ISSN 0012-2661, Differential Equations, 2021, Vol. 57, No. 6, pp. 768–779. © Pleiades Publishing, Ltd., 2021.
Russian Text © The Author(s), 2021, published in Differentsial'nye Uravneniya, 2021, Vol. 57, No. 6, pp. 784–795.

INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

On Positive Bounded Solutions of One Class of Nonlinear Integral Equations with the Hammerstein–Nemytskii Operator

A. Kh. Khachatryan[1*], Kh. A. Khachatryan[2,3], and H. S. Petrosyan[1,3***]**

[1] *Armenian National Agrarian University, Yerevan, 0009 Armenia*
[2] *Yerevan State University, Yerevan, 0025 Armenia*
[3] *Lomonosov Moscow State University, Moscow, 119991 Russia*
*e-mail: *Aghavard59@mail.ru, **khachatur.khachatryan@ysu.am, Khach82@rambler.ru, ***Haykuhi25@mail.ru*

Received February 28, 2021; revised April 16, 2021; accepted April 27, 2021

Abstract—We study a class of nonlinear integral equations with a noncompact Hammerstein–Nemytskii operator on the entire line. Some special cases of such equations have specific applications in various fields of natural science. The combination of a method for constructing invariant cone segments for the corresponding nonlinear monotone operator with methods of the theory of functions of a real variable allows one to prove a constructive theorem on the existence of bounded positive solutions of equations of the class under consideration. The asymptotic behavior of the solution at $\pm\infty$ is studied as well. In particular, we prove that the solution constructed in the paper is an integrable function on the negative half-line and that the difference between the limit at $+\infty$ and the solution is integrable on the positive half-line. In one special case, we show that our solution generates a one-parameter family of bounded positive solutions. At the end of the paper, we give specific applied examples of nonlinearities to illustrate the results.

DOI: 10.1134/S0012266121060069

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Consider the following class of nonlinear integral equations with a Hammerstein–Nemytskii type operator on the entire line:

$$f(x) = G_0\big(x, f(x)\big) + \int\limits_{\mathbb{R}} K(x-t)G\big(f(t)\big)\,dt, \quad x \in \mathbb{R} := (-\infty, +\infty), \tag{1}$$

for the unknown measurable bounded nonnegative function $f(x)$ on $\mathbb{R}$. The nonlinearities $G_0(x,u)$ and $G(u)$ in Eq. (1) are defined on the sets $\mathbb{R} \times \mathbb{R}^+$ and $\mathbb{R}^+$, respectively (where $\mathbb{R}^+ := [0, +\infty)$), take real values, and satisfy the conditions stated below.

The kernel K has the following main properties:

$$K(x) > 0, \quad x \in \mathbb{R}, \quad \int\limits_{\mathbb{R}} K(x)\,dx = 1, \tag{2}$$

$$K \in M(\mathbb{R}), \quad \int\limits_{\mathbb{R}} x^2 K(x)\,dx < +\infty, \tag{3}$$

$$\nu(K) := \int\limits_{\mathbb{R}} xK(x)\,dx > 0, \tag{4}$$

where $M(\mathbb{R})$ is the space of essentially bounded functions on $\mathbb{R}$.

The nonlinearity G satisfies the following conditions on the set $\mathbb{R}^+$:

1. $G(0) = 0$, and $G(u)$ is concave and monotone increasing.

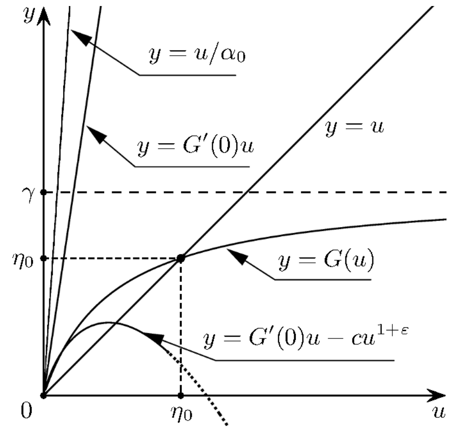

Fig. 1.

2. There exists a number $\eta_0 > 0$ such that $G(\eta_0) = \eta_0$, $G(u) > u$ for $u \in (0, \eta_0)$, and $G(u) < u$ for $u \in (\eta_0, +\infty)$.
3. There exists a limit $\lim\limits_{u\to+\infty} G(u) =: \gamma < +\infty$.
4. There exists a finite derivative $G'(0) > 1$ at zero with
$$G(u) \le G'(0)u \quad \text{for all} \quad u \ge 0.$$
5. There exist numbers $c > 0$ and $\varepsilon > 0$ such that $G(u) \ge G'(0)u - cu^{1+\varepsilon}$, $u \in [0, \eta_0]$.

Figure 1 graphically illustrates conditions 1–5.

Equation (1) arises in the mathematical modeling of various processes in many fields of natural science. In particular, such equations are encountered in the mathematical theory of space-time propagation of an epidemic taking into account the emergence of the second wave, in the kinetic gas theory, and in the theory of radiation transfer in spectral lines (see [1–6] and the bibliography therein). In the special case where $G_0 \equiv 0$, Eq. (1) was studied in detail in the papers [1–3]. In the case of $\nu(K) \le 0$, Eq. (1) was analyzed in the papers [7–9]. Note that these papers heavily rely on methods of the linear theory of integral equations of the convolution type.

In the present paper, we impose other conditions on the nonlinearities G_0 and G, use methods of the theory of nonlinear monotone operators, and apply specially selected iterations to prove the existence of positive bounded solutions of Eq. (1). The corresponding a priori estimates in particular imply the integrability of the solution on the negative half-line. Further, in one important special case we use the convexity of the nonlinearity G and some geometric inequalities to establish the existence of a limit $f(+\infty)$ of the solution at infinity and the inclusion $f(+\infty) - f(x) \in L_1(\mathbb{R}^+)$. At the end of the paper, particular applied examples of the functions G_0 and G are given. Before imposing conditions on the function $G_0(x, u)$, we introduce some notation and present corollaries, to be used in what follows, of the properties of the function $G(u)$.

2. NOTATION, AUXILIARY FACTS, AND MAIN CONDITIONS ON THE FUNCTION G_0

2.1. Notation and Auxiliary Facts

Set
$$\alpha_0 := \int\limits_{-\infty}^{0} K(t)\,dt > 0. \tag{5}$$

Along with Eq. (1), consider the following auxiliary equation of the Hammerstein type on the entire line:
$$\varphi(x) = \int\limits_{\mathbb{R}} K(x-t)G\big(\varphi(t)\big)\,dt, \quad x \in \mathbb{R}, \tag{6}$$

for the unknown bounded continuous positive function $\varphi(x)$ on $\mathbb{R}$.

It was proved in the paper [3] that a necessary condition for Eq. (6) to have such a solution is given by the inequality

$$G'(0)\alpha_0 < 1. \tag{7}$$

In what follows, we assume that condition (7) holds unless specified otherwise.

On the set $\mathbb{R}^+$, we define the function

$$\chi(u) := (uG'(0) - G(u))\alpha_0 + G(u) - u, \quad u \geq 0. \tag{8}$$

Since there exists a finite derivative $G'(0)$ by condition 4 and α_0 is finite by property (2), we see that χ is well defined. It follows from the equality in condition 1 that $\chi(0) = 0$ and from the definition of the number η_0 in condition 2 and the inequality $G'(0) > 1$ in condition 4 that $\chi(\eta_0) = \eta_0(G'(0) - 1)\alpha_0 > 0$. By virtue of condition 3 and inequality (7), we have $\chi(+\infty) = -\infty$.

Consequently, by the Bolzano–Cauchy theorem there exists a number $\eta > \eta_0$ such that

$$\chi(\eta) = 0. \tag{9}$$

Let us verify that the number $\eta > \eta_0$ is uniquely determined from the equation $\chi(u) = 0$. Assume the contrary: there exists an $\tilde{\eta} > \eta_0$, $\tilde{\eta} \neq \eta$, such that $\chi(\tilde{\eta}) = 0$. Then definition (8) of the function $\chi(u)$ implies the relation

$$\frac{G(\tilde{\eta}) - G(\eta)}{\tilde{\eta} - \eta} = \frac{1 - G'(0)\alpha_0}{1 - \alpha_0}. \tag{10}$$

On the other hand, since the function $G(u)$ is concave on $\mathbb{R}^+$, it follows that

$$\frac{G(\widetilde{\eta}) - G(\eta)}{\widetilde{\eta} - \eta} < \frac{G(\eta)}{\eta} \tag{11}$$

for any $\widetilde{\eta}, \eta \in \mathbb{R}^+$. Indeed, consider the points $O(0,0)$, $A(\eta, G(\eta))$, and $\widetilde{A}(\widetilde{\eta}, G(\widetilde{\eta}))$ of the graph of $G(u)$. If $\widetilde{\eta} - \eta > 0$, then inequality (11) is obviously equivalent to the inequality $G(\widetilde{\eta})/\widetilde{\eta} < G(\eta)/\eta$. However, the latter inequality holds, because, by virtue of the concavity of G, the point A lies above the segment $O\widetilde{A}$ and hence the slope $G(\widetilde{\eta})/\widetilde{\eta}$ of the segment $O\widetilde{A}$ to the abscissa axis is less than the slope $G(\eta)/\eta$ of the segment OA to the abscissa axis. In exactly the same way, if $\eta - \widetilde{\eta} > 0$, then inequality (11) is obviously equivalent to the inequality $G(\widetilde{\eta})/\widetilde{\eta} > G(\eta)/\eta$, which holds, because in this case, by virtue of the concavity of G, the point $\widetilde{A}$ lies above the segment OA.

By the definition of the function χ (or by setting $\widetilde{\eta} = 0$ in (10)), one has the relation

$$\frac{G(\eta)}{\eta} = \frac{1 - G'(0)\alpha_0}{1 - \alpha_0},$$

which contradicts relations (10) and (11). Therefore, the equation $\chi(u) = 0$ has a unique solution for $u \in (\eta_0, +\infty)$.

2.2. Main Conditions on the Function G_0

Now we are in a position to present the conditions that will be imposed on the function $G_0(x, u)$.

(a) The function $G_0(x, u)$ satisfies the Carathéodory condition with respect to the argument u on the set $\mathbb{R} \times [0, \eta]$; i.e., the function $G_0(x, u)$ is measurable with respect to x on $\mathbb{R}$ for any $u \in [0, \eta]$ and continuous with respect to u on $[0, \eta]$ for almost all $x \in \mathbb{R}$.

(b) For each x, the function $G_0(x, u)$ is monotone increasing with respect to u on the set $\mathbb{R}^+$.

(c) There exists a number $\xi \in (\eta_0, \eta)$ such that

$$G_0(x, u) \geq u \int_{-\infty}^{x} K(y)\, dy, \quad u \in [0, \xi], \quad x \in \mathbb{R}.$$

(d) One has the upper bound

$$G_0(x,\eta) \le \big(\eta G'(0) - G(\eta)\big)\alpha_0 \int\limits_{-\infty}^{x} K(y)\,dy, \quad x \in \mathbb{R},$$

where the number $\eta > \eta_0$ is the unique root of the equation $\chi(u) = 0$ and the number α_0 is determined by relation (5).

3. CONSTRUCTION OF A POSITIVE BOUNDED SOLUTION OF EQUATION (1)

3.1. Diekmann Function

Consider the Diekmann function [2]

$$L(\lambda) := G'(0) \int\limits_{\mathbb{R}} K(t) e^{-\lambda t}\,dt, \quad \lambda \ge 0,$$

under the condition that this integral converges for $\lambda \in [0,\lambda_1]$, $\lambda_1 > 0$. In view of the fact that $G'(0) > 1$, properties (2)–(4) imply the inequalities

$$L(0) = G'(0) > 1, \tag{12}$$

$$\left.\frac{dL}{d\lambda}\right|_{\lambda=0} = -G'(0) \int\limits_{\mathbb{R}} K(t) t\,dt < 0, \tag{13}$$

$$\frac{d^2L}{d\lambda^2} = G'(0) \int\limits_{\mathbb{R}} K(t) t^2 e^{-\lambda t}\,dt > 0 \tag{14}$$

(where the last integral may be infinite).

It follows from (14) that the function $L(\lambda)$ is convex on $[0,\lambda_1]$. Since $L(\lambda)$ is continuous, it follows from inequality (13) by the Cauchy theorem that there exists a number $\lambda_0 \in (0,\lambda_1]$ such that the inequality

$$\frac{dL}{d\lambda} < 0 \tag{15}$$

holds for all $\lambda \in [0,\lambda_0]$. Assume that

$$L(\lambda_0) < 1. \tag{16}$$

Then it follows from inequalities (12) and (16) by the Bolzano–Cauchy theorem that there exists a unique $\sigma \in (0,\lambda_0)$ such that

$$L(\sigma) = 1. \tag{17}$$

Now consider the auxiliary equation (6). It follows from the results in the paper [3] that under conditions 1–5 Eq. (6) has a positive bounded continuous nondecreasing solution φ with the properties

$$\varphi(-\infty) = 0, \quad \varphi(+\infty) = \eta_0, \tag{18}$$

$$\varphi \in L_1(-\infty,0), \quad \eta_0 - \varphi \in L_1(0,+\infty). \tag{19}$$

Moreover, the following upper bound holds for the solution φ:

$$\varphi(x) \le \begin{cases} \eta_0 e^{\sigma x} & \text{for} \quad x \le 0 \\ \eta_0 & \text{for} \quad x > 0. \end{cases} \tag{20}$$

Properties (18)–(20) are important in the subsequent argument.

3.2. Successive Approximations to the Solution of Equation (1)

Let $\psi(x)$ be any measurable "test" function defined on the set $\mathbb{R}$ and satisfying the following conditions:

$$0 \le \psi(x) \le \varphi(x) \int\limits_{-\infty}^{x} K(y)\, dy, \quad x \in \mathbb{R}; \tag{21}$$

there exists a number $r > 0$ such that

$$\inf_{x \in [r, +\infty)} \psi(x) > 0. \tag{22}$$

Recall that $\varphi(x)$ is a solution of the nonlinear equation (6).

We introduce the following special iterations for Eq. (1):

$$\begin{gathered} f_{n+1}(x) = G_0(x, f_n(x)) + \int\limits_{\mathbb{R}} K(x-t) G\big(f_n(t)\big)\, dt, \\ f_0(x) = \varepsilon\psi(x) + \varphi(x), \quad n \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}, \quad x \in \mathbb{R}, \end{gathered} \tag{23}$$

where

$$\varepsilon := \min\big\{1, (\eta - \eta_0)/\eta_0, (\xi - \eta_0)/\sup_{x \in \mathbb{R}} \psi(x)\big\}. \tag{24}$$

By induction on n we verify that the sequence of functions $\{f_n(x)\}_{n \in \mathbb{Z}_+}$ possesses the properties

$$\{f_n(x)\}_{n \in \mathbb{Z}_+} \quad \text{is nondecreasing; i.e.} \quad f_n(x) \le f_{n+1}(x), \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R}; \tag{25}$$

$$f_n(x) \le \eta, \quad n \in \mathbb{Z}_+, \quad x \in \mathbb{R}. \tag{26}$$

The inequality $f_0(x) \le \eta$, $x \in \mathbb{R}$, readily follows from (23), (24), (20), and (21),

$$f_0(x) \le (\varepsilon + 1)\varphi(x) \le \eta_0(\varepsilon + 1) \le \eta_0\left(\frac{\eta - \eta_0}{\eta_0} + 1\right) = \eta.$$

Now let us prove that

$$f_1(x) \ge f_0(x), \quad x \in \mathbb{R}.$$

We take into account conditions (b) and 1, property (2), condition (c), inequality (21), Eq. (24), and the fact that the function $\varphi(x)$ is a solution of the nonlinear equation (6), use definition (23), and obtain

$$\begin{aligned} f_1(x) &\ge G_0\big(x, \varepsilon\psi(x) + \varphi(x)\big) + \int\limits_{\mathbb{R}} K(x-t) G\big(\varphi(t)\big)\, dt \ge \big(\varepsilon\psi(x) + \varphi(x)\big) \int\limits_{-\infty}^{x} K(y)\, dy + \varphi(x) \\ &\ge \varepsilon\psi(x) \int\limits_{-\infty}^{x} K(y)\, dy + \psi(x) + \varphi(x) \ge \varepsilon\psi(x) + \varphi(x) = f_0(x). \end{aligned}$$

Assume that the following inequalities hold for some positive integer n:

$$\begin{aligned} &f_n(x) \ge f_{n-1}(x), \qquad x \in \mathbb{R}, \\ &f_n(x) \le \eta, \qquad\qquad\; x \in \mathbb{R}. \end{aligned}$$

Then, once again, in view of the monotonicity of the functions G_0 and G as well as properties (2) and condition (d), it follows from definition (23) by virtue of relations (8) and (9) that

$$\begin{aligned} f_{n+1}(x) &\ge G_0\big(x, f_{n-1}(x)\big) + \int\limits_{\mathbb{R}} K(x-t) G\big(f_{n-1}(t)\big)\, dt = f_n(x), \\ f_{n+1}(x) &\le G_0(x, \eta) + G(\eta) \le \big(\eta G'(0) - G(\eta)\big)\alpha_0 + G(\eta) = \eta. \end{aligned}$$

Using the Carathéodory condition for the function G_0 and the continuity of the function G, by induction on n one can readily verify that each element of the sequence $\{f_n(x)\}_{n\in\mathbb{Z}_+}$ is a measurable function on $\mathbb{R}$. It follows from the already proved properties (25) and (26) that the sequence of measurable functions $\{f_n(x)\}_{n\in\mathbb{Z}_+}$ has a pointwise limit $\lim_{n\to\infty} f_n(x) = f(x)$ as $n\to\infty$. In view of conditions 1 and (a), by virtue of Krasnosel'skii's (see [10, p. 340]) and Lévy's (see [11, p. 303]) limit theorems, the function $f(x)$ satisfies Eq. (1). It also follows from properties (25) and (26) that one has the two-sided inequality

$$\varepsilon\psi(x)+\varphi(x)\le f(x)\le \eta,\quad x\in\mathbb{R}. \tag{27}$$

3.3. Asymptotic Behavior of the Solution at $-\infty$

In this section, we prove that the solution $f(x)$ thus constructed satisfies the inequality

$$f(x)\le \eta e^{\sigma x},\quad x\le 0, \tag{28}$$

where the number σ is determined from Eq. (17) and $\sigma\in(0,\lambda_0)$. To this end, by induction on n we first prove the inequality

$$f_n(x)\le \eta e^{\sigma x},\quad x\le 0,\quad n\in\mathbb{Z}_+. \tag{29}$$

For $n=0$, this inequality readily follows from (24), (20), and the definition of the zero approximation,

$$\begin{aligned} f_0(x)&=\varepsilon\psi(x)+\varphi(x)\le\varphi(x)\left(\varepsilon\int_{-\infty}^{x}K(y)\,dy+1\right)\\ &\le \eta_0 e^{\sigma x}(\varepsilon+1)\le \eta_0 e^{\sigma x}\left(\frac{\eta-\eta_0}{\eta_0}+1\right)=\eta e^{\sigma x},\quad x\le 0.\end{aligned}$$

Assume that $f_n(x)\le \eta e^{\sigma x}$, $x\le 0$, for some positive integer n. Then, using condition (d), the monotonicity of the functions $G_0(x,u)$ and $G(u)$ with respect to u, condition 4, and Eq. (17), in view of definition (23) for $x\le 0$ we obtain

$$\begin{aligned} f_{n+1}(x)&\le G_0(x,\eta e^{\sigma x})+\int_{\mathbb{R}}K(x-t)G\big(f_n(t)\big)\,dt\\ &\le G_0(x,\eta)+\int_{-\infty}^{0}K(x-t)G(\eta e^{\sigma t})\,dt+\int_{0}^{\infty}K(x-t)G(\eta)\,dt\\ &\le \big(\eta G'(0)-G(\eta)\big)\alpha_0\int_{-\infty}^{x}K(y)\,dy+\eta G'(0)\int_{-\infty}^{0}K(x-t)e^{\sigma t}\,dt+G(\eta)\int_{-\infty}^{x}K(y)\,dy\\ &\le \big(\eta G'(0)-G(\eta)\big)\int_{-\infty}^{x}K(y)\,dy+\eta G'(0)\int_{x}^{\infty}K(y)e^{\sigma(x-y)}\,dy+G(\eta)\int_{-\infty}^{x}K(y)\,dy\\ &=\eta G'(0)\int_{-\infty}^{x}K(y)\,dy+\eta G'(0)e^{\sigma x}\int_{x}^{\infty}K(y)e^{-\sigma y}\,dy\\ &=\eta G'(0)\int_{-\infty}^{x}K(y)\,dy+\eta e^{\sigma x}\left(G'(0)\int_{\mathbb{R}}K(y)e^{-\sigma y}\,dy-G'(0)\int_{-\infty}^{x}K(y)e^{-\sigma y}\,dy\right)\\ &=\eta G'(0)\int_{-\infty}^{x}K(y)\,dy+\eta e^{\sigma x}L(\sigma)-\eta e^{\sigma x}G'(0)\int_{-\infty}^{x}K(y)e^{-\sigma y}\,dy\end{aligned}$$

$$= \eta e^{\sigma x} - \eta G'(0) \int\limits_{-\infty}^{x} K(y)\big(e^{\sigma(x-y)} - 1\big)\, dy \le \eta e^{\sigma x}.$$

The proof of inequality (29) is complete. Passing to the limit as $n \to \infty$ on both sides in this inequality, we arrive at the estimate (28). In particular, it follows from (28) that

$$\lim_{x\to-\infty} f(x) = 0 \quad \text{and} \quad f \in L_1(-\infty, 0). \tag{30}$$

Thus, based on the above, we conclude that the following assertion holds.

Theorem 1. *Under properties* (2)–(4), *conditions* 1–5 *and* (a)–(d), *and inequalities* (7) *and* (16), *Eq.* (1) *has a positive essentially bounded solution f satisfying relations* (30). *Moreover, the following two-sided estimate holds*:

$$\varepsilon\psi(x) + \varphi(x) \le f(x) \le \Phi(x) := \begin{cases} \eta e^{\sigma x} & \textit{for} \quad x \le 0 \\ \eta & \textit{for} \quad x > 0, \end{cases}$$

where the number ε is determined by relation (24), *$\psi(x)$ is any measurable "test" function satisfying conditions* (21) *and* (22), *and $\varphi(x)$ is a continuous monotone nondecreasing positive bounded solution of Eq.* (6) *on $\mathbb{R}$ with properties* (18) *and* (19).

In the next section, we prove that the solution f has some additional properties in one special case.

4. ASYMPTOTIC BEHAVIOR OF THE SOLUTION AT $+\infty$ IN ONE SPECIAL CASE

4.1. Main Conditions. Statement of the Theorem

In this section, we assume that the nonlinearity $G_0(x, u)$ admits a representation of the form

$$G_0(x, u) = G_1(u) \int\limits_{-\infty}^{x} K(y)\, dy, \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^+, \tag{31}$$

where $G_1(u)$ is a real continuous function defined on $\mathbb{R}^+$ and satisfying the conditions

(A) $G_1(u)$ is concave and monotone increasing on $\mathbb{R}^+$.

(B) $G_1(u) \ge u$, $u \in [0, \xi]$, $\quad \xi \in (\eta_0, \eta)$.

(C) $G_1(\eta) = (G'(0)\eta - G(\eta))\alpha_0$.

Remark 1. It can readily be verified that if the function G_0 admits the representation (31) and the function G_1 has properties (A)–(C), then conditions (a)–(d) are satisfied automatically.

We use the concavity of the functions G_1 and G_0 and some geometric inequalities to prove the following assertion.

Theorem 2. *Let the kernel K and the nonlinearity G satisfy properties* (2)–(4), *inequalities* (7) *and* (16), *and conditions* 1–5, *and let the function $G_0(x, u)$ admit the representation* (31) *in which the function $G_1(u)$ has properties* (A)–(C). *Then the solution $f(x)$ has the following additional properties*:

$$\lim_{x\to+\infty} f(x) = \eta \quad \textit{and} \quad \eta - f \in L_1(0, +\infty).$$

4.2. Proof of Theorem 2

Note that (18) implies the existence of a number $r_0 > 0$ such that

$$\eta_0 - \varphi(x) < \varepsilon \inf_{x\ge r} \psi(x)$$

for $x \geq r_0$. Set $r^* := \max(r, r_0)$. Then, by virtue of inequality (27), the solution $f(x)$ has the following lower bound for all $x \in [r^*, +\infty)$:

$$f(x) \geq \varepsilon \inf_{x \geq r^*} \psi(x) + \varphi(x) \geq \varepsilon \inf_{x \geq r} \psi(x) + \varphi(x) > \eta_0. \tag{32}$$

By virtue of relations (31), (2), and (9) and condition (C), Eq. (1) implies that

$$\begin{aligned}
0 \leq \eta - f(x) &= \eta - \int_{-\infty}^{x} K(y)\,dy \cdot G_1(f(x)) - \int_{\mathbb{R}} K(x-t) G\big(f(t)\big)\,dt \\
&= \big(\eta G'(0) - G(\eta)\big)\alpha_0 + G(\eta) - G_1\big(f(x)\big) \\
&\quad + \int_{x}^{\infty} K(y)\,dy \cdot G_1\big(f(x)\big) - \int_{-\infty}^{0} K(x-t) G\big(f(t)\big)\,dt - \int_{0}^{\infty} K(x-t) G\big(f(t)\big)\,dt \\
&\leq G_1(\eta) \int_{x}^{\infty} K(y)\,dy + G_1(\eta) - G_1\big(f(x)\big) \\
&\quad + \int_{-\infty}^{0} K(x-t)\Big(G(\eta) - G\big(f(t)\big)\Big)\,dt + \int_{0}^{\infty} K(x-t)\Big(G(\eta) - G\big(f(t)\big)\Big)\,dt \\
&\leq G_1(\eta) \int_{x}^{\infty} K(y)\,dy + G(\eta) \int_{x}^{\infty} K(y)\,dy + G_1(\eta) - G_1\big(f(x)\big) \\
&\quad + \int_{0}^{r^*} K(x-t)\Big(G(\eta) - G\big(f(t)\big)\Big)\,dt + \int_{r^*}^{\infty} K(x-t)\Big(G(\eta) - G\big(f(t)\big)\Big)\,dt \\
&\leq \eta \int_{x}^{\infty} K(y)\,dy + G(\eta) \int_{x-r^*}^{x} K(y)\,dy + G_1(\eta) - G_1\big(f(x)\big) + \int_{r^*}^{\infty} K(x-t)\Big(G(\eta) - G\big(f(t)\big)\Big)\,dt.
\end{aligned}$$

Note that by virtue of the estimate (32) and conditions (A), 1, and 2, the inequalities

$$0 \leq G_1(\eta) - G_1(f(x)) \leq \frac{G_1(\eta)}{\eta}\big(\eta - f(x)\big), \tag{33}$$

$$0 \leq G(\eta) - G(f(x)) \leq \frac{G(\eta) - \eta_0}{\eta - \eta_0}\big(\eta - f(x)\big) \tag{34}$$

hold for all $x \geq r^*$. Indeed, by virtue of the estimate (32), one has the inequality $\eta_0 < f(x) < \eta$. Since the function G_1 is concave according to condition (A), we see that, in view of condition (B), inequality (33) can be proved in exactly the same manner as inequality (11). Taking $\widetilde{\eta} = \eta_0$ in (11), we obtain the inequality

$$\frac{G(\eta) - G(\eta_0)}{\eta - \eta_0} < \frac{G(\eta)}{\eta}. \tag{35}$$

Let us prove inequalities (34). Since the function G is monotone increasing by condition 1, we see that the left inequality in (34) is obvious. To prove the right inequality, we replace η_0 by $G(\eta_0)$ in the numerator of the fraction according to condition 2 and take into account the fact that $\eta > f(x)$; then the right inequality in (34) acquires the form

$$\frac{G(\eta) - G\big(f(x)\big)}{\eta - f(x)} \leq \frac{G(\eta) - G(\eta_0)}{\eta - \eta_0}. \tag{36}$$

Writing the numerator of the fraction on the left-hand side as $(G(\eta)-G(\eta_0))+(G(\eta_0)-G(f(x)))$, we see that inequality (36) is equivalent to the inequality

$$\big(G(\eta)-G(\eta_0)\big)(\eta-\eta_0)+\Big(G(\eta_0)-G\big(f(x)\big)\Big)(\eta-\eta_0)\le\big(G(\eta)-G(\eta_0)\big)\big(\eta-f(x)\big),$$

i.e., the inequality $(G(\eta_0)-G(f(x)))(\eta-\eta_0)\le(G(\eta)-G(\eta_0))(\eta_0-f(x))$, or

$$\frac{G\big(f(x)\big)-G(\eta_0)}{f(x)-\eta_0}\ge\frac{G(\eta)-G(\eta_0)}{\eta-\eta_0}. \tag{37}$$

Inequality (37) holds true and can be proved in the same way as inequality (11). Indeed, consider the points $E(\eta_0,G(\eta_0))$, $A(\eta,G(\eta))$, and $F(f(x),G(f(x)))$. Since the points E, A, and F belong to the graph of the function $G(u)$ and this function is concave, it follows that the point F lies above the segment EA; therefore, the slope of the segment EF to the abscissa axis is greater than the slope of the segment EA to the abscissa axis; i.e., inequality (37) is satisfied, and then so is the right inequality in (34), equivalent to it.

Taking into account the estimates (33) and (34) in the inequality derived above,

$$\begin{aligned}0\le\eta-f(x)\le\eta\int\limits_x^\infty K(y)\,dy+G(\eta)\int\limits_{x-r^*}^{x}K(y)\,dy\\+G_1(\eta)-G_1\big(f(x)\big)+\int\limits_{r^*}^\infty K(x-t)\Big(G(\eta)-G\big(f(t)\big)\Big)\,dt,\end{aligned} \tag{38}$$

we arrive the following estimate for $x\ge r^*$:

$$\begin{aligned}0\le\eta-f(x)\le\eta\int\limits_x^\infty K(y)\,dy+G(\eta)\int\limits_{x-r^*}^{x}K(y)\,dy\\+\frac{G_1(\eta)}{\eta}\big(\eta-f(x)\big)+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^\infty K(x-t)\big(\eta-f(t)\big)\,dt,\quad x\ge r^*,\end{aligned} \tag{39}$$

or

$$\begin{aligned}0\le\frac{\eta-G_1(\eta)}{\eta}\big(\eta-f(x)\big)\le\eta\int\limits_x^\infty K(y)\,dy+G(\eta)\int\limits_{x-r^*}^{x}K(y)\,dy\\+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^\infty K(x-t)\big(\eta-f(t)\big)\,dt,\quad x\ge r^*.\end{aligned} \tag{40}$$

It follows from inequality (37) that $\eta-f\in L_1^{\mathrm{loc}}(\mathbb{R}^+)$. Let us verify that $\eta-f\in L_1(r^*,+\infty)$. Let $\delta>r^*$ be an arbitrary number. Let us integrate both parts of inequality (38) from r^* to δ. Then, taking into account properties (2)–(4), by the Fubini theorem (see [11, p. 317]) we have

$$\begin{aligned}0&\le\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx\le\eta\int\limits_{r^*}^{\delta}\int\limits_x^\infty K(y)\,dy\,dx+G(\eta)\int\limits_{r^*}^{\delta}\left(\int\limits_{x-r^*}^\infty K(y)\,dy-\int\limits_x^\infty K(y)\,dy\right)\\&+\frac{G_1(\eta)}{\eta}\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^{\delta}\int\limits_{r^*}^\infty K(x-t)\big(\eta-f(t)\big)\,dt\,dx\\&\le\eta\int\limits_0^\infty yK(y)\,dy+G(\eta)\int\limits_0^\infty yK(y)\,dy+\frac{G_1(\eta)}{\eta}\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx\end{aligned}$$

$$+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^{\delta}\int\limits_{r^*}^{\delta}K(x-t)\big(\eta-f(t)\big)\,dt\,dx+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^{\delta}\int\limits_{\delta}^{\infty}K(x-t)\big(\eta-f(t)\big)\,dt\,dx$$

$$\le\big(\eta+G(\eta)\big)\int\limits_0^{\infty}yK(y)\,dy+\frac{G_1(\eta)}{\eta}\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx$$

$$+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^{\delta}\big(\eta-f(t)\big)\,dt+\eta\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^{\delta}\int\limits_{\delta}^{\infty}K(x-t)\,dt\,dx$$

$$=\big(\eta+G(\eta)\big)\int\limits_0^{\infty}yK(y)\,dy+\frac{G_1(\eta)}{\eta}\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx$$

$$+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{r^*}^{\delta}\big(\eta-f(t)\big)\,dt+\eta\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{0}^{\delta}\int\limits_{-\infty}^{x-\delta}K(y)\,dy\,dx$$

$$\le\big(\eta+G(\eta)\big)\int\limits_0^{\infty}yK(y)\,dy+\left(\frac{G_1(\eta)}{\eta}+\frac{G(\eta)-\eta_0}{\eta-\eta_0}\right)\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx+\eta\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{-\infty}^{0}K(y)(-y)\,dy,$$

or

$$\left(1-\frac{G_1(\eta)}{\eta}-\frac{G(\eta)-\eta_0}{\eta-\eta_0}\right)\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx$$
$$\le\big(\eta+G(\eta)\big)\int\limits_0^{\infty}yK(y)\,dy+\eta\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{-\infty}^{0}K(y)(-y)\,dy.$$

By virtue of inequality (35) and the relation $G_1(\eta)+G(\eta)=\eta$, we have

$$\lambda_0:=1-\frac{G_1(\eta)}{\eta}-\frac{G(\eta)-\eta_0}{\eta-\eta_0}>1-\frac{G_1(\eta)+G(\eta)}{\eta}=0.$$

Consequently,

$$\int\limits_{r^*}^{\delta}\big(\eta-f(x)\big)\,dx\le\frac{1}{\lambda_0}\left(\big(\eta+G(\eta)\big)\int\limits_0^{\infty}yK(y)\,dy+\eta\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{-\infty}^{0}K(y)(-y)\,dy\right).$$

Letting δ tend to infinity in this inequality, we conclude that $f\in L_1(r^*,+\infty)$ and

$$\int\limits_{r^*}^{\infty}\big(\eta-f(x)\big)\,dx\le\frac{1}{\lambda_0}\left(\big(\eta+G(\eta)\big)\int\limits_0^{\infty}yK(y)\,dy+\eta\frac{G(\eta)-\eta_0}{\eta-\eta_0}\int\limits_{-\infty}^{0}K(y)(-y)\,dy\right).$$

To complete the proof, it remains to verify that $\lim\limits_{x\to+\infty}f(x)=\eta$. Indeed, since $K\in L_1(\mathbb{R})\cap M(\mathbb{R})$ and $\eta-f\in L_1(\mathbb{R}^+)\cap M(\mathbb{R}^+)$, it follows by Lemma 5 in the paper [12] that

$$\lim_{x\to+\infty}\int\limits_{r^*}^{\infty}K(x-t)\big(\eta-f(t)\big)\,dt=0. \qquad (41)$$

Taking into account inequality (40) and the limit relation (41), we conclude that $\lim\limits_{x\to+\infty}f(x)=\eta$. The proof of Theorem 2 is complete.

Let us make an important remark.

Remark 2. If the function $G_0(x,u)$ is periodic in x with principal period $T>0$ and jointly continuous on $\mathbb{R}\times\mathbb{R}^+$, then the solution $f(x)$ constructed here gives rise to a one-parameter family of new solutions of the form $\{f(x+cT)\}_{c\in\mathbb{R}}$.

This assertion follows from the obvious equation

$$G_0\big(x, f(x+cT)\big) + \int\limits_{\mathbb{R}} K(x-t)G\big(f(x+cT)\big)\,dt$$
$$= G_0\big(x+cT, f(x+cT)\big) + \int\limits_{\mathbb{R}} K(x+cT-y)G\big(f(y)\big)\,dy = f(x+cT).$$

5. EXAMPLES

As an application and illustration of the results obtained, we give applied examples of the nonlinearities G_0 and G. The following functions are well-known examples in the mathematical theory of the space-time propagation of an epidemic (see [2, 3]):

$$G_0(x,u) = \int\limits_{-\infty}^{x} K(y)\,dy\cdot q\sqrt{u},\quad x\in\mathbb{R},\quad u\in\mathbb{R}^+, \tag{42}$$

$$G(u) = \gamma(1-e^{-u}),\quad u\in\mathbb{R}^+, \tag{43}$$

where $q := \alpha_0(\eta G'(0) - G(\eta))/\sqrt{\eta}$ and $\gamma > 1$ is a numerical parameter. In this theory, the inequality $\gamma > 1$ is called the "threshold condition" and represents the critical value of the number of infected persons, above which the epidemic cannot be stopped without serious medical intervention. From the mathematical viewpoint, if $\gamma \le 1$, then, in the framework of the model considered by Diekmann [2], the corresponding nonlinear equation has only the trivial solution in the class of bounded functions. The latter means that the epidemic will not die out over time.

In the example (42), for ξ we can take, say, the number

$$\xi := \max\left\{\alpha_0^2(\eta G'(0) - G(\eta))^2/\eta, \eta_0+\varepsilon_0\right\},$$

where $\varepsilon_0 > 0$ is an arbitrary sufficiently small parameter.

Let us also provide an applied example of the kernel K for which properties (2)–(4), as well as the related inequalities (7) and (16), are satisfied. For such a kernel K we take the following antisymmetric Gaussian distribution:

$$K(x) = \frac{1}{\sqrt{\pi}}e^{-(x-1)^2}. \tag{44}$$

Obviously, the kernel (44) satisfies properties (2)–(4). In this case, the Diekmann function $L(\lambda)$ has the form

$$L(\lambda) = G'(0)\int\limits_{\mathbb{R}} K(t)e^{-\lambda t}\,dt = G'(0)e^{\lambda^2/4-\lambda}.$$

For the nonlinearity $G(u)$ we take a function of the form (43) and assume that $\gamma\in(1,e)$. Obviously, $L(\lambda)$ decays on the interval $[0,2]$, is concave on $\mathbb{R}^+$, and if for λ_0 we take $\lambda_0 = 2$, then $L(\lambda_0) = \gamma/e < 1$; i.e., condition (16) is satisfied. Now let us verify condition (7). By virtue of the choice in (44), we have

$$G'(0)\alpha_0 = \gamma\int\limits_{-\infty}^{0} K(y)\,dy = \frac{\gamma}{\sqrt{\pi}}\int\limits_{-\infty}^{0} e^{-(y-1)^2}\,dy = \frac{\gamma\big(1-\mathrm{erf}(1)\big)}{2} \approx 0.079\gamma < 1,$$

because $\gamma\in(1,e)$. Condition (7) is satisfied.

It is of interest to note that the answer to the question of the uniqueness of the solution in a particularly chosen cone segment for Eq. (1) still remains an open problem.

FUNDING

The research by Kh.A. Khachatryan and H.S. Petrosyan was supported by the Russian Science Foundation, project no. 19-11-00223.

REFERENCES

1. Barbour, A.D., The uniqueness of Atkinson and Reuter's epidemic waves, *Math. Proc. Cambridge Phil. Soc.*, 1977, vol. 82, no. 1, pp. 127–130.
2. Diekmann, O., Thresholds and travelling waves for the geographical spread of infection, *J. Math. Biol.*, 1978, vol. 6, no. 2, pp. 109–130.
3. Khachatryan, Kh.A. and Petrosyan, H.S., On the solvability of a class of nonlinear Hammerstein–Stieltjes integral equations on the whole line, *Proc. Steklov Inst. Math.*, 2020, vol. 308, pp. 238–249.
4. Khachatryan, A.Kh. and Khachatryan, Kh.A., Solvability of a nonlinear model Boltzmann equation in the problem of a plane shock wave, *Theor. Math. Phys.*, 2016, vol. 189, pp. 1609–1623.
5. Engibaryan, N.B. and Khachatryan, A.Kh., Exact linearization of the sliding problem for a dilute gas in the Bhatnagar–Gross–Krook model, *Theor. Math. Phys.*, 2000, vol. 125, pp. 1589–1592.
6. Engibaryan, N.B., On one radiation transfer problem, *Astrofizika*, 1966, vol. 2, no. 1, pp. 31–36.
7. Khachatryan, Kh.A. and Petrosyan, H.S., On the construction of a summable solution of one class of nonlinear integral equations of the Hammerstein–Nemytskii type on the entire line, *Tr. Inst. Mat. Mekh. Ural. Otd. Ross. Akad. Nauk*, 2020, vol. 26, no. 2, pp. 278–287.
8. Khachatryan, Kh.A., On the solution of one system of nonlinear integral equations of the Hammerstein–Nemytskii type on the entire line, *Tr. Inst. Mat. Nats. Akad. Nauk Belarusi*, 2013, vol. 21, no. 2, pp. 154–161.
9. Khachatryan, Kh.A., Positive solubility of some classes of non-linear integral equations of Hammerstein type on the semi-axis and on the whole line, *Izv. Math.*, 2015, vol. 79, no. 2, pp. 411–430.
10. Krasnosel'skii, M.A., Zabreiko, P.P., Pustyl'nik, E.I., and Sobolevskii, P.E., *Integral'nye operatory v prostranstvakh summiruemykh funktsii* (Integral Operators in Spaces of Integrable Functions), Moscow: Nauka, 1966.
11. Kolmogorov, A.N. and Fomin, V.S., *Elementy teorii funktsii i funktsional'nogo analiza* (Elements of the Theory of Functions and Functional Analysis), Moscow: Nauka, 1976.
12. Arabadzhyan, L.G. and Khachatryan, A.S., A class of integral equations of convolution type, *Sb. Math.*, 2007, vol. 198, no. 7, pp. 949–966.