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PARTIAL DIFFERENTIAL EQUATIONS

On Multiwave Solutions of One Nonlinear Schrödinger Equation

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Abstract—We consider a nonlinear Schrödinger equation arising in a number of physical problems. It is shown that when the real part is separated in this equation, there arises a nonlinear differential equation that has at least two types of solutions, a multiwave solution and a solution in the form of standing waves. Numerical examples of the multiwave solution and its transition to the standing wave solution are presented.

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INTRODUCTION

Nonlinear Schrödinger equations arise when solving various physical and technical problems. The most frequently studied nonlinear Schrödinger equation is the one with a cubic nonlinearity; it has special types of solutions in the form of solitons [1, Ch. 1, Sec. 1.7a; 2, Ch. 4, Sec. 4.1]. For such an equation, single- and multisoliton solutions are found by various methods, in particular, by the inverse scattering transform. However, there arise nonlinear Schrödinger equations with other types of nonlinearities in physical and technical problems. One such equation is studied in the present paper.

1. NONLINEAR SCHRÖDINGER EQUATION

Consider a nonlinear Schrödinger equation of the form

$$i\mu\frac{\partial\Phi}{\partial t}+V^2\frac{\partial^2\Phi}{\partial X^2}=\left(\frac{\partial\ln|\Phi|}{\partial t}\right)^2\Phi, \tag{1}$$

where μ and V are real constants and $\Phi=\Phi(X,t)$ is the unknown function.

Although Eq. (1) is not usually discussed in the mathematical literature, it arises, for example, in the analysis of self-induced transparency [3].

Let us study progressing waves satisfying Eq. (1) and described by the relation

$$\Phi(X,t)=(\varphi-\varphi_0)\exp\{i(kX-\omega t)\}, \tag{2}$$

where k is the wave number of the progressing wave, ω is its cyclic frequency, φ_0 is a constant, and $\varphi=\varphi(X,t)$ is a real-valued function that has the second X-derivative and the first t-derivative and satisfies $\varphi>\varphi_0$ for all X and t; in particular, $|\Phi|=\varphi-\varphi_0$.

Substituting the expression (2) for the function Φ into Eq. (1) and taking into account the fact that

$$\frac{\partial\ln|\Phi|}{\partial t}=\frac{1}{(\varphi-\varphi_0)}\frac{\partial\varphi}{\partial t},$$

we arrive at the following equation for the function φ:

$$V^2\frac{\partial^2\varphi}{\partial X^2}+i\left(\mu\frac{\partial\varphi}{\partial t}+2V^2k\frac{\partial\varphi}{\partial X}\right)+(\mu\omega-\omega^2)(\varphi-\varphi_0)=\frac{1}{\varphi-\varphi_0}\left(\frac{\partial\varphi}{\partial t}\right)^2. \tag{3}$$

The function φ is real-valued; therefore,

$$\mu\frac{\partial\varphi}{\partial t}+2V^2k\frac{\partial\varphi}{\partial X}=0. \tag{4}$$

Denote the factor multiplying $\varphi-\varphi_0$ on the left-hand side in Eq. (3) by ν; i.e.,

$$\nu=\mu\omega-\omega^2. \tag{5}$$

Then Eq. (3) (under condition (4)) is written in the form

$$V^2\frac{\partial^2\varphi}{\partial X^2}+\nu(\varphi-\varphi_0)=\frac{1}{\varphi-\varphi_0}\left(\frac{\partial\varphi}{\partial t}\right)^2. \tag{6}$$

An equation of the form (6) also arises when studying a nervous impulse [4] and the electromagnetic field propagation in a chiral medium [5].

Thus, the Schrödinger equation (1) has a solution of the form (2) with a real-valued function φ if and only if the function φ satisfies Eqs. (4) and (6) and the inequality $\varphi>\varphi_0$.

Equation (4) is a linear homogeneous first-order partial differential equation with constant coefficients and hence is easy to solve; its general real-valued solution has the form $\varphi-\varphi_0=f(\xi)$, where $f\colon\mathbb{R}\to\mathbb{R}$ is an arbitrary differentiable function and

$$\xi=\mu X-2V^2kt.$$

The minus sign in ξ corresponds to a wave propagating from left to right, just as in the exponential (2).

Consequently, the Schrödinger equation (1) has a solution of the form (2) if and only if there exists a twice differentiable positive function f defined on some (possibly, infinite) interval such that the function

$$\varphi(X,t)=\varphi_0+f(\mu X-2V^2kt) \tag{7}$$

satisfies Eq. (6).

Replacing the function φ by its representation (7) in Eq. (6), we arrive at the ordinary differential equation

$$V^2\mu^2f''+\nu f=4V^4k^2\frac{1}{f}(f')^2 \tag{8}$$

(where the prime stands for the derivative with respect to ξ).

For simplicity, we assume that $V\mu\neq 0$. Equation (8) does not contain the independent variable and hence can be integrated by quadratures in a standard way (see, e.g., [6, p. 169]); namely, the function $p=df/d\xi$ is taken to be the new unknown function, and f is taken for the independent variable. Then $f''=pdp/df$, and in this notation Eq. (8) acquires the form

$$p\frac{dp}{df}+\frac{a}{f}p^2+bf=0,$$

where $a=-4V^2k^2/\mu^2$ and $b=\nu/(V^2\mu^2)$. This equation can be reduced to the Bernoulli equation.

We do not write and analyze the general solution of Eq. (8) in the case of arbitrary coefficients, because our goal is to indicate some parametric family of solutions (2) of Eq. (1) as well as solutions of Eqs. (1) and (6) in the form of progressing waves. Therefore, we assume that the number k in the representation (2) is taken to satisfy

$$4V^2k^2=\mu^2. \tag{9}$$

Then the numerical coefficients multiplying f'' and $(f')^2$ in Eq. (8) coincide, and dividing both sides of the equation by $4V^4k^2$, after obvious transformations, in view of relation (5), we arrive at the equation

$$\left(\frac{f'}{f}\right)'=-\frac{\mu\omega-\omega^2}{4V^4k^2},$$

whence we find

$$f(\xi) = C_1 \exp\left\{ -\frac{\mu\omega - \omega^2}{4V^4k^2}\frac{\xi^2}{2} + C_2\xi \right\}, \tag{10}$$

where C_1 and C_2 are arbitrary real constants.

Thus, if $V\mu \neq 0$, then for either $k = -\mu/(2V)$ and $k = \mu/(2V)$ the Schrödinger equation (1) has the three-parameter family of solutions (2), where $\varphi = f(\mu X - 2V^2kt)$ and the function $f(\xi)$ is given by relation (10) with arbitrary real constants $C_1 > 0$, C_2, and ω.

2. SOLUTIONS IN THE FORM OF PROGRESSING WAVES

Let us proceed to constructing solutions of Eqs. (1) and (6) in the form of progressing waves. In this section, we assume that the number ν in relation (5) is nonnegative; i.e.,

$$\omega_0^2 = \mu\omega - \omega^2. \tag{11}$$

(If we seek a particular solution of Eq. (6) in the form proposed below, then, as is shown, $\mu\omega - \omega^2$ must be nonnegative; therefore, we have denoted it by ω_0^2 straight away.) Thus, in what follows we seek solutions of Eq. (1) and the equation

$$V^2\frac{\partial^2\varphi}{\partial X^2} + \omega_0^2(\varphi - \varphi_0) = \frac{1}{\varphi - \varphi_0}\left(\frac{\partial\varphi}{\partial t}\right)^2 \tag{12}$$

in the form of progressing waves.

We seek a solution of the nonlinear equation (12) in the form of a running solitary wave

$$\varphi - \varphi_0 = \varphi_{\max} \exp\left\{-(k_0(X - X_0) \pm \omega_0(t - t_0))^2/2\right\}, \tag{13}$$

where k_0, X_0, and t_0 are real constants and the physical meaning of the quantities occurring in the representation (13) is as follows: $\varphi_{\max}$ is the amplitude value of the function $\varphi - \varphi_0$, X_0 is the coordinate of the maximum (center) of the wave impulse, and t_0 is the time of attaining this maximum. The minus sign refers to a wave running from left to right; and the plus sign, from right to left.

By a straightforward substitution of the function φ defined by relation (13) into Eq. (12), it is easy to make sure that this function is a solution of this equation only if

$$V^2k_0^2 = \omega_0^2. \tag{14}$$

Moreover, the function (13) a solution of Eq. (4) if we select the minus sign in (13) in front of ω_0 and if the coefficients k_0 and ω_0 are proportional with one and the same factor to the coefficients μ and $2V^2k$, respectively, of the linear form ξ; i.e.,

$$2V^2kk_0 = \mu\omega_0. \tag{15}$$

Indeed, in this case, by virtue of (13), (15), and (11), we obtain

$$\varphi - \varphi_0 = \varphi_{\max} \exp\left\{ -\left(\frac{\omega_0^2}{4V^4k^2}\frac{\xi^2}{2} + \frac{\omega_0\delta_0\xi}{2V^2k} + \frac{\delta_0^2}{2}\right) \right\},$$

where $\delta_0 = -k_0X_0 + \omega_0t_0$; i.e., since relations (14) and (15) imply relation (9), we obtain the above solution corresponding to the function (10) under the condition that $\mu\omega - \omega^2 \geq 0$.

Figure 1 shows the graph of a solitary impulse $\Phi(X, t)$ constructed using formula (2) with the function (13) substituted into it under the following conditions: $\omega_0 = \omega = 0$, the dependence on time is lacking, $\varphi_{\max} = 1$, $X_0 = 0$, and the ratio of the wave numbers is $k/k_0 = 5$.

The nonlinear equation (12), and hence also Eq. (1) admit a multiwave solution.

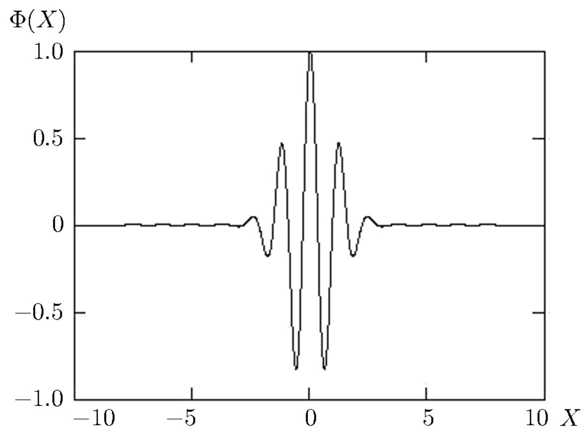

Fig. 1. Solitary impulse of the function $\Phi(X)$.

Multiwave solutions have been found for a very narrow range of nonlinear equations [1, 2]. We seek a multiwave solution of Eq. (12) in the form

$$\varphi = \varphi_0 + \varphi_{\max} \sum_{n=1}^{N} \varphi_n(X,t), \tag{16}$$

where

$$\varphi_n(X,t) = \varphi_n = \exp\{-\Delta_n^2(X,t)/2\} \quad \text{and} \quad \Delta_n(X,t) = \Delta_n = k_0(X - X_{0n}) - \omega_0(t - t_{0n}). \tag{17}$$

In the representation (16), (17), the number N is the total number of wave impulses, n is the current impulse number, X_{0n} are the coordinates of the maxima of wave impulses, and t_{0n} are the times when they attain these maxima.

Replacing the function φ in Eq. (12) by its representation (16), (17), we obtain the relation

$$\sum_{n=1}^{N} \varphi_n \sum_{n=1}^{N} \varphi_n \Delta_n^2 = \left(\sum_{n=1}^{N} \varphi_n \Delta_n \right)^2. \tag{18}$$

Consider two consecutive identical impulses with $n = 1, 2$. Writing relation (18) for this case ($N = 2$), we have

$$(\varphi_1 + \varphi_2)(\varphi_1 \Delta_1^2 + \varphi_2 \Delta_2^2) = (\varphi_1 \Delta_1 + \varphi_2 \Delta_2)^2. \tag{19}$$

Obviously, relation (19) is equivalent to the equation $\Delta_1 - \Delta_2 = 0$, or

$$k_0(X_{02} - X_{01}) - \omega_0(t_{02} - t_{01}) = 0. \tag{20}$$

Relation (20) shows that the distance between the impulses $\delta = X_{02} - X_{01}$ is covered by a wave in time $t_{02} - t_{01}$ with velocity $V = \omega_0/k_0$.

If we take $t_{0n} = X_{0n}/V = k_0 X_{0n}/\omega_0$ in Δ_n for each n, then we conclude that $\Delta_n = k_0 X - \omega_0 t$ for all $n = 1, \dots, N$; i.e., then $\Delta_1 = \Delta_2 = \ldots = \Delta_N$ and they can be taken outside the sum and canceled. As a result, (18) turns into an identity.

Consequently, with this choice of the values of t_{0n}, $n = 1, \dots, N$, the function (16) is a multiwave solution of the nonlinear equation (12).

The multiwave solution (16) acquires the simplest form in the case of the same distance δ between all wave impulses. In this case, for the coordinates of the maxima of impulses one has the formula $X_{0n} = n\delta$ and for the times of attaining maxima, the formula $t_{0n} = k_0 X_{0n}/\omega_0 = k_0 n\delta/\omega_0$.

Figure 2 shows several consecutive impulses constructed using formula (16) under the conditions

$$\omega_0 = 0, \quad \varphi_0 = 0, \quad \varphi_{\max} = 1, \quad k_0 = 2, \quad \delta = 4.$$

Let us consider another type of a wave representing a solution of Eq. (16).

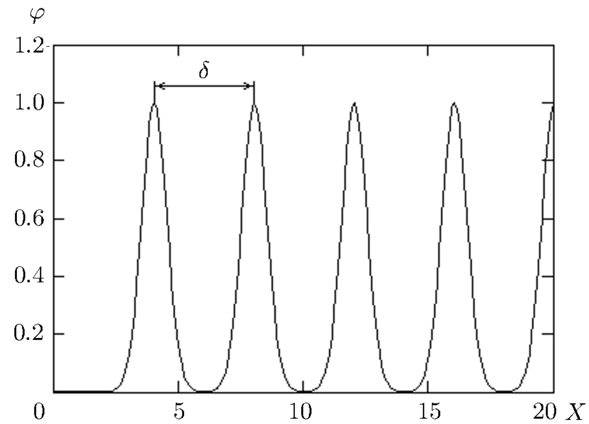

Fig. 2. Consecutive impulses in a multiwave solution.

3. SOLUTION IN THE FORM OF A STANDING WAVE

Standing waves are most often formed in linear systems as a result of superposition (interference) of forward and reflected traveling waves. However, it is well known that standing waves can also arise in nonlinear systems [7, Ch. 4, Sec. 2]. Many physical processes are fundamentally nonlinear, and the process of the appearance of standing waves in such systems is nontrivial. We will not discuss the physical mechanisms of the appearance of standing waves in these systems. Let us investigate the mathematical possibility of the occurrence of standing waves obeying the nonlinear equation (12).

Some solutions of the nonlinear equation (12) can be found by separation of variables (see, e.g., [8, Ch. 2, Sec. 3]). Consider a solution of Eq. (12) of the form

$$\varphi - \varphi_0 = \phi(X)T(t), \tag{21}$$

where $\phi(X)$ is a function of the coordinate X alone and $T(t)$ is a function only of time t.

Substituting (21) into (12), we obtain

$$V^2\phi(X)T^2(t)\frac{d^2\phi(X)}{dX^2} = \left(\phi(X)\frac{dT(t)}{dt}\right)^2 - \phi^2(X)T^2(t)\omega_0^2. \tag{22}$$

Let us divide both sides of Eq. (22) by $\phi^2(X)T^2(t)$ to obtain

$$V^2\frac{1}{\phi(X)}\frac{d^2\phi(X)}{dX^2} + \omega_0^2 = \left(\frac{1}{T(t)}\frac{dT(t)}{dt}\right)^2 = -\alpha^2, \tag{23}$$

where α is a constant.

Equations (23) split into two mutually independent equations. One of them is an equation for the function $\phi(X)$ and has the form

$$\frac{d^2\phi(X)}{dX^2} + \left(k_0^2 + \frac{\alpha^2}{V^2}\right)\phi(X) = 0; \tag{24}$$

here we have used relation (14), by virtue of which $\omega_0^2/V^2 = k_0^2$.

Denote $k_S^2 = k_0^2 + \alpha^2/V^2$. The general solution of Eq. (24) is

$$\phi(X) = \phi(0)\exp(ik_SX), \tag{25}$$

where $\phi(0)$ is the value of the function $\phi(X)$ at the origin.

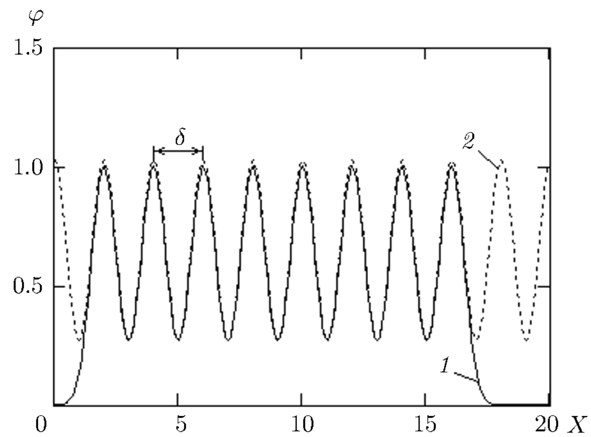

Fig. 3. Transition of the multipulse solution into a solution in the form of a standing wave: (*1*) multiwave solution, (*2*) standing wave.

Another equation following from (23) is an equation for the function $T(t)$; it has the form

$$\frac{dT(t)}{dt} = i\alpha T(t). \tag{26}$$

The number α can be positive as well as negative; this does not affect further reasoning. The general solution of Eq. (26) is

$$T(t) = T(0)\exp(i\alpha t), \tag{27}$$

where $T(0)$ is the initial value of the function $T(t)$.

Substituting the general solutions (25) and (27) into the representation (21), we obtain

$$\varphi - \varphi_0 = \varphi_A \exp(i\alpha t)\exp(ik_S X), \tag{28}$$

where we have denoted $\varphi_A = T(0)\phi(0)$.

As shown above, the function $\varphi - \varphi_0$ must be real-valued. However, the use of exponentials with imaginary exponents is introduced solely for the ease of transformations. In fact, only real terms should be taken into account in these exponentials. Therefore, formula (28) describes a solution of Eq. (12) in the form of standing waves

$$\varphi - \varphi_0 = \varphi_A \cos(\alpha t)\cos(k_S X) = \varphi_A \cos(\alpha t)\cos(2\pi X/\delta), \tag{29}$$

where φ_A is the amplitude value of standing waves and δ is the wavelength.

It is of interest to trace the transition of the multiwave solution (16) into the solution in the form of standing waves (29) graphically. This transition occurs when the pulses approach each other (see Fig. 2), i.e. with decreasing δ.

Figure 3 shows two graphs. Graph *1* has been constructed using formula (16) under the conditions $\omega_0 = 0$, $\varphi_0 = 0$, $\varphi_{\max} = 1$, $k_0 = 2$, and $\delta = 2$ for $N = 8$ impulses. Graph *2* (dashed line) has been constructed by formula (29) under the conditions $\varphi_0 = 0.65$ and $\varphi_A \cos(\alpha t) = 0.38$ for some time t.

CONCLUSIONS

In the present paper, the problem of finding solutions of a nonlinear Schrödinger equation in the form of traveling waves is reduced to solving a system of two partial differential equations for the same function. It is shown how to find all the solutions of this system in closed form. One of the equations of this system is nonlinear and arises in various problems of an applied nature. It has been established that this nonlinear equation has both multiwave and standing-wave solutions. As the distance between the wave impulses decreases, the multiwave solution transforms into a solution in the form of standing waves.

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