h)$ and using the summation by parts formula

$$(u, v_x) = -(u_{\overline{x}}, v] + u_N v_N - u_0 v_1$$

as well as the identity

$$z^{n+1} = 0.5(z^n + z^{n+1}) + 0.5\tau z_t,$$

we obtain the energy relation

$$\tau^2\|z_t\|_h^2 + \|z^{n+1}\|_h^2 + 2\tau(a(y), \hat{z}_{\overline{x}}^2] = \|z^n\|_h^2 + 2\tau(a(y) - a(u), \hat{u}_{\overline{x}}\hat{z}_{\overline{x}}] + 2\tau(\hat{z}, \psi).$$

Further, following the paper [19] with the use of the technique of negative norms, we arrive at the following estimate of the accuracy of the method in the mesh L_2-norm:

$$\|y^n - u^n\|_h \le c_6(\sqrt{h} + \sqrt{\tau}), \quad n = 0, \dots, N_0,$$

This estimate implies the unconditional convergence of the difference solution to the generalized solution of the differential problem (13), (10).

Obviously, a similar estimate also holds for the solution $\tilde{y}$ of the difference scheme (18), (19) with the perturbed initial condition

$$\|\tilde{y}^n - \tilde{u}^n\|_h \le c_7(\sqrt{h} + \sqrt{\tau}), \quad n = 0, \dots, N_0.$$

To apply the theorem proved in this paper, we should assume that the generalized solution of problem (13), (10) exists, is unique, and continuously depends on the initial condition,

$$\max_{0\le t\le T}\|\tilde{u}-u\|_{L_2(0,l)} \le c_8\|\tilde{u}_0-u_0\|_{L_2(0,l)}.$$

Since we do not perturb the boundary condition ($\delta u(0,t)=\delta u(l,t)=0$), we see that the error

$$R(u)=\|u\|_h^2-\|u\|_{L_2(0,l)}$$

is the approximation error of the generalized quadrature trapezoid rule. Owing to the lack of existence of the second derivative $\partial^2 u/\partial x^2$, the norm consistency condition looks as follows:

$$\big|R(u)\big| \le c_8 h.$$

Now, applying similar estimates, just as in the proof of sufficiency in the theorem, we obtain the stability estimate

$$\max_{t\in\overline{\omega}_\tau}\big\|\tilde{y}(t)-y(t)\big\|_h \le c_9\|\tilde{u}_0-u_0\|_{L_2(0,l)},$$

which holds for all sufficiently small $h\le h_0$ and $\tau\le\tau_0$ satisfying the condition

$$\sqrt{h}+\sqrt{\tau}\le c_{10}\|\tilde{u}_0-u_0\|_{L_2(0,l)}.$$

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