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PARTIAL DIFFERENTIAL EQUATIONS

Biquaternionic Wave Equations and the Properties of Their Generalized Solutions

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Abstract—We consider biquaternionic wave (biwave) equations. They are biquaternionic generalizations of the Maxwell and Dirac equations and are equivalent to a system of eight differential equations of hyperbolic type. Using the theory of generalized functions, we construct fundamental and generalized solutions of such equations, including discontinuous ones, describing shock waves and obtain conditions on the fronts. A solution of the Cauchy problem for a biwave equation is constructed, and so are analogs of Kirchhoff–Green's formulas that permit one to determine the solution inside a bounded domain given the boundary and initial values of the solution in this domain.

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INTRODUCTION

In this paper, we construct solutions of boundary value problems for biquaternionic wave (*biwave*) equations. These equations are biquaternionic generalizations of the Maxwell and Dirac equations. Note that the quaternionic representation of the Maxwell equations began with the works of James Maxwell himself and is dealt with in a fairly extensive literature (see, e.g., [1–10] etc.). Biquaternionic wave equations belong to the class of hyperbolic equations and describe solutions of hyperbolic systems of eight first-order differential equations. The theory of boundary value problems for such systems of equations has not yet received such significant development as the theory of boundary value problems for equations and systems of elliptic and parabolic types.

Here a similar theory is developed for biwave equations using methods of the theory of generalized functions. The simplest of the boundary value problems is the Cauchy problem with initial conditions. Its solution for the wave equation is well known and is determined by the d'Alembert, Poisson, and Kirchhoff formulas for the space dimensions 1, 2, and 3, respectively. The solution of the wave equation for any right-hand sides and initial conditions in the class of generalized functions was proposed by Vladimirov [11, 12]. Here we use Vladimirov's method to solve the corresponding Cauchy problem for the biwave equation.

In the present paper, we develop a method of generalized functions to solve initial–boundary value problems. The main ideas of the method for the classical wave equation in spaces of dimension not exceeding three were presented by the author in [13, 14]. The method of generalized functions is based on the representation of the boundary value problem in the space of generalized functions, which allows the initial and boundary conditions to be transferred to the right-hand side of the differential equations using singular generalized functions—simple and double layers on the boundary of the domain where the solution is defined. The densities of these layers are determined by the boundary values of the solution and of its derivatives. The properties of the fundamental solution—the Green's function of the equation—permit one to construct the solution of the resulting equation in the space of generalized functions in the form of the convolution of the Green's function with the right-hand side of the equation. A regular integral representation of the generalized solution is given by the classical solution of the boundary value problem, which allows finding the solution in the interior of the domain based on the boundary values of the solution, some of which are known and some are to be determined. These formulas are similar to the well-known Green's formula for the Laplace equation, which is used to calculate the solution in the domain based on the boundary value and the normal derivative of the solution. The resolving boundary integral equations (as a rule, singular) are constructed to determine the unknown boundary functions using the limit properties of the solution when approaching the boundary of the domain. The method allows one to construct solutions taking into account shock waves, characteristic of hyperbolic equations, at the fronts of which the derivatives undergo jumps. This method is used here to construct generalized solutions of boundary value problems and their integral representations.

1. MUTUAL BIGRADIENTS. THE BIWAVE EQUATION

Consider the biquaternionic wave equation

$$\nabla^{\pm}\mathbf{B} + \mathbf{F} \circ \mathbf{B} = \mathbf{G}(\tau, x), \quad (\tau, x) \in \mathbb{M}, \tag{1.1}$$

where $\mathbb{M}$ is the Minkowski space. Here and in the following, we adhere to Hamilton's scalar-vector notation for biquaternions, in which same-name lowercase and uppercase letters are used to denote a scalar and a vector, respectively:

$$\mathbf{B} = b(\tau, x) + B(\tau, x), \quad \mathbf{G} = g(\tau, x) + G(\tau, x),$$

and the structural coefficient $\mathbf{F}$ is a constant biquaternion. It is assumed that $\mathbf{B}(\tau, x)$ and $\mathbf{G}(\tau, x)$ belong to the space $\mathbb{B}'(\mathbb{M})$ of *generalized* biquaternions on $\mathbb{M}$. By these we mean biquaternions whose components belong to the class of generalized functions of tempered growth [12, Sec. 8].

Differential biquaternionic operators—*mutual bigradients*—have the form [14]

$$\nabla^{+} = \partial_\tau + i\nabla, \quad \nabla^{-} = \partial_\tau - i\nabla. \tag{1.2}$$

Their action is determined according to the quaternionic multiplication in the algebra of biquaternions,

$$\mathbf{F} \circ \mathbf{B} = (f + F) \circ (b + B) = \big(fb - (F, B)\big) + \big(fB + bF + [F, B]\big), \tag{1.3}$$

where

$$(F, B) = \sum_{j=1}^{3} F_j B_j \quad \text{and} \quad [F, B] = \sum_{k,l,m=1}^{3} \varepsilon_{klm} F_k B_l e_m$$

are the scalar and vector products of the indicated vectors, respectively, ε_{klm} is the Levi-Civita pseudotensor, and the e_m are the basis elements of the algebra of biquaternions $(m = 0, 1, 2, 3)$. Hence we have

$$\begin{aligned}\nabla^{\pm}\mathbf{B} \triangleq (\partial_\tau \pm i\nabla) \circ \big(b(\tau, x) + B(\tau, x)\big) &= \Big(\partial_\tau b \mp i(\nabla, B) + \partial_\tau B \pm i\big(\nabla b + [\nabla, B]\big)\Big) \\ &= (\partial_\tau b \mp i\operatorname{div} B) + \partial_\tau B \pm i\operatorname{grad} b \pm i \operatorname{rot} B\end{aligned}$$

(according to the upper and lower signs).

Equation (1.1) belongs to the class of biwave equations of the general form

$$\mathbf{A} \circ \nabla^{\pm}\mathbf{B} + \mathbf{C} \circ \mathbf{B} = \mathbf{H}(\tau, x),$$

reducible to (1.1) if there exists an inverse biquaternion $\mathbf{A}^{-1}$,

$$\mathbf{A}^{-1} = \mathbf{A}^{-} \circ \mathbf{A}/\big(a^2 + (A, A)\big),$$

where $\mathbf{A}^{-} = a - A$ (the *mutual* biquaternion). In this case, multiplying relation (1.3) on the left by $\mathbf{A}^{-1}$, we obtain Eq. (1.1), where

$$\mathbf{F} = \mathbf{A}^{-1} \circ \mathbf{C}, \quad \mathbf{G} = \mathbf{A}^{-1} \circ \mathbf{H}.$$

We also introduce the *conjugate* biquaternion $\mathbf{A}^{*} = \bar{\mathbf{A}}^{-}$. Here and in what follows, the bar above a symbol stands for the complex conjugation of the scalar and vector parts of a biquaternion.

We have previously considered special cases where $\mathbf{F}$ is a scalar or a vector. Equation (1.1) is equivalent to the modified system of Maxwell equations for $\mathbf{F} = 0$ and to the Dirac equations for a pure imaginary $\mathbf{F} = i\rho$ (see the papers [15, 16]). These papers use the theory of generalized functions to construct elementary and general solutions of Eq. (1.1) that describe time-varying, time-harmonic, and static biquaternionic fields.

Let us construct generalized solutions of (1.1) for an arbitrary right-hand side $\mathbf{G}(\tau, x) \in \mathbb{B}'(\mathbb{M})$.

2. MUTUAL MD-OPERATORS AND THEIR PROPERTIES

Let us introduce the differential biquaternionic operators

$$\mathbf{D}^+_\mathbf{F} = \nabla^+ + \mathbf{F} = \nabla^+ + f + F, \quad \mathbf{D}^-_\mathbf{F} = \nabla^- + \mathbf{F}^- = \nabla^- + f - F, \tag{2.1}$$

whose properties will further be used to solve the problem posed. In connection with the above, we refer to these operators as the *mutual MD-operators* (*Maxwell–Dirac operators*). We use the following notation for the classical wave operator (d'Alembertian):

$$\Box = \frac{\partial^2}{\partial \tau^2} - \Delta,$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplacian.

The mutual bigradients and MD-operators have properties useful in applications.

Lemma. *The mutual bigradients commute with each other, and for their composition one has*

$$\nabla^+\nabla^- = \nabla^-\nabla^+ = \Box.$$

The MD-operators commute with each other, and for their composition one has

$$\mathbf{D}^+_\mathbf{F} \circ \mathbf{D}^-_\mathbf{F} = \mathbf{D}^-_\mathbf{F} \circ \mathbf{D}^+_\mathbf{F} = \Box + 2f\partial_\tau + f^2 + (F, F) - 2i(F, \nabla).$$

Proof. Indeed, according to definition (1.2), we have

$$\nabla^+\nabla^- = (\partial_\tau + i\nabla) \circ (\partial_\tau - i\nabla) = \partial_\tau\partial_\tau - (\nabla, \nabla) - i\partial_\tau\nabla + i\partial_\tau\nabla - [\nabla, \nabla] = \nabla^- \circ \nabla^+ = \Box.$$

In a similar way, using definition (2.1), we obtain

$$\mathbf{D}^+_\mathbf{F} \circ \mathbf{D}^-_\mathbf{F} = \mathbf{D}^-_\mathbf{F} \circ \mathbf{D}^+_\mathbf{F} = (\nabla^+ + f + F) \circ (\nabla^- + f - F) = \Box + 2f\partial_\tau + f^2 + (F, F) + 2i(F, \nabla).$$

In what follows, we drop the quaternionic multiplication sign between operators.

3. SHOCK WAVES AS GENERALIZED SOLUTIONS OF THE BIWAVE EQUATION

Consider the solution of Eq. (1.1) for the upper sign of bigradient,

$$\mathbf{D}^+_\mathbf{F}\mathbf{B} = \mathbf{G}(\tau, x). \tag{3.1}$$

The solution for the lower sign of bigradient can be constructed by analogy with what is shown below or simply by using the complex conjugation operation.

Note that, by the Lemma, the biwave equation (1.1) is hyperbolic and admits solutions discontinuous on characteristic surfaces $F \subset \mathbb{M}$ whose normal satisfies the characteristic equation for the d'Alembert wave equation,

$$n_0^2 - n_1^2 - n_2^2 - n_3^2 = 0. \tag{3.2}$$

These surfaces are associated with wave fronts $F_\tau \subset \mathbb{R}^3$ propagating in the direction of the wave vector $n(\tau, x) = (n_1, n_2, n_3)$ with the velocity

$$v = -\frac{n_0}{\sqrt{n_1^2 + n_2^2 + n_3^2}} = 1.$$

It follows from relation (3.2) that the surface $\tau = \mathrm{const}$ cannot be characteristic. If we normalize the wave vector,

$$\|n\| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1,$$

then these relations imply that

$$n_0 = -1. \tag{3.3}$$

In view of the differentiation rules for discontinuous regular functions [12, Sec. 6], the action of the bigradient on the corresponding biquaternion has the form

$$\nabla^+ \hat{\mathbf{B}} = \nabla^+ \mathbf{B} + \{n_0 + n\} \circ \big[\mathbf{B}(\tau, x)\big]_F \delta_F(\tau, x),$$

where $[\mathbf{B}]_F \delta_F(\tau, x)$ is a simple layer on the surface F with density equal to the jump of the biquaternion on F; i.e.,

$$[\mathbf{B}]_F = \lim_{\varepsilon \to +0} \big\{\mathbf{B}(\tau + n_0\varepsilon, x + \varepsilon n) - \mathbf{B}(\tau - n_0\varepsilon, x - \varepsilon n)\big\}.$$

Consequently, discontinuous solutions of the biwave equation (1.1) must satisfy the following condition (*condition on shock fronts*):

$$\{n_0 + n\} \circ \big[\mathbf{B}(\tau, x)\big]_F = 0.$$

Expanding the scalar and vector parts in this relation and taking into account (3.3), we obtain

$$\big[b(\tau, x)\big]_F + i\Big(n(\tau, x), \big[B(\tau, x)\big]_F\Big) = 0$$

and

$$\big[B(\tau, x)\big]_F = i\Big\{n\big[b(\tau, x)\big]_F + \Big[n, \big[B(\tau, x)\big]_F\Big]\Big\}.$$

It follows that if $[b(\tau, x)]_F = 0$, then the vector $[B(\tau, x)]_F$ is perpendicular to the wave front; i.e., if the field ahead of the front is zero, then the wave is transverse. This fact is well known for electromagnetic waves described by a pure vector biquaternion with zero scalar part.

4. CONSTRUCTING SOLUTIONS OF THE MD-EQUATION

The biwave equation (1.1) will be called the *MD-equation*. To construct its solutions, we use definition (2.1). As a result, it follows from (3.1) that

$$\mathbf{D}_\mathbf{F}^- \mathbf{D}_\mathbf{F}^+ \mathbf{B} = \big\{\Box + 2f\partial_\tau + f^2 + (F, F) + 2i(F, \nabla)\big\}\mathbf{B} = \mathbf{D}_\mathbf{F}^- \mathbf{G} \triangleq \mathbf{Q}.$$

In other words, each component of the biquaternion $\mathbf{B}$ satisfies the scalar equation

$$\Box u + 2f\partial_\tau u + 2i(F, \nabla u) + f^2 u + (F, F)u = q(\tau, x) \tag{4.1}$$

with the appropriate component of the biquaternion $\mathbf{Q}$ on the right-hand side.

Note that for $m^2 = f^2 + (F, F)$ this equation contains the Klein–Gordon–Fock operator $\Box + m^2$ together with the additional term $2f\partial_\tau + 2i(F, \nabla)$. If $f = ik$ is pure imaginary, then one can also see the Schrödinger operator $2ik\partial_\tau - \Delta$ in this equation. Therefore, Eq. (4.1) will be referred to as the KGFS-*equation*.

Theorem 1. *The solution of the biwave equation* (1.1) *can be represented in the form*

$$\mathbf{B} = \mathbf{D}_\mathbf{F}^-(\psi * \mathbf{G}) = \mathbf{D}_\mathbf{F}^- \psi * \mathbf{G} + \mathbf{B}^0 = \psi * \mathbf{D}_\mathbf{F}^- \mathbf{G} + \mathbf{B}^0; \tag{4.2}$$

here $\psi(\tau, x)$ is the fundamental solution of Eq. (4.1) *(for $q = \delta(\tau)\delta(x)$) and $\mathbf{B}^0(\tau, x)$ is a solution of the homogeneous equation* (3.1) *(for $\mathbf{G} \equiv \mathbf{0}$),*

$$\mathbf{B}^0 = \sum_{\psi^0} \mathbf{D}_\mathbf{F}^- \psi^0 * \mathbf{C}^0 = \sum_{\psi^0} \psi^0 * \mathbf{D}_\mathbf{F}^- \mathbf{C}^0 = \sum_{\psi^0} \mathbf{D}_\mathbf{F}^-(\psi^0 * \mathbf{C}^0), \tag{4.3}$$

where the $\psi^0(\tau,x)$ *are solutions of the homogeneous equation* (1.1) (*for* $q=0$) *and the* $\mathbf{C}^0\in\mathbb{B}'(\mathbb{M})$ *are arbitrary biquaternions that admit such a convolution.*

Proof. By virtue of the linearity of the equation, it suffices to prove the assertion for each term in formula (4.1). Let us substitute the first term into Eq. (3.1) and, using definition (2.1) and the property of differentiation of the convolution, obtain

$$\mathbf{D}_\mathbf{F}^-\mathbf{D}_\mathbf{F}^+(\psi*\mathbf{G})=\big\{\Box\psi+2f\partial_\tau\psi+f^2\psi+(F,F)\psi+2i(F,\nabla\psi)\big\}*\mathbf{G}=\delta(\tau)\delta(x)*\mathbf{G}=\mathbf{G}.$$

In a similar way, for each term in the second sum we have the relations

$$\mathbf{D}_\mathbf{F}^-\mathbf{D}_\mathbf{F}^+(\psi^0*\mathbf{C}^0)=\big\{\Box+2f\partial_\tau+f^2+(F,F)+2i(F,\nabla)\big\}\psi^0*\mathbf{C}^0=0.$$

Obviously, by virtue of the linearity of Eq. (1.1), any of its solutions is representable in a similar form. To construct a solution, in formulas (4.2) and (4.3) in the theorem we can take any of the relations depending on convenience in calculating the convolutions; the latter depends only on the particular form of the functions appearing in the convolution.

Consequently, the solution of the biwave equation (3.1) is determined by the scalar functions $\psi(\tau,x)$ and $\psi^0(\tau,x)$—solutions of Eq. (4.1)—which we refer to as the *scalar potentials* of solutions of the MD-equation.

5. CONSTRUCTING THE GREEN'S FUNCTION FOR MD-EQUATION

Consider the fundamental solutions of Eq. (1.1),

$$\nabla^\pm\mathbf{U}+\mathbf{F}\circ\mathbf{U}=\delta(\tau)\delta(x);$$

here the right-hand side contains singular delta functions. The fundamental solutions are determined up to solutions of the homogeneous biwave equation (with zero right-hand side).

Definition. We refer to a fundamental solution of Eq. (4.2) satisfying the radiation conditions

$$\mathbf{U}(\tau,x)=0\ \text{ for }\ \tau<0\quad\text{and}\quad \mathbf{U}(\tau,x)=0\ \text{ for }\ \|x\|>\tau$$

as the *Green's function*.

The properties of fundamental solutions allow constructing particular solutions of Eq. (1.1) in the form of the functional convolution

$$\begin{aligned}\mathbf{B}&=\mathbf{G}*\mathbf{U}=(g+G)*(u+U)\\&=\left\{g*u-\sum_{k=1}^{3}G_k*U_k\right\}+\left\{u*G+g*U+\sum_{j,k,l=1}^{3}\varepsilon_{jkl}e_j(G_k*U_l)\right\},\end{aligned}\tag{5.1}$$

where the right-hand side contains componentwise convolutions taken according to the rules of the theory of generalized functions [12, Sec. 7]. The conditions for the existence of such convolutions define the class of biquaternions on the right-hand side in (1.1) for which there exist solutions of the equation.

Using the formula in Theorem 1, we construct fundamental solutions of Eq. (5.1),

$$\mathbf{U}(\tau,x)=\mathbf{D}_\mathbf{F}^-\big(\psi*\delta(\tau)\delta(x)\big)=\mathbf{D}_\mathbf{F}^-\psi+\mathbf{B}^0=\mathbf{D}_\mathbf{F}^-\psi+\mathbf{B}^0.\tag{5.2}$$

Here we have used the property of convolutions with the delta function, $\psi*\delta=\psi$.

6. GREEN'S FUNCTION FOR THE KGFS-EQUATION

To construct the Green's function of the biwave equation, let us find the Green's function of the KGFS-equation. This function satisfies the equation

$$\Box\psi+2f\partial_\tau u+2i(F,\nabla\psi)+f^2\psi+(F,F)\psi=\delta(\tau)\delta(x)\tag{6.1}$$

and the radiation conditions

$$\psi(\tau, x) = 0 \ \text{ for } \ \tau < 0 \quad \text{and} \quad \psi(\tau, x) = 0 \ \text{ for } \ \|x\| > \tau.$$

Theorem 2. *The Green's function of Eq.* (5.1) *can be represented in the form*

$$\psi = \frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta(\tau - \|x\|), \tag{6.2}$$

where $\delta(\tau - \|x\|)$ is a singular generalized function—a simple layer on the light cone $\|x\| = \tau$.

Proof. To prove formula (6.2), we use the Fourier transform of generalized functions. The Fourier variables corresponding to (τ, x) are denoted by (ω, ξ) below. Consider the Fourier transform of the KGFS-equation (6.1),

$$\big(\|\xi\|^2 - \omega^2 - 2if\omega + 2(F, \xi) + f^2 + (F, F)\big)\bar{\psi}(\omega, \xi) = 1,$$

which can be written as

$$\big\{(\xi + F, \xi + F) + (f - i\omega)^2\big\}\bar{\psi} = 1.$$

Based on the above, we obtain the Fourier transform of the scalar potential in the form

$$\bar{\psi}(\omega, \xi) = \frac{1}{(\xi + F, \xi + F) - (\omega + if)^2} \tag{6.3}$$

To construct the inverse Fourier transform, we use the fundamental solution of the d'Alembert equation

$$\Box\chi = \delta(x, t)$$

with the radiation conditions. This solution has the form

$$\chi(x, \tau) = \frac{1}{4\pi\|x\|}\delta\big(\tau - \|x\|\big).$$

Its Fourier transform is equal to the following regularization of the function $(\|\xi\|^2 - \omega^2)^{-1}$:

$$\mathrm{F}\left[\frac{1}{4\pi\|x\|}\delta\big(\tau - \|x\|\big)\right] = \frac{1}{\|\xi\|^2 - (\omega + i0)^2}. \tag{6.4}$$

In view of the properties of the shift of the Fourier transform, relations (6.3) and (6.4) imply the representation (6.2). The proof of the theorem is complete.

Note that ψ is a singular generalized function whose support is a expanding-in-time sphere, i.e., a spherical diverging wave propagating in $\mathbb{R}^3$ with the unit velocity (τ is time).

7. GREEN'S FUNCTION AND SOLUTIONS OF THE MD-EQUATION

Using relations (6.2) and (6.4), we can now give a representation of the Green's function. Using formula (5.2) for the generalized solution, we obtain a representation of the solution in the form

$$\begin{aligned}\mathbf{U}(\tau, x) &= \mathbf{D}^-_{\mathbf{F}}\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau - \|x\|\big)\right\} = (\partial_\tau - i\nabla + \mathbf{F}^-)\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau - \|x\|\big)\right\}\\ &= (\partial_\tau - i\nabla)\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau - \|x\|\big)\right\} + \left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau - \|x\|\big)\right\}\mathbf{F}^-,\end{aligned}$$

where $\mathbf{F}^- = f - F$. Let us state the result in the form of a theorem.

Theorem 3. *The Green's function of the MD-equation* (1.1) *has the form*

$$\mathbf{U} = (\partial_\tau - i\nabla)\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau - \|x\|\big)\right\} + \left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau - \|x\|\big)\right\}\mathbf{F}^-,$$

and its general solution can be represented in the form

$$\mathbf{B} = \frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau - \|x\|\big) * \mathbf{D}^{-}_{\mathbf{F}}\mathbf{G} + \mathbf{B}^0, \tag{7.1}$$

where $\mathbf{B}^0$ *is a solution of the homogeneous equation* (1.1).

For $\mathbf{B}^0 = 0$, this solution describes diverging waves generated by the source $\mathbf{G}$. For the regular differentiable biquaternions

$$\begin{aligned}\mathbf{K}(\tau,x) &= k(\tau,x) + K(\tau,x) = \mathbf{D}^{-}_{\mathbf{F}}\mathbf{G}(\tau,x) = (\partial_\tau - i\nabla)(g+G) + (f-F)\circ(g+G)\\ &= \big(\partial_\tau g + fg + (F,G) + i\operatorname{div} G\big) - i\nabla g + \partial_\tau G - i\operatorname{rot} G + fG - gF - [F,G],\end{aligned}$$

formula (7.1) can be represented in the integral form

$$\begin{aligned}\mathbf{B}(\tau,x) &= \frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau-\|x\|\big) * (k+K) + \mathbf{B}^0\\ &= \int\limits_{\|x-y\|<\tau} \frac{e^{-i(F,x-y)-\|x-y\|f}}{4\pi\|x-y\|}\mathbf{K}\big(y,\tau-\|x-y\|\big)\, dy_1\, dy_2\, dy_3 + \mathbf{B}^0.\end{aligned}$$

For calculating convolutions of singular biquaternions, one should use the definition of convolutions in the space of generalized functions.

8. CAUCHY PROBLEM. ANALOG OF THE KIRCHHOFF FORMULA

We use this term to refer to the solution of the Cauchy problem for the biwave equation by analogy with the Kirchhoff formulas that provide a representation of the solution of the wave equation with given initial data in three-dimensional space [12, Sec. 13, p. 233]. Here the initial conditions have the form

$$\mathbf{B}(0,x) = \mathbf{B}0(x),$$

where the initial data biquaternion $\mathbf{B}0(x)$ is a regular biquaternion whose components belong to the class of continuous differentiable functions. Suppose that its support is bounded,

$$\Omega_a = \operatorname{supp}\mathbf{B}0(x) \subset \big\{x\in\mathbb{R}^3 : \|x\| < a\big\}. \tag{8.1}$$

For this problem, we need to construct a solution satisfying the radiation condition

$$\mathbf{B}(\tau,x) = 0 \quad \text{for} \quad \|x\| > \tau + a. \tag{8.2}$$

To construct the solution of the Cauchy problem, we use Vladimirov's method [12, Sec. 13, p. 229]. Consider Eq. (1.1) on the space of generalized biquaternions supported on the positive time half-line, which we represent in the form of the generalized biquaternion

$$\hat{\mathbf{B}}(\tau,x) = \mathbf{B}(\tau,x)H(\tau),$$

where $\mathbf{B}(\tau,x)$ is the solution of the Cauchy problem and $H(\tau)$ is the Heaviside function. In this space, we have

$$\nabla^{+}\hat{\mathbf{B}} + \mathbf{F}\circ\hat{\mathbf{B}} = (\nabla^{+}\mathbf{B} + \mathbf{F}\circ\mathbf{B})H(\tau) + \mathbf{B}0(x)\delta(\tau) = \mathbf{G}(\tau,x)H(\tau) + \mathbf{B}0(x)\delta(\tau).$$

Using a property of the Green's function, we represent the solution in the form of the convolution of the Green's function with the right-hand side,

$$\begin{aligned}\mathbf{B}(\tau,x) &= \mathbf{U} * \mathbf{G}(\tau,x)H(\tau) + \mathbf{U} * \mathbf{B}0(x)\delta(\tau) = \mathbf{U} * \mathbf{G}(\tau,x)H(\tau)\\ &\quad + (\partial_\tau - i\nabla)\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau-\|x\|\big)\right\} * \mathbf{B}0(x)\delta(\tau)\\ &\quad + \left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta\big(\tau-\|x\|\big)\right\}\mathbf{F}^{-} * \mathbf{B}0(x)\delta(\tau)\\ &= \mathbf{U} * \mathbf{G}(\tau,x)H(\tau) + (\partial_\tau - i\nabla)\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta_{S_\tau}(x) \underset{x}{*} \mathbf{B}0(x)\right\} = \hat{\mathbf{B}}_1(\tau,x) + \hat{\mathbf{B}}_2(\tau,x),\end{aligned} \tag{8.3}$$

where $\delta_{S_\tau}(x)$ is a simple layer on the sphere S_τ of radius τ centered at the origin; i.e., $S_\tau = \{x : \|x\| = \tau\}$. Here, in view of the radiation conditions, the first term on the right-hand side has the form

$$\hat{\mathbf{B}}_1(\tau, x) = \int\limits_{\|x-y\|<\tau} \frac{e^{-i(F,x-y)-\|x-y\|f}}{4\pi\|x-y\|}\mathbf{K}\big(y, \tau - \|x-y\|\big)\, dy_1\, dy_2\, dy_3.$$

The second term $\hat{\mathbf{B}}_2(\tau, x)$, containing an incomplete convolution only in x, is represented as follows:

$$\begin{aligned}
\hat{\mathbf{B}}_2(\tau, x) &= (\partial_\tau - i\nabla)\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta_{S_\tau}(x) \underset{x}{*} \mathbf{B}0(x)\right\} \\
&= \left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\partial_\tau\big(\delta_{S_\tau}(x) * \mathbf{B}0(x)\big)\right\} - i\left\{\frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\delta_{S_\tau}(x) \underset{x}{*} \nabla\mathbf{B}0(x)\right\} \\
&= \frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\frac{\partial}{\partial\tau}\left(\int\limits_{\|x-y\|\le\tau} \mathbf{B}0(x-y)\, dy_1\, dy_2\, dy_3\right) - i\int\limits_{\|x-y\|=\tau}\frac{e^{-i(F,x-y)-f\|x-y\|}}{4\pi\|x-y\|}\nabla\mathbf{B}0(y)dS(y) \\
&= \frac{e^{-i(F,x)-\|x\|f}}{4\pi\|x\|}\left(\int\limits_{\|x-y\|=\tau} b0(x-y) + B0(x-y)dS(y)\right) \\
&\quad + ie^{-i(F,x)}\int\limits_{\|x-y\|=\tau}\frac{e^{i(F,y)-f\|x-y\|}}{4\pi\|x-y\|}\big(\operatorname{div} B0(y) - \operatorname{rot} B0(y)\big)dS(y).
\end{aligned}$$

Here the integrals are surface ones and are taken over the sphere of radius τ centered at the point x.

Formulas (8.1)–(8.3) are an analog of the Kirchhoff formula for the biwave equation (1.1). They represent the solution of the Cauchy problem for the biwave MD-equation.

9. DYNAMIC ANALOG OF GREEN'S FORMULA

We mean a representation of a solution of the biwave equation with zero initial data in a bounded open domain $S^- \subset \mathbb{R}^3$ via its boundary values on the boundary S, by analogy with the representation of solutions of the Laplace equation via the boundary values of the solutions and their derivatives [12]. To this end, we use the characteristic function $H_S^-(x)$ of the domain, the Heaviside function $H(\tau)$, and the characteristic function $H_S^-(x)H(\tau)$ of the spatial-temporal cylinder $\mathrm{C}_+ = \{(\tau, x) \in \mathbb{M} : \tau \ge 0,\, x \in S^- + S\}$. Their generalized derivatives have the form

$$\partial_j H_S^-(x) = -n_j(x)\delta_S(x), \quad \partial_t H(t) = \delta(t),$$
$$\partial_j\big(H_S^-(x)H(t)\big) = -n_j(x)\delta_S(x)H(t), \quad \partial_t\big(H_S^-(x)H(t)\big) = H_S^-(x)\delta(t),$$

where the singular generalized function $n_j(x)\delta_S(x)$ is a simple layer on the surface S and $n(x) = (n_1, n_2, n_3)$ is the outward unit normal on the boundary S.

We introduce the regular biquaternion

$$\hat{\mathbf{B}}(\tau, x) = \mathbf{B}(\tau, x)H_S^-(x)H(\tau),$$

which coincides in the domain and on the boundary with the solution $\mathbf{B}(\tau, x)$ of this equation with zero initial conditions and is zero outside the closure of the domain. The generalized partial derivatives of this biquaternion in $\mathbb{B}'(\mathbb{M})$ are equal to

$$\begin{aligned}
\partial_\tau\hat{\mathbf{B}}(\tau, x) &= \frac{\partial\mathbf{B}}{\partial\tau}H_S^-(x)H(\tau) + \mathbf{B}0H_S^-(x)\delta(\tau), \\
\partial_j\hat{\mathbf{B}}(\tau, x) &= \frac{\partial\mathbf{B}}{\partial x_j}H_S^-(x)H(\tau) - \mathbf{B}_S(\tau, x)n_j(x)\delta_S(x)H(\tau).
\end{aligned} \tag{9.1}$$

Here the first terms on the right-hand side are the classical partial derivatives of the biquaternion, and $\mathbf{B}_S(\tau, x)$ is the restriction of $\mathbf{B}(\tau, x)$ to S. In view of the representations (9.1) and the zero initial conditions ($\mathbf{B}0(x) = 0$), the action of the bigradient on this biquaternion has the form

$$\begin{aligned}\nabla^+\hat{\mathbf{B}} = \big\{(\partial_\tau b - i \operatorname{div} B) + \partial_\tau B + i(\operatorname{grad} b + \operatorname{rot} B)\big\} H_S^-(x)H(\tau) \\ + i\Big\{\big(n(x), B\big) - bn(x) - \big[n(x), B\big]\Big\}\delta_S(x)H(t).\end{aligned} \tag{9.2}$$

Then the action of the MD-operators on this biquaternion in the space of generalized biquaternions, in view of Eq. (1.1) and relation (9.2), is written as

$$\begin{aligned}\mathbf{D}^+\hat{\mathbf{B}}(\tau, x) = \nabla^+\hat{\mathbf{B}} + \mathbf{F} \circ \hat{\mathbf{B}} = \mathbf{G}(\tau, x)H_S^-(x)H(\tau) \\ + i\Big\{\big(n(x), B_S\big) - b_S n(x) - \big[n(x), B_S\big]\Big\}\delta_S(x)H(t).\end{aligned}$$

Set $\hat{\mathbf{G}}(\tau, x) = \mathbf{G}(\tau, x)H_S^-(x)H(\tau)$ and introduce the singular boundary biquaternion

$$\begin{aligned}\hat{\mathbf{\Gamma}}(\tau, x) &= \mathbf{\Gamma}(\tau, x)(\tau, x)\delta_S(x)H(t) \\ &= i\Big\{\big(n(x), B_S\big) - b_S n(x) - \big[n(x), B_S\big]\Big\}\delta_S(x)H(t).\end{aligned}$$

Here $\mathbf{\Gamma}(\tau, x)$ is the density of a simple layer on the spatial-temporal cylindrical surface, $\tau \geq 0$, $x \in S$. As a result, we arrive at the equation

$$\mathbf{D}^+\hat{\mathbf{B}}(\tau, x) = \nabla^+\hat{\mathbf{B}} + \mathbf{F} \circ \hat{\mathbf{B}} = \hat{\mathbf{G}}(\tau, x) + \hat{\mathbf{\Gamma}}(\tau, x),$$

whose solution has the form of the biquaternionic convolution of the right-hand side with the Green's function,

$$\hat{\mathbf{B}}(\tau, x) = \mathbf{U} * \hat{\mathbf{B}} = \mathbf{U}^+ * \hat{\mathbf{G}}(\tau, x) + \mathbf{U} * \hat{\mathbf{\Gamma}}(\tau, x). \tag{9.3}$$

The first term on the right-hand side is calculated by the formula

$$\begin{aligned}\hat{\mathbf{B}}_1(\tau, x) &= \mathbf{U} * \hat{\mathbf{G}}(\tau, x) = (u + U) * (\hat{g} + \hat{G}) \\ &= \left(u * \hat{g} - \sum_{k=1}^{3} U_k * \hat{G}_k\right) + \sum_{k=1}^{3}(u * \hat{G}_k + \hat{g} * U_k)e_k + \sum_{k=1}^{3} e_k \sum_{l,m=1}^{3} \varepsilon_{klm}(U_l * \hat{G}_m),\end{aligned}$$

where the componentwise convolutions are taken according to the definition of convolutions in the space of generalized functions.

For regular $\hat{\mathbf{G}}(\tau, x) = \mathbf{G}(\tau, x)$, these convolutions have the integral representation

$$\begin{aligned}4\pi\mathbf{B}_1 = \mathbf{U} * \hat{\mathbf{G}}(\tau, x) = \mathbf{D}_{\mathbf{F}}^-\left\{\frac{e^{-(f\|x-y\| + i(F, x-y))}}{\|x\|}\delta(\tau - \|x\|) * \hat{\mathbf{G}}\right\} \\ = \nabla^- \int\limits_{r(x,y)\leq\tau} \frac{e^{-(fr(x,y) + i(F, x-y))}}{r(x,y)} \mathbf{G}(\tau - r(x,y), y)H_S^-(y)\,dy_1\,dy_2\,dy_3 \\ + \mathbf{F}^- \circ \int\limits_{r(x,y)\leq\tau} \frac{e^{-(fr(x,y) + i(F, x-y))}}{r(x,y)} \mathbf{G}(\tau - r(x,y), y)H_S^-(y)\,dy_1\,dy_2\,dy_3,\end{aligned} \tag{9.4}$$

$r(x,y) = \|x-y\|$. Calculating the second term, we obtain

$$\begin{aligned}
4\pi\hat{\mathbf{B}}_2(\tau,x) &= \mathbf{U} * \hat{\mathbf{\Gamma}}(\tau,x) \\
&= i\nabla^- \frac{e^{-(fr(x,y)+i(F,x-y))}}{\|x\|}\delta\big(\tau-\|x\|\big) * \Big(\big(n(x),B_S\big) - b_S n(x) - \big[n(x),B_S\big]\Big)\delta_S(x)H(t) \\
&\quad + i\mathbf{F}^- \circ \left\{\frac{e^{-(fr(x,y)+i(F,x-y))}}{\|x\|}\delta\big(\tau-\|x\|\big) * \right\}\Big(\big(n(x),B_S\big) - b_S n(x) - \big[n(x),B_S\big]\Big)\delta_S(x)H(t) \\
&= i\nabla^- \int\limits_S \frac{\Big(n(y),B_S\big(\tau-r(x,y),y\big)\Big) - b_S\big(\tau-r(x,y),y\big)n(y) - \Big[n(x),B_S\big(\tau-r(x,y),y\big)\Big]}{r(x,y)\exp\big(r(x,y)f+i\big(F,x-y\big)\big)}\,dS(y) \\
&\quad + i\mathbf{F}^-\circ \int\limits_S \frac{i\Big(n(y),B_S\big(\tau-r(x,y),y\big)\Big) - b_S\big(\tau-r(x,y),y\big)n(y) - \Big[n(x),B_S\big(\tau-r(x,y),y\big)\Big]}{r(x,y)\exp\big(r(x,y)f+i(F,x-y)\big)}\,dS(y).
\end{aligned}$$

Here we first take the integrals and then apply the MD-operator to the resulting biquaternion.

Formulas (9.3) and (9.4) are similar to the Green's formula. They allow one to calculate the biquaternion inside the domain based on its boundary values. Note that all integrals and their derivatives exist only for $x \notin S$. For boundary points, the integrals themselves are weakly singular and convergent, but their derivatives are not. Here one can observe the same features as in the solutions of the wave equation in three-dimensional space [13].

The analysis of the integral representations of the biquaternion on the boundary allows one to obtain boundary singular integral equations for the solutions of initial–boundary value problems for the biwave equations and state well-posed boundary conditions for their solutions. This can be done by passing to the limit with respect to $x \notin S$ in an analog of the Green's formula towards the boundary S in the same way as in boundary value problems for the wave equation [13].

CONCLUSIONS

Using the constructed analogs of Kirchhoff's and Green's formulas, one can obtain analogs of the Green's formula for nonzero initial conditions by decomposing the solution of Eq. (1.1) into two biquaternions, one of which satisfies the initial conditions and the other has zero initial conditions. Formulas (9.3) and (9.4) give its representation with allowance for the acting sources. We obtain the boundary conditions for the second biquaternion using the boundary values of the original biquaternion and taking into account the corrections due to the first constructed biquaternion. Then, using these formulas, we construct the second biquaternion.

Since the Maxwell and Dirac equations are special cases of biwave equations, the constructed solutions can be used to solve problems of electrodynamics and field theory. They can be used in experiments, since the field characteristics of EM fields at the boundary can be measured experimentally without solving the singular boundary integral equations.

We also note that the transformation of electro-gravimagnetic (EGM) charges and currents under the action of external EGM fields is described by biquaternion differential equations similar to (1.1) (see [17, 18]). The solutions constructed here can be used to solve boundary value problems in EGM fields.

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