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CONTROL THEORY

Feedback in a Control Problem for a System with Discontinuous Right-Hand Side

Yu. S. Osipov[1,2*] **and V. I. Maksimov**[3**]

[1]*Lomonosov Moscow State University, Moscow, 119991 Russia*
[2]*Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow, 119991 Russia*
[3]*Krasovskii Institute of Mathematics and Mechanics, Ural Branch, Russian Academy of Sciences, Yekaterinburg, 620108 Russia*
*e-mail: *osipov@pran.ru, **maksimov@imm.uran.ru*

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Abstract—On a finite time horizon, we consider a control system described by a vector differential equation with right-hand side that changes its structure at some times spaced by a distance that cannot be less than a certain given value. In between two adjacent structure change instants, the right-hand side is a function that is Lipschitz in state variables, continuous in time, and linear in the control and perturbation, which take values in some convex closed sets. It is assumed that at the structure change instants the solution of the system may experience a jump by a certain vector of which only the direction is known. A uniform mesh is specified on the system operation interval. The values of the state vector are measured (with an error) at the mesh points. We solve the problem of constructing an algorithm for the formation of a system control that ensures bringing the system trajectory to the minimum possible neighborhood of the goal set at the end time. A solution algorithm is indicated that is based on the constructions of positional control theory and is resistant to information interferences and computational errors.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

We consider an optimal control problem for the system of differential equations

$$\ddot{x}(t) = f\big(t, x(t), \dot{x}(t), u(t), v(t), V(t)\big), \quad t \in T = [0, \vartheta], \tag{1}$$

with the initial state

$$x(0) = x_0, \quad \dot{x}(0) = y_0. \tag{2}$$

Here $\vartheta = \mathrm{const} \in (0, +\infty)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$ is the control, $v(t) \in \mathbb{R}^p$ and $V(t) \in \mathbb{N}$ are disturbances, and $\mathbb{N}$ is the set of positive integers. Using the terminology of the theory of positional differential games [1], we will say that player 1 is in charge of designing the control $u(\cdot)$. In turn, the disturbances $v(\cdot)$ and $V(\cdot)$ are formed by player 2. The function $V(t)$ is piecewise constant and has the form

$$V(t) = k \quad \text{for} \quad t \in [a_k^*, a_{k+1}^*), \quad k \in [0 : r], \quad a_k^* < a_{k+1}^*, \quad a_0^* = 0, \quad a_{r+1}^* = \vartheta,$$

where the number $r \in \mathbb{N}$ and the times a_k^* are at the disposal of player 2. The right-hand side of system (1) has the structure

$$f(t, x, y, u, v, k) = f_k(t, x, u, u, v), \quad k \in [0 : r].$$

Thus,

$$f\big(t, x, y, u, v, V(t)\big) = f_k(t, x, y, u, v) \quad \text{for} \quad t \in [a_k^*, a_{k+1}^*), \quad k \in [0 : r].$$

We will also write system (1) in the form

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f\big(t, x(t), y(t), u(t), v(t), V(t)\big). \tag{3}$$

The initial state of the latter system is

$$x(0) = x_0, \quad y(0) = y_0. \tag{4}$$

The problem in question is essentially as follows. Assume that $u(t) \in P(t) \subset \mathbb{R}^q$ and $v(t) \in Q(t) \subset \mathbb{R}^p$, where $P(t)$ and $Q(t)$ are convex bounded closed sets—the "resources" of players 1 and 2, respectively. A uniform mesh $\Delta = \{\tau_i\}_{i=0}^m$, $\tau_0 = 0$, $\tau_{i+1} = \tau_i + \delta$, $\tau_m = \vartheta$, is selected on the interval T. We measure (with an error) the state of system (1), (2) (system (3), (4)) at the mesh nodes τ_i; i.e., we find vectors ψ_i^h and ξ_i^h such that

$$\left|\xi_i^h - x(\tau_i)\right|_n \le h, \quad \left|\psi_i^h - \dot{x}(\tau_i)\right|_n \le h. \tag{5}$$

Here and in what follows, $h \in (0,1)$ is the value of *information error*, and by $|x|_n$ we denote the Euclidean norm of a vector x. Moreover, the system structure changes (switching occurs) at the times a_k^*, $k \in [1:r]$, and the sets P and Q also change, $P(t) = P_k$ and $Q(t) = Q_k$ for $t \in [a_k^*, a_{k+1}^*)$. We assume that the functions f_k as well as the sets P_k and Q_k are known to player 1, whereas the switching times a_k^* remain unknown to him. The choice of these times (i.e., the control $V(t)$) is at the disposal of player 2. We also assume that the "jumps" of states occur at the times a_k^*, $k \in [1:r]$. Namely, if a state $\{x(a_k^*), y(a_k^*-)\}$ must be realized at time a_k^*, where $x(a_k^*) = \lim\limits_{t\to a_k^*-} x(t)$ and $y(a_k^*-) = \lim\limits_{t\to a_k^*-} y(t)$, then we take

$$x(a_k^*) = x(a_k^*+), \quad y(a_k^*) = y(a_k^*+) = y(a_k^*-) + b_k^* e_k,$$

where the vectors $e_k \in \mathbb{R}^n$, $|e_k|_n = 1$, and the quantities $b_k^* \in \mathbb{R}$ are selected by player 2. In this case, the structure of the "jump" is presumed to be partly known to player 1. Namely, player 1 knows the vectors e_k but does not known the quantities b_k^*. In what follows, the times a_k^* will be called the *switching times*. The functions f_k will be assumed to be Lipschitz in x and y and continuous in t, u, and v.

The problem discussed in the present paper is to design a control $u(t) = u(\tau_i, \xi_i^h, \psi_i^h)$, $t \in [\tau_i, \tau_{i+1})$, ensuring bringing the state trajectory of system (1), (2) onto a closed set $M \subset \mathbb{R}^{2n}$ (at the time ϑ) or its "minimum admissible" neighborhood. The meaning of the last term will be explained below.

In the case where the system structure remains unchanged ($f = f_0$ at all $t \in T$ and there are no "jumps"), the problem under consideration can be solved within the approach proposed in the monograph [1]. According to this approach, one needs to proceed as follows. At the initial time, having the initial state known, one can determine the least neighborhood (ε-neighborhood, i.e., M^ε) of the goal set into which player 1 can surely transfer the system state vector at time ϑ. (Speaking of one or another neighborhood of the set M in what follows, we mean a closed neighborhood.) Then one can construct some family of u-stable sets $W^\varepsilon(t)$, $t \in T$, that stops at time ϑ on the set M^ε ($W^\varepsilon(\vartheta) \subset M^\varepsilon$) and such that the initial state of the system resides in the set $W^\varepsilon(0)$. For such sets one can take the broadest possible family of sets (the family of positional absorption sets) or a narrower family, for example, stable tracks. After this, we organize the procedure of positional control of a given system that ensures that the state trajectory of this system follows the state trajectory of the so-called guide, which moves over the selected family of u-stable sets. The strategy (rule of selection) of the control ensuring the above-indicated tracking property is called the *extremal strategy*. If $\{x_0, y_0\} \in W(0)$, then, as established in [1, Sec. 57], the extremal strategy solves the problem of guaranteed guidance to the set M at time ϑ for any admissible realization of the control by player 2.

We say that a control design strategy ensures a solution of the problem of guidance to the "minimum admissible" neighborhood of the set M if it is defined as follows. (In what follows, we will refer to this strategy as the *strategy of guaranteed guidance*—SGG.) At the initial time, we construct a family of u-stable sets $W_0(t)$, $t \in T$, that ensure the solution of the problem of guaranteed guidance of system (3) with right-hand side $f = f_0$ from the initial state $\{x_0, y_0\}$ into the least neighborhood of the set M. After this, for the SGG on the half-open interval $[0, a_1^*)$ we select the strategy of extremal aiming at the sets $W_0(t)$. At the time a_1^*, a state $\{x(a_1^*), y(a_1^*-)\}$ is realized as a result of application of this strategy and some admissible control $v(\cdot)$ of player 2. In view of a jump and a change in the system structure, starting from the time a_1^* (up to the time a_2^*), system (3) is described by the relations

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f_1\big(t, x(t), y(t), u(t), v(t)\big) \tag{6}$$

with the initial state

$$x(a_1^*), \quad y(a_1^*) = y(a_1^*+) = y(a_1^*-) + b_1^* e_1. \tag{7}$$

For system (6) with the initial (at time $t = a_1^*$) state (7), we construct the system of u-stable sets $W_1(t)$, $t \in [a_1^*, \vartheta]$, that ensure solution of the problem of guaranteed guidance to the least neighborhood of the set M at time ϑ. For the SGG on the half-open interval $[a_1^*, a_2^*)$ we choose the strategy of extremal aiming at the sets $W_1(t)$. The SGG on the half-open intervals $[a_k^*, a_{k+1}^*)$, $k \in [2 : r]$, is defined in a similar way. Let an SGG be defined on a half-open interval $[0, a_k^*)$. The state $\{x(a_k^*), y(a_k^*-)\}$ is realized at the time $t = a_k^*$ as a result of application of this strategy and some admissible control $v(\cdot)$ of player 2. In view of a change in the system structure and a jump, starting from the time a_k^* (up to the time a_{k+1}^*), system (3) is described by the relations

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f_k\big(t, x(t), y(t), u(t), v(t)\big) \tag{8}$$

with the initial state

$$x(a_k^*), \quad y(a_k^*) = y(a_k^*+) = y(a_k^*-) + b_k^* e_k. \tag{9}$$

For system (8) with the initial (at time $t = a_k^*$) state (9) we construct the system of u-stable sets $W_k(t)$, $t \in [a_k^*, \vartheta]$, that ensure the solution of the problem of guaranteed guidance from the state $\{x(a_k^*), y(a_k^*)\}$ into the least neighborhood of the set M. For the SGG on the half-open interval $[a_k^*, a_{k+1}^*)$, we select the strategy of extremal aiming at the set $W_k(t)$.

We have introduced the notion of SGG under the assumption that the times of jumps a_k^* are known to player 1 and that player 1 also knows the states $\{x(a_k^*), y(a_k^*)\}$. In reality, this is not the case. Namely, both the times a_k^* and the states $\{x(a_k^*), y(a_k^*)\}$ of the form (9) are unknown to player 1 and are to be determined. Suppose that, when constructing the SGG, instead of the times of jumps a_k^* as well as the states $\{x(a_k^*), y(a_k^*)\}$ one takes their approximate values determined using some algorithm. Calculating these values will take some time. Therefore, when constructing the SGG, instead of unknown jump times a_k^*, it is natural to use other times, slightly exceeding a_k^*. Such a modification of SGG leads to a new strategy of selection of the control by player 1, which we will call the *ε-strategy of guaranteed guidance* (ε-SGG). The present paper is aimed at constructing an ε-SGG.

Note that the foundations of guaranteed control theory in a formalization that goes back to the works by N.N. Krasovskii were laid in the papers [1–7]. However, these papers discussed guaranteed control problems for systems with a fixed right-hand side (with a given structure). In addition, the case of measuring all state coordinates was considered. The case of measuring part of the coordinates was investigated in [8–11]. In this paper, we study the problem of guidance for systems with variable structure in the presence of jumps in states. Note that jumps of this type appear, for example, in impulse control problems.

In this paper, for simplicity, we restrict ourselves to the case of functions f_k linear in the controls; i.e., we set

$$f_k = f_{1k}(t, x, \dot{x}) + B_k u - C_k v,$$

where B_k and C_k are constant matrices of appropriate sizes, and stable tracks are taken for stable sets. In this case, it is natural to choose the strategy of aiming at the corresponding tracks as the extremal aiming strategy.

Remark 1. If the maximum stable bridges, i.e., the sets of positional absorption, are taken as stable sets, then it is convenient to choose the strategy of aiming at the guide moving along the corresponding bridge as the extremal aiming strategy.

Systems with discontinuous right-hand side are a special case of hybrid systems. The latter include systems with variable structure [12] as well as impulse systems [13, 14]. The theory of control of hybrid systems has received rapid development in recent years [15–18]. Switched systems are an important subclass of hybrid systems [19, 20]. The latter include the systems considered in this paper.

We will need the following two conditions in the sequel.

Condition 1. There exist convex and closed sets $E_k \subset \mathbb{R}^n$, $k \in [0 : r]$, such that $B_k P_k = C_k Q_k + E_k$.

Here $B_k P_k = \{B_k u : u \in P_k\}$, $C_k Q_k = \{C_k v : v \in Q_k\}$, and $C_k Q_k + E_k = \{u_1 + u_2 : u_1 \in C_k Q_k, u_2 \in E_k\}$.

Condition 2. Numbers $b^* > 0$, $d_0^* > 0$, and $d^* > 0$ are given such that

$$b^* \le b_k^* \quad \text{for all} \quad k \in [1:r],$$
$$d_0^* \le a_{k+1}^* - a_k^* \quad \text{for all} \quad k \in [1:r-1], \quad a_1^* > d^*, \quad a_r^* < \vartheta.$$

2. AUXILIARY RESULTS

Consider the problem of constructing an algorithm for finding the points as well as sizes of discontinuities of the derivative of an n-dimensional function $x(\cdot)$ given with an error. The essence of the problem is as follows. We have some n-dimensional function $x(\cdot)$ given on a finite time interval $T = [0, \vartheta]$. The interval T is divided into finitely many half-open intervals

$$[\tau_i, \tau_{i+1}), \quad i \in [0 : m-1], \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = 0, \quad \tau_m = \vartheta.$$

The values $x(\tau_i)$ of the function $x(\cdot)$ are measured (approximately) at the times $\tau_i \in \Delta = \{\tau_i\}_{i=0}^m$; i.e., vectors $\Xi_i^h \in \mathbb{R}^n$ with the properties

$$\left|x(\tau_i) - \Xi_i^h\right|_n \le h \tag{10}$$

are found. The function $x(\cdot)$ itself is unknown. It is necessary to indicate a dynamic algorithm for calculating the points as well as the sizes of discontinuities of the derivative of the function $\dot{x}(\cdot)$ based on an imprecise measurement of the quantity $x(\tau_i)$. Such an algorithm is characterized by two properties:

(a) Calculation of the points of discontinuities (as well as the corresponding sizes of discontinuities) of the derivative of the function $x(\cdot)$ smaller than the current value of t is carried out based on the results of measuring the state $x(\tau)$ at the times τ preceding t.

(b) Only after the points and sizes of discontinuities of the function $\dot{x}(\cdot)$ on the interval $0 \le \tau \le t$ have been calculated, it becomes possible to use new information to calculate them at the subsequent times (for $\tau > t$).

To solve this problem, we will use the method of positional control with a model developed in the papers [1–4, 6–12]. In accordance with this method, the problem under consideration is replaced by another problem, namely, the feedback control problem for some system. In the sequel, this system will be called the *model*.

Consider the case where $\dot{x}(\cdot)$ is a piecewise continuous function. Namely, let $\{a_k\}_{k=1}^r$ be the (unknown) points of discontinuity of the function $\dot{x}(\cdot)$ arranged in ascending order; i.e., $a_{k+1} > a_k$. To be definite, we assume that the function $\dot{x}(\cdot)$ is right continuous at these points,

$$\dot{x}(a_k) = \dot{x}(a_k+) = \lim_{\substack{t\to a_k \\ t > a_k}} \dot{x}(t).$$

By b_k we denote the (unknown) sizes of discontinuities; i.e.,

$$b_k = \left|\dot{x}(a_k+) - \dot{x}(a_k-)\right|_n, \quad \dot{x}(a_k-) = \lim_{\substack{t\to a_k \\ t < a_k}} \dot{x}(t).$$

Let three numbers $b > 0$, $d_0 > 0$, and $d > 0$ be given, and assume that it is known that

$$b \le b_k \quad \text{for all} \quad k \in [1:r],$$
$$d_0 \le a_{k+1} - a_k \quad \text{for all} \quad k \in [1:r-1], \quad a_1 > d_0 \quad a_r < \vartheta,$$
$$\left|\dot{x}(t)\right|_n \le d \quad \text{for a.a.} \quad t \in T.$$

(The value of r may be unknown.) Assume also that the function $\dot{x}(\cdot)$ is continuously differentiable everywhere except for the points $\{a_k\}_{k=1}^r$, and that a number $F > 0$ is known such that

$$\left|\ddot{x}(t)\right|_n \le F$$

at all points where the function $\dot{x}(\cdot)$ is differentiable.

Let us fix a family of partitions of the interval T,

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h^3}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h^3} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h),$$

where $\delta(h) = \vartheta m_h^{-3}$, $m_h \in \mathbb{N}$.

Fix some function $\alpha = \alpha(h) : (0,1) \to (0,1)$. Introduce a controllable system (model) described by a vector differential equation ($w \in \mathbb{R}^n$, $u^h \in \mathbb{R}^n$) of the form

$$\dot{w}(t) = u^h(t) \tag{11}$$

(system M) with control $u^h(t)$. Let

$$u^h(t) = -\frac{1}{\alpha}\big[w^h(\tau_i) - \Xi_i^h\big] \quad \text{for} \quad t \in \delta_i \equiv [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{h,i}, \quad i \in [0 : m_h^3 - 1], \tag{12}$$

where $\alpha = \alpha(h)$. In Eq. (11), we define the control $u^h(t)$ according to (12). Thus, the control $u^h(\cdot)$ in system (11) will be found based on the feedback principle,

$$u^h(t) = u^h\big(\tau_i; w^h(\tau_i), \Xi_i^h\big), \quad t \in \delta_i.$$

In this case, system (11) acquires the form

$$\dot{w}^h(t) = -\frac{1}{\alpha}\big[w^h(\tau_i) - \Xi_i^h\big] \quad \text{for a.a.} \quad t \in \delta_i, \quad i \in [0 : m_h^3 - 1]. \tag{13}$$

Its initial state is

$$w^h(0) = \Xi_0^h.$$

We introduce the notation

$$\mu(t) = \max_{0 \le \tau \le t} \big|w^h(\tau) - x(\tau)\big|_n. \tag{14}$$

By $\Xi(x(\cdot), h)$ we denote the set of admissible measurement results, i.e., the set of all piecewise constant functions $\Xi^h(\cdot) : T \to \mathbb{R}^n$ with the structure

$$\Xi^h(t) = \Xi_i^h \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{h,i}, \quad i \in [0 : m_h^3 - 1],$$

which satisfy inequalities (10).

We introduce the following condition.

Condition 3. One has the relations

$$\delta(h) \to 0, \quad \alpha(h) \to 0, \quad \frac{h + \delta(h)}{\alpha(h)} \to 0 \quad \text{as} \quad h \to 0.$$

Taking into account this condition, we can claim that there exists an $h_* \in (0,1)$ such that for any $h \in (0, h_*)$ one has the inclusions

$$\alpha(h) \in (0,1), \quad \delta(h) \in (0,1), \quad h/\alpha(h) \in (0,1), \quad \delta(h)/\alpha(h) \in (0, 1/2). \tag{15}$$

Lemma 1. *Let $\dot{x}(\cdot) \in L_\infty(T; \mathbb{R}^n)$, $|\dot{x}(t)|_n \le d$ for a.a. $t \in T$, let $\mu(a) \le q$ for some $a \in T$, and let condition 3 be satisfied. Then for all $h \in (0, h_*)$, $\Xi^h(\cdot) \in \Xi(x(\cdot), h)$, and $\tau_{i+1} > a$ the inequalities*

$$\mu(t) \le 2q + (2 + 3d)(\alpha + \delta), \quad t \in [a, \vartheta], \tag{16}$$

$$\left(\int_{\tilde{\tau}_i}^{\tau_{i+1}} \big|\dot{w}^h(s)\big|_n^2 \, ds\right)^{1/2} \le \sqrt{2}(4 + 4.5d)\delta^{1/2} + 2\sqrt{2}\delta^{1/2}\alpha^{-1} q \tag{17}$$

hold, where $\tilde{\tau}_i = \tau_i$ if $\tau_i \ge a$ and $\tilde{\tau}_i = a$ if $\tau_i < a$.

Proof. Using relation (13), we conclude that the relations

$$\begin{aligned}\frac{d}{dt}\big[w^h(t)-x(t)\big] &= -\frac{1}{\alpha}\big[w^h(\tau_i)-\Xi_i^h\big]-\dot{x}(t)\\ &= -\frac{1}{\alpha}\big[w^h(t)-x(t)\big]+\Psi_h^{(1)}(t) \text{ for a.a. } t\in\delta_i=[\tau_i,\tau_{i+1}),\ \ i\in[0:m_h^3-1],\end{aligned} \tag{18}$$

hold, where

$$\Psi_h^{(1)}(t)=\Psi_h(t)+\frac{1}{\alpha}[w^h(t)-w^h(\tau_i)],$$
$$\Psi_h(t)=-\frac{1}{\alpha}\big[x(t)-\Xi_i^h\big]-\dot{x}(t) \quad \text{for a.a.} \quad t\in\delta_i.$$

By virtue of the inclusions (15), the inequalities $h\alpha^{-1}\le 1$ and $\delta\alpha^{-1}\le 1/2$ hold for $h\in(0,h_*)$. In this case, the family of functions $\Psi_h(\cdot)$ is bounded (uniformly with respect to all $h\in(0,h_*)$),

$$\begin{aligned}\big|\Psi_h(t)\big|_n &\le \frac{1}{\alpha}\Big(h+\big|x(t)-x(\tau_i)\big|_n\Big)+\big|\dot{x}(t)\big|_n\\ &\le \frac{h}{\alpha}+\frac{1}{\alpha}\int\limits_{\tau_i}^{\tau_{i+1}}\big|\dot{x}(\tau)\big|_n\,d\tau+\big|\dot{x}(t)\big|_n\le 1+1,5d \quad \text{for a.a.} \quad t\in\delta_i.\end{aligned} \tag{19}$$

The representation (18) implies the equality

$$w^h(t)-x(t)=w^h(a)-x(a)+\int\limits_a^t e^{-(t-s)/\alpha}\Psi_h^{(1)}(s)\,ds, \quad t\in[a,\vartheta]. \tag{20}$$

Further, the following estimates hold (see (13), (14)):

$$\begin{aligned}\frac{1}{\alpha}\int\limits_{\tau_i}^{\tau_{i+1}}\big|\dot{w}^h(s)\big|_n\,ds &\le \frac{1}{\alpha}\int\limits_{\tau_i}^{\tau_{i+1}}\bigg|\frac{1}{\alpha}\big[w^h(\tau_i)-\Xi_i^h\big]\bigg|_n\,ds\\ &\le \frac{\delta}{\alpha^2}\big(\mu(\tau_i)+h\big), \quad \mu(\tau_i)\le\mu(\tau_{i+1}), \quad i\in[0:m_h^3-1].\end{aligned} \tag{21}$$

Note that one has the inequality

$$\big|\Psi_h^{(1)}(t)\big|_n\le\big|\Psi_h(t)\big|_n+\frac{1}{\alpha}\int\limits_{\tau_i}^{\tau_{i+1}}\big|\dot{w}^h(s)\big|_n\,ds \quad \text{for} \quad t\in\delta_i. \tag{22}$$

Taking into account relations (20)–(22), we obtain

$$\begin{gathered}\mu(t)\le q+\left(\frac{\delta}{\alpha^2}\mu(\tau_i)+\frac{\delta h}{\alpha^2}\right)\int\limits_a^t e^{-(t-s)/\alpha}\,ds+\int\limits_a^t e^{-(t-s)/\alpha}\big|\Psi_h(s)\big|_n\,ds,\\ t\in[\tilde{\tau}_i,\tau_{i+1}], \quad \tau_{i+1}>a.\end{gathered} \tag{23}$$

It can readily be seen that the inequality

$$\int\limits_a^t e^{-(t-s)/\alpha}\,ds\le\alpha(1-e^{-(t-a)/\alpha})\le\alpha \tag{24}$$

holds. Using inequalities (19) and (24), we have

$$\int_a^t e^{-(t-s)/\alpha}\big|\Psi_h(s)\big|_n\,ds \le (1+1.5d)\left(\int_a^t e^{-(t-s)/\alpha}\,ds\right) \le \alpha K_1, \tag{25}$$

where $K_1 = 1 + 1,5d$. In turn, taking $t = \tilde{\tau}_i$ in (23) and taking into account inequality (24) as well as the inequality $\mu(\tau) \le \mu(\tilde{\tau}_i)$ for $\tau \in [0, \tilde{\tau}_i]$, from (23) and (25) we derive the estimate

$$\left(1 - \frac{\delta}{\alpha}\right)\mu(\tilde{\tau}_i) \le q + \frac{\delta h}{\alpha} + \alpha K_1 \le q + K_1\left(\alpha + \frac{\delta h}{\alpha}\right),$$

which implies, by virtue of the inequalities $1 - \delta/\alpha \ge 1/2$ and $h\alpha^{-1}(h) \le 1$ (see (15)), that

$$\mu(\tilde{\tau}_i) \le 2q + 2K_1\left(\alpha + \frac{\delta h}{\alpha}\right) \le 2q + 2K_1(\alpha + \delta). \tag{26}$$

Further, we have

$$\mu(\tilde{\tau}_i) \ge \mu(\tau_i). \tag{27}$$

Considering (23) and (25), we obtain

$$\mu(t) \le q + \frac{\delta h}{\alpha} + \frac{\delta}{\alpha}\mu(\tau_i) + \alpha K_1.$$

Hence, in view of (26) and (24), one has the inequality

$$\mu(t) \le q + \frac{\delta h}{\alpha} + 2\frac{\delta}{\alpha}q + 2\frac{\delta}{\alpha}K_1(\alpha + \delta) + \alpha K_1,$$

which implies inequality (16).

Let us check if inequality (17) holds. We have

$$\big|u^h(t)\big|_n \le \frac{1}{\alpha}\big|w^h(\tau_i) - \Xi_i^h\big|_n \quad \text{for a.a.} \quad t \in \delta_i.$$

Therefore, considering (15), (26), and (27), we obtain

$$\big|u^h(t)\big|_n \le \frac{1}{\alpha}\big(\mu(\tau_i) + h\big) \le \frac{h}{\alpha} + 2\frac{q}{\alpha} + 2K_1\left(1 + \frac{\delta}{\alpha}\right) \le 2\frac{q}{\alpha} + (4 + 4.5d) \quad \text{for a.a.} \quad t \in \delta_i. \tag{28}$$

From (28) we derive the inequality

$$\int_{\tilde{\tau}_i}^{\tau_{i+1}} \big|\dot{w}^h(s)\big|_n^2 ds \le \int_{\tau_i}^{\tau_{i+1}} \big|\dot{w}^h(s)\big|_n^2\,ds = \int_{\tau_i}^{\tau_{i+1}} \big|v^h(s)\big|_n^2\,ds \le 8\frac{q^2}{\alpha^2}\delta + 2(4 + 4.5d)^2\delta,$$

which implies inequality (17). The proof of the lemma is complete.

The symbol $W^{1,\infty}([a,b];\mathbb{R}^n)$ will denote the space of differentiable n-dimensional functions whose derivatives are elements of the space $L_\infty([a,b];\mathbb{R}^n)$.

Lemma 2. *Let the assumptions of Lemma 1 be satisfied. If* $\dot{x}(\cdot) \in W^{1,\infty}([a,\vartheta];\mathbb{R}^n)$, $a \in [0,\vartheta)$, *then for* $t \in [a,\vartheta]$ *one has the inequality*

$$\begin{aligned}\big|u^h(t) - \dot{x}(t)\big|_n &\le \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{t-a}, \frac{\delta q}{\alpha^2}\right)\\ &\equiv \frac{\alpha}{t-a}d + \tilde{c}_1\alpha(h) + \tilde{c}_2\big(h + \delta(h)\big)\alpha^{-1}(h) + \tilde{c}_3\delta(h)q\alpha^{-2}(h),\end{aligned}$$

where $\tilde{c}_1 = F$, $\tilde{c}_2 = 2\sqrt{2}(4 + 4.5d) + 2\max\{1, d\}$, $\tilde{c}_3 = 4\sqrt{2}$, $\operatorname{vrai}\max\limits_{t\in[a,\vartheta]} |\ddot{x}(t)|_n \le F$, *and* $|\dot{x}(t)|_n \le d$ *for a.a.* $t \in [a, \vartheta]$.

Proof. Taking into account the representation (20), we arrive at the relation

$$\begin{aligned}\alpha^{-1}\big[w^h(t) - x(t)\big] - \alpha^{-1}\big[w^h(a) - x(a)\big] &= \int\limits_a^t \frac{d}{ds}\big(\varrho_\alpha(t-s)\big)\Psi_h^{(1)}(s)\,ds \\ &= -\int\limits_a^t \frac{d}{ds}\big(\varrho_\alpha(t-s)\big)\dot{x}(s)\,ds + \sum_{j=1}^{2}\int\limits_a^t \frac{d}{ds}\big(\varrho_\alpha(t-s)\big)\gamma_\delta^{(j)}(s)\,ds, \quad t \in [a, \vartheta],\end{aligned} \tag{29}$$

where

$$\varrho_\alpha(t) = \exp(-\alpha^{-1}t), \quad \gamma_\delta^{(1)}(s) = \alpha^{-1}\big[w^h(s) - w^h(\tau_i)\big],$$
$$\gamma_\delta^{(2)}(s) = -\alpha^{-1}\big[x(s) - \Xi_i^h\big] \quad \text{for a.a.} \quad s \in [\tau_i, \tau_{i+1}].$$

By virtue of Lemma 1 (see (17)), the relations

$$\begin{aligned}\big|\gamma_\delta^{(1)}(s)\big|_n &\le \frac{1}{\alpha}\int\limits_{\tilde{\tau}_i}^{s} \big|\dot{w}^h(s)\big|_n\,ds \le \frac{\delta^{1/2}}{\alpha}\left(\int\limits_{\tilde{\tau}_i}^{\tau_{i+1}} \big|\dot{w}^h(s)\big|_n^2\,ds\right)^{1/2} \\ &\le \frac{\delta^{1/2}}{\alpha}\left\{\sqrt{2}(4 + 4.5d)\delta^{1/2} + 2\sqrt{2}\frac{\delta^{1/2}}{\alpha}q\right\} \\ &= \sqrt{2}(4 + 4.5d)\frac{\delta}{\alpha} + 2\sqrt{2}\frac{\delta}{\alpha^2}q, \quad s \in [\tilde{\tau}_i, \tau_{i+1}],\end{aligned} \tag{30}$$

hold. Using condition (10) and the inequality $|\dot{x}(t)|_n \le d$, we have

$$\big|\gamma_\delta^{(2)}(s)\big|_n \le c_0(\delta + h)\alpha^{-1}, \quad s \in [a, \vartheta], \tag{31}$$

where $c_0 = \max\{1, d\}$. In this case, taking into account inequality (24), from (30) and (31) we derive the estimate

$$\left|\sum_{j=1}^{2}\int\limits_a^t \frac{d}{ds}\varrho_\alpha(t-s)\gamma_\delta^{(j)}(s)\,ds\right|_n \le \varrho(h, \alpha, \delta) + 2\sqrt{2}\frac{\delta}{\alpha^2}q, \tag{32}$$

where $\varrho(h, \alpha, \delta) = c_1(\delta + h)/\alpha$, $c_1 = \sqrt{2}(4 + 4.5d) + c_0$. Integrating by parts in the first term on the right-hand side in relation (29), we obtain

$$-\int\limits_a^t \left(\frac{d}{ds}\varrho_\alpha(t-s)\right)\dot{x}(s)\,ds = \varrho_\alpha(t-a)\dot{x}(a) - \dot{x}(t) + \int\limits_a^t \varrho_\alpha(t-s)\ddot{x}(s)\,ds, \quad t \in [a, \vartheta]. \tag{33}$$

In turn, it follows from relation (29) with regard to relations (32) and (33), that

$$\begin{aligned}&\left| -\frac{1}{\alpha}\big[w^h(t) - x(t)\big] + \frac{1}{\alpha}\big[w^h(a) - x(a)\big] - \dot{x}(t)\right|_n \\ &\qquad \le 2\sqrt{2}\frac{\delta}{\alpha^2}q + \varrho(h, \alpha, \delta) + \big|\varrho_\alpha(t-a)\dot{x}(a)\big|_n + \int\limits_a^t e^{-(t-s)/\alpha}\big|\ddot{x}(s)\big|_n\,ds.\end{aligned} \tag{34}$$

Since the inequalities

$$\begin{gathered}\big|\varrho_\alpha(t-a)\dot{x}(a)\big|_n = e^{-(t-a)/\alpha}\big|\dot{x}(a)\big|_n \le \frac{\alpha}{t-a}\big|\dot{x}(a)\big|_n, \quad t\in[a,\vartheta],\\ \int\limits_a^t e^{-(t-s)/\alpha}\big|\ddot{x}(s)\big|_n\,ds \le \alpha F\end{gathered} \tag{35}$$

hold, it follows from them and inequality (34) that

$$\left|-\frac{1}{\alpha}\big[w^h(t)-x(t)\big]+\frac{1}{\alpha}\big[w^h(a)-x(a)\big]-\dot{x}(t)\right|_n \le 2\sqrt{2}\frac{\delta}{\alpha^2}q+\varrho(h,\alpha,\delta)+\alpha F+\frac{\alpha}{t-a}\big|\dot{x}(a)\big|_n.$$

Moreover, by virtue of (10) and (30), for $t\in[\tilde{\tau}_i,\tau_{i+1}]$ we have the estimate

$$\begin{aligned}\left|\alpha^{-1}\Big\{\big[w^h(t)-x(t)\big]-\big[w^h(\tau_i)-\Xi_i^h\big]\Big\}\right|_n &\le \frac{1}{\alpha}\left\{\int\limits_{\tau_i}^t\big|\dot{w}^h(s)\big|_n\,ds+h+\int\limits_{\tau_i}^t\big|\dot{x}(s)\big|_n\,ds\right\}\\ &\le (h+d\delta)\alpha^{-1}+\delta\alpha^{-1}\left(\sqrt{2}(4+4.5d)+2\sqrt{2}\frac{q}{\alpha}\right).\end{aligned} \tag{36}$$

In view of the boundedness of the second derivative $\ddot{x}(\cdot)$ ($|\ddot{x}(t)|_n \le F$ for a.a. $t\in[a,\vartheta]$), relations (34)–(36) imply (for $t\in\delta_i$) the inequality

$$\begin{aligned}&\left|\frac{1}{\alpha}\big[w^h(\tau_i)-\Xi_i^h\big]+\frac{1}{\alpha}\big[w^h(a)-x(a)\big]-\dot{x}(t)\right|_n\\ &\qquad\le 4\sqrt{2}\frac{\delta}{\alpha^2}q+\frac{h}{\alpha}+\varrho(h,\delta,\alpha)+\Big(d+\sqrt{2}(4+4.5d)\Big)\frac{\delta}{\alpha}+F\alpha+\frac{\alpha}{t-a}\big|\dot{x}(a)\big|_n,\end{aligned}$$

which implies the assertion of the lemma. The proof of the lemma is complete.

We introduce functions $\alpha=\alpha(h)$, $\gamma=\gamma(h)$, and $N=N(h)$ as follows:

$$\alpha(h)=\delta^{2/3}(h), \quad \gamma(h)=\delta(h)m_h^2=\frac{\vartheta}{m_h}<\frac{d_0}{2}, \quad N(h)=\frac{\gamma(h)}{\delta(h)}=m_h^2.$$

Here $\delta(h)$ is the step of the partition Δ_h, i.e., $\delta(h)=\vartheta m_h^{-3}$, $m_h=[(\vartheta/h)^{1/3}]$, and $[a]$ stands for the integer part of a number a. Note that with this choice of α, δ, and γ one has the relations

$$h\le\delta(h), \quad \frac{h}{\alpha(h)}\le\frac{\delta(h)}{\alpha(h)}=\frac{\vartheta^{2/3}\alpha(h)}{\gamma(h)}=\frac{\vartheta^{1/3}}{m_h}\to 0 \quad \text{as} \quad h\to 0. \tag{37}$$

Let $\delta(1+N)<\vartheta-a_r$ and

$$\begin{aligned}\chi_1(\alpha,\delta,h)&=F(\delta+\gamma)+\Psi\left(\frac{h}{\alpha},\frac{\delta}{\alpha},\alpha,\frac{\alpha}{d_0},\frac{\delta h}{\alpha^2}\right)+\Psi\left(\frac{h}{\alpha},\frac{\delta}{\alpha},\alpha,\frac{\alpha}{\gamma},\frac{\delta\big(2h+(2+3d)(\alpha+\delta)\big)}{\alpha^2}\right),\\ \chi(\alpha,\delta,h)&=F(\delta+\gamma)+\Psi\left(\frac{h}{\alpha},\frac{\delta}{\alpha},\alpha,\frac{3\alpha}{d_0},\frac{\delta\big(2h+(2+3d)(\alpha+\delta)\big)}{\alpha^2}\right)\\ &\qquad+\Psi\left(\frac{h}{\alpha},\frac{\delta}{\alpha},\alpha,\frac{\alpha}{\gamma},\frac{\delta\big(4h+(6+9d)(\alpha+\delta)\big)}{\alpha^2}\right).\end{aligned} \tag{38}$$

By virtue of relations (37), there exists a number $h_1\in(0,h_*)$ such that the following inequalities hold for all $h\in(0,h_*)$:

$$\delta(h)=\vartheta m_h^{-3}\le d_0/4, \quad \chi_1\big(\alpha(h),\delta(h),h\big)\le b/2, \quad \chi\big(\alpha(h),\delta(h),h\big)\le b/2. \tag{39}$$

The number h_* has been defined above. (We assume Condition 3 to be satisfied.)

Let us describe the algorithm for solving the problem considered in this section. For the model we consider a system of the form (11) with the initial state $w^h(0) = \Xi_0^h$. The control $u^h(\cdot)$ will be calculated in the model by the rule (12). Prior to starting the operation of the algorithm, we fix a value of $h \in (0, h_1)$ and a partition Δ_h with diameter $\delta = \delta(h) = \vartheta m_h^{-3}$. First, we determine the half-open interval where the first discontinuity point resides. To this end, for each time $\tau_i \geq d_0$ we calculate the value of

$$\nu_i = \left|u^h(\tau_{i-N-1}) - u^h(\tau_i)\right|_n.$$

Lemma 3. *Suppose that the inequality*

$$\nu_i > b/2 \tag{40}$$

is satisfied for the first time for some $i \in [1 : m_h^3 - 1]$ *such that* $\tau_i > d_0$, *i.e., for all* $j \leq i-1$, $d_0 \leq \tau_j$ *the inequalities* $\nu_j \leq b/2$ *hold. Then the first discontinuity point* a_1 *resides on the half-open interval* $\gamma_i = (\tau_{i-N-1}, \tau_{i-N}]$, *with the discontinuity size* b_1 *being such that*

$$|b_1 - \nu_i| \leq \chi_1(\alpha, \delta, h). \tag{41}$$

Assume that k $(1 \leq k)$ half-intervals, that is, the first k discontinuity points have been calculated; i.e., $a_j \in (\tau_{i_j-1}, \tau_{i_j}]$, $j \in [1 : k]$, $\tau_{i_j+1} < \tau_{i_{j+1}-1}$. The last inequality follows from the estimate $\delta(h) \leq d_0/4$. At each time $\tau_i \geq \tau_{i_k} + d_0$, we calculate the quantity ν_i.

Lemma 4. *Assume that inequality* (40) *is satisfied for the first time for some* i *such that* $\tau_i > \tau_{i_k} + d_0$; *i.e., for all* $j \leq i-1$, $\tau_{i_k-1} + d_0 \leq \tau_j$, *the inequalities* $\nu_j \leq b/2$ *hold. Then the* $(k+1)$*st point of discontinuity of the function* $x(\cdot)$ *lies on the half-open interval* γ_i, *and the size* b_{k+1} *of the discontinuity obeys the inequality*

$$|b_{k+1} - \nu_i| \leq \chi(\alpha, \delta, h).$$

If the number (r) of points of discontinuity is known, then, after calculating the quantity a_r, i.e., after finding a half-interval γ_i such that $a_r \in \gamma_i$, the algorithm halts. If the number r is unknown, then the algorithm continues operating up to the time ϑ. In this case, by virtue of the condition $\delta(1+N) < \vartheta - a_r$, the last point of discontinuity (a_r) will be determined.

Proof of Lemma 3. Let $\tau_{i_1} = \tau_{i_1(h)} \in \Delta_h$, $a_1 \in (\tau_{i_1-1}, \tau_{i_1}]$. The function $\dot{x}(\cdot)$ is continuous on the interval $[0, \tau_{i_1-1}]$. Hence $\dot{x}(\cdot) \in W^{1,\infty}([0, \tau_{i_1-1}]; \mathbb{R}^n))$. Therefore, by virtue of Lemma 2, the inequality

$$\left|u^h(\tau_{i_1-1}) - \dot{x}(\tau_{i_1-1})\right|_n \leq \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{d_0}, \frac{\delta h}{\alpha^2}\right) \tag{42}$$

holds. Moreover, taking into account the fact that $|\ddot{x}(t)|_n \leq F$ for a.a. $t \in T$, we have

$$\left|\dot{x}(a_1-) - \dot{x}(\tau_{i_1-1})\right|_n \leq F(a_1 - \tau_{i_1-1}) \quad \text{and} \quad \left|\dot{x}(a_1+) - \dot{x}(\tau_{i_1})\right|_n \leq F(\tau_{i_1} - a_1). \tag{43}$$

In turn, it follows from inequalities (43) that

$$\left|\left|\dot{x}(\tau_{i_1}) - \dot{x}(\tau_{i_1-1})\right|_n - b_1\right| \leq F\delta, \tag{44}$$

where $b_1 = |\dot{x}(a_1+) - \dot{x}(a_1-)|_n$. Using Lemma 1 and the inequality $|x(0) - w^h(0)|_n \leq h$, we establish the estimate

$$\mu(\tau_{i_1}) \leq 2h + (2+3d)(\alpha+\delta). \tag{45}$$

Since $\gamma = m_h^2\delta \leq 0.5d_0$, we have $\dot{x}(\cdot) \in W^{1,\infty}([\tau_{i_1}, \tau_{i_1+N}]; \mathbb{R}^n)$. Therefore, by Lemma 2, in view of the estimate (45), we have the inequality

$$\left|u^h(\tau_{i_1+N}) - \dot{x}(\tau_{i_1+N})\right|_n \leq \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{\gamma}, \frac{\delta\big(2h + (2+3d)(\alpha+\delta)\big)}{\alpha^2}\right). \tag{46}$$

Moreover, by virtue of the relation $N(h)\delta(h) = \gamma(h)$, we have the estimate

$$\left|\dot{x}(\tau_{i_1+N}) - \dot{x}(\tau_{i_1})\right|_n \leq F\gamma, \tag{47}$$

which has been derived using the continuity of the function $\dot{x}(\cdot)$ on the interval $[\tau_{i_1}, \tau_{i_1+N}]$, which follows from the inequality $2\gamma \le d_0$. From (46) and (47) we derive the inequality

$$\big|u^h(\tau_{i_1+N}) - \dot{x}(\tau_{i_1})\big|_n \le F\gamma + \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{\gamma}, \frac{\delta\big(2h + (2+3d)(\alpha+\delta)\big)}{\alpha^2}\right). \tag{48}$$

In turn, it follows from inequalities (42) and (44) that

$$\Big|\big|u^h(\tau_{i_1-1}) - \dot{x}(\tau_{i_1})\big|_n - b_1\Big| \le F\delta + \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{d_0}, \frac{\delta h}{\alpha^2}\right). \tag{49}$$

Combining inequalities (48) and (49), we obtain $||u^h(\tau_{i_1+N}) - u^h(\tau_{i_1-1})|_n - b_1| \le \chi_1(\alpha, \delta, h)$. Thus, taking $i = i_1 + N$, we have $|b_1 - |u^h(\tau_i) - u^h(\tau_{i-N-1})|_n| \le \chi_1(\alpha, \delta, h)$; i.e., $0.5b \le b_1 - \chi_1(\alpha, \delta, h) \le \nu_i \le b_1 + \chi_1(\alpha, \delta, h)$. Inequality (41) has thus been established.

Note that if the function $\dot{x}(\cdot)$ were continuous on the half-interval $(\tau_{i_1-1}, \tau_{i_1}]$, then, by virtue of (42), (46), (47), and the right continuity of the function $\dot{x}(\cdot)$ at the points of discontinuity, the inequality

$$\begin{aligned}\nu_{i_1+N} \equiv \big|u^h(\tau_{i_1+N}) - u^h(\tau_{i_1-1})\big|_n &\le \big|u^h(\tau_{i_1+N}) - \dot{x}(\tau_{i_1+N})\big|_n \\ &+ \big|\dot{x}(\tau_{i_1+N}) - \dot{x}(\tau_{i_1})\big|_n + \big|\dot{x}(\tau_{i-1}) - \dot{x}(\tau_i)\big|_n + \big|u^h(\tau_{i_1-1}) - \dot{x}(\tau_{i_1-1})\big|_n \\ &\le \chi_1(\alpha, \delta, h) \le 0.5b\end{aligned} \tag{50}$$

would be satisfied, because $\tau_{i_1+N} - \tau_{i_1-1} = \gamma + \delta < d_0$.

Inequalities (50) will also be satisfied if we replace $i_1 + N$ by any value $i \in [i_1^* : i_1 + N - 1]$, where $i_1^* = [d_0/\delta(h)] + 1$. Hence the inequalities $\nu_i \le 0.5b$ hold for all such i. The proof of Lemma 3 is complete.

The **proof of Lemma 4** follows the scheme of proof of Lemma 3.

Assume that the k half-intervals to which the first k points of discontinuity belong have been calculated; i.e., $a_j \in (\tau_{i_j-1}, \tau_{i_j}]$, $j \in [1:k]$, $\tau_{i_j+1} < \tau_{i_{j+1}}$. Then $a_{k+1} \in (\tau_{i_{k+1}-1}, \tau_{i_{k+1}}]$. The function $\dot{x}(\cdot)$ is continuous on the interval $[\tau_{i_k+1}, \tau_{i_{k+1}-1}]$. Moreover, $\tau_{i_{k+1}-1} - \tau_{i_k+1} \ge 0.5d_0$, because $2\delta(h) \le 0.5d_0$ and $a_{k+1} - a_k \ge d_0$.

Therefore, by Lemmas 1 and 2, the inequality

$$\big|u^h(\tau_{i_{k+1}-1}) - \dot{x}(\tau_{i_{k+1}-1})\big|_n \le \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{3\alpha}{d_0}, \frac{\delta\big(2h + (2+3d)(\alpha+\delta)\big)}{\alpha^2}\right) \tag{51}$$

holds, because $\tau_{i_{k+1}-1} > a_k + d_0/3$ and the function $\dot{x}(\cdot)$ is continuous on the interval $[a_k, a_k - \delta]$. In addition, we have the inequalities

$$\big|\dot{x}(a_{k+1}-) - \dot{x}(\tau_{i_{k+1}-1})\big|_n \le F(a_{k+1} - \tau_{i_{k+1}-1}) \quad \text{and} \quad \big|\dot{x}(a_{k+1}+) - \dot{x}(\tau_{i_{k+1}})\big|_n \le F(\tau_{i_{k+1}} - a_{k+1}).$$

Taking into account these inequalities, we obtain

$$\Big|\big|\dot{x}(\tau_{i_{k+1}}) - \dot{x}(\tau_{i_{k+1}-1})\big|_n - b_{k+1}\Big| \le F\delta, \tag{52}$$

where $b_{k+1} = |\dot{x}(a_{k+1}+) - \dot{x}(a_{k+1}-)|_n$.

Note that (see Lemma 1) $\mu(\tau_{i_k}) \le 2h + (2+3d)(\alpha+\delta)$. Therefore,

$$\mu(\tau_{i_{k+1}}) \le 2\mu(\tau_{i_k}) + (2+3d)(\alpha+\delta) \le 4h + (6+9d)(\alpha+\delta). \tag{53}$$

By Lemma 2 (we take $a = \tau_{i_k+1}$ and $q = 4h + (6+9d)(\alpha+\delta)$) and inequality (53), we have

$$\big|u^h(\tau_{i_{k+1}+N}) - \dot{x}(\tau_{i_{k+1}+N})\big|_n \le \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{\gamma}, \frac{\delta\big(4h + (6+9d)(\alpha+\delta)\big)}{\alpha^2}\right). \tag{54}$$

Moreover, the estimate

$$\left|\dot{x}(\tau_{i_{k+1}+N}) - \dot{x}(\tau_{i_{k+1}})\right|_n \le F\gamma \tag{55}$$

is satisfied, because $\tau_{i_{k+1}+N} < a_{k+2}$. From (54) and (55) we derive the inequality

$$\left|u^h(\tau_{i_{k+1}+N}) - \dot{x}(\tau_{i_{k+1}})\right|_n \le F\gamma + \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{\gamma}, \frac{\delta\big(4h + (6+9d)(\alpha+\delta)\big)}{\alpha^2}\right). \tag{56}$$

In turn, it follows from (51) and (52) that

$$\left|\left|u^h(\tau_{i_{k+1}-1}) - \dot{x}(\tau_{i_{k+1}})\right|_n - b_{k+1}\right| \le F\delta + \Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, \frac{\alpha}{d_0}, \frac{\delta\big(2h + (2+3d)(\alpha+\delta)\big)}{\alpha^2}\right). \tag{57}$$

Combining inequalities (56) and (57), we obtain

$$\left|\left|u^h(\tau_{i_{k+1}+N}) - u^h(\tau_{i_{k+1}-1})\right|_n - b_{k+1}\right| \le \chi(\alpha,\delta,h).$$

Thus, taking $i = i_{k+1} + N$, we have

$$\left|b_{k+1} - \left|u^h(\tau_i) - u^h(\tau_{i-N-1})\right|_n\right| \le \chi(\alpha,\delta,h);$$

i.e.,

$$0.5b \le b_{k+1} - \chi(\alpha,\delta,h) \le \nu_i \le b_{k+1} + \chi(\alpha,\delta,h).$$

Note that if the function $\dot{x}(\cdot)$ were continuous on the half-open interval $(\tau_{i_k+1-1}, \tau_{i_k}]$, then, by virtue of (51), (54), and (55), we would have the inequality

$$\begin{aligned}\nu_{i_{k+1}+N} &\equiv \left|u^h(\tau_{i_{k+1}+N}) - u^h(\tau_{i_{k+1}-1})\right|_n \\ &\le \left|u^h(\tau_{i_{k+1}+N}) - \dot{x}(\tau_{i_{k+1}+N})\right|_n + \left|\dot{x}(\tau_{i_{k+1}+N}) - \dot{x}(\tau_{i_{k+1}})\right|_n \\ &\quad + \left|\dot{x}(\tau_{i_{k+1}}) - \dot{x}(\tau_{i_{k+1}-1})\right|_n + \left|u^h(\tau_{i_{k+1}-1}) - \dot{x}(\tau_{i_{k+1}-1})\right|_n \le \chi(\alpha,\delta,h) \le 0.5b.\end{aligned} \tag{58}$$

Inequalities (58) will also hold if we replace $i_{k+1} + N$ by any value $i \in [i_k^* : i_{k+1} + N - 1]$, where $i_k^* = i_k + [d_0/\delta(h)]$. Consequently, for all such i the inequalities $\nu_i \le 0.5b$ will be satisfied. The proof of Lemma 4 is complete.

3. SOLUTION ALGORITHM

Suppose that for all possible actions of players 1 and 2 system (1), (2) stays in the domain

$$\left|f(t,x,y,u,v,V)\right|_n \le F, \quad |y|_n \le d. \tag{59}$$

Let us proceed to describing the algorithm for solving the problem under consideration. Fix a family of partitions of the interval T,

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h^3}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h^3} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h),$$

where $\delta(h) = \vartheta m_h^{-3}$, $m_h \in \mathbb{N}$, $m_h = [(\vartheta/h)^{1/3}]$ (the value of $h \in (0,1)$ has been defined in inequalities (5)), as well as the functions $\chi_1(\alpha,\delta,h)$, $\chi(\alpha,\delta,h)$ (see the definitions in (38) in which, instead of d_0, one should take d_0^*), and

$$\Psi\left(\frac{h}{\alpha}, \frac{\delta}{\alpha}, \alpha, e^{(1)}, e^{(2)}\right) \equiv e^{(1)}d + \tilde{c}_1\alpha(h) + \tilde{c}_2(h + \delta(h))\alpha^{-1}(h) + \tilde{c}_3 e^{(2)}.$$

Here $\tilde{c}_1 = F$, $\tilde{c}_2 = 2\sqrt{2}(4 + 4.5d) + 2\max\{1,d\}$, and $\tilde{c}_3 = 4\sqrt{2}$.

We introduce functions $\alpha = \alpha(h)$, $\gamma = \gamma(h)$, and $N = N(h)$ as follows:

$$\alpha(h) = \delta^{2/3}(h), \quad \gamma(h) = \delta(h)m_h^2 = \frac{\vartheta}{m_h} \le \frac{d_0^*}{2}, \quad N(h) = \frac{\gamma(h)}{\delta(h)} = m_h^2.$$

Consider the system

$$\dot{w}^h(t) = v^h(t), \quad t \in T \quad (w^h, v^h \in \mathbb{R}^n), \tag{60}$$

with the initial state $w^h(0) = \xi_0^h$.

Fix the value of error in the measurement of $h \in (0, h_1)$. Here $h_1 \in (0, h_*)$ is such that for $h \in (0, h_1)$ one has the inequalities $\delta(h) \le d_0^*/4$ and inequalities (39). Together with the value of h, we fix the partition $\Delta_h = \{\tau_{i,h}\}_{i=0}^{m_h^3}$ of the interval T. Consider the system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f_0\big(t, x(t), y(t)\big) + \tilde{u}(t), \quad t \in T, \tag{61}$$

with the initial state

$$x(0) = \xi_0^h, \quad y(0) = \psi_0^h \tag{62}$$

and the control $\tilde{u}(\cdot) \in \{u(\cdot) \in L_2(T; \mathbb{R}^n) : u(t) \in E_0$ for a.a. $t \in T\}$. We solve the problem of optimal numerical control, which consists in bringing the state trajectory of system (61), (62) at time ϑ into the minimum neighborhood of the set M. Let $u_0(\cdot)$ be an optimal control solving this problem, and let M^{ε_0} be the corresponding closed ε_0-neighborhood of the set M. In particular, if the problem on bringing the trajectory at time ϑ to the set M is solvable, then we take $\varepsilon_0 = 0$. For the family of stable sets $W_0(t)$, $t \in T$, we take the solution of system (61), (62) for $\tilde{u}(t) = u_0(t)$, $t \in T$. Denote this solution by $\{x_0(t), y_0(t)\}$. Thus, $W_0(t) = \{x_0(t), y_0(t)\}$. On the half-interval $\delta_0 = [0, \tau_1)$, we feed the constant control

$$u(t) = u_0$$

to system (1), where u_0 is an arbitrary element of the set P_0. Under the action of this control and an unknown disturbance $v(t) \in Q_0$, $t \in \delta_0$, $(V(t) = 0)$, a trajectory $\{x_p(t), y_p(t)\}$, $t \in [0, \tau_1]$, of system (1) is realized. On the intervals $\delta_i = [\tau_i, \tau_{i+1})$, $i > 0$, we proceed as follows. We specify vectors u_i and v_i^h at the times $t = \tau_i$ according to the rules

$$\big(\psi_i^h - y_0(\tau_i), B_0 u_i\big) = \min\Big\{\big(\psi_i^h - y_0(\tau_i), B_0 u\big) : u \in P_0\Big\}, \quad \big|\psi_i^h - y_p(\tau_i)\big|_n \le h, \tag{63}$$

$$v_i^h = -\alpha^{-1}\big[w^h(\tau_i) - \xi_i^h\big], \quad \big|\xi_i^h - x_p(\tau_i)\big|_n \le h. \tag{64}$$

After this, in (1) and (60) we assume

$$u(t) = u_i, \quad v^h(t) = v_i^h, \quad t \in [\tau_i, \tau_{i+1}). \tag{65}$$

Then we calculate the trajectories $\{x_p(\cdot), y_p(\cdot)\}$ (of system (1), (2)) and $w^h(\cdot)$ (of system (61), (62)) on the interval $[\tau_i, \tau_{i+1}]$. Now let us determine the half-interval to which the first discontinuity point belongs. To this end, at each time $\tau_i \ge d_0^*$ we calculate $\tilde{\nu}_i = |v^h(\tau_{i-N-1}) - v^h(\tau_i)|_n$. Assume that for some $i \in [1 : m_h^3 - 1]$ such that $\tau_i > d_0^*$ the inequality

$$\tilde{\nu}_i > b^*/2 \tag{66}$$

is satisfied for the first time; i.e., for all $j \le i - 1$, $d_0^* \le \tau_j$ the inequalities $\tilde{\nu}_j \le b^*/2$ hold. Denote the time corresponding to this i by τ_{i_1+N}. Then the first jump point a_1^* belongs to the half-interval $(\tau_{i_1-1}, \tau_{i_1}]$. Here the size of discontinuity b_1^* is such that

$$|b_1^* - \tilde{\nu}_{i_1+N}| \le \chi_1(\alpha, \delta, h).$$

Now let us determine the half-interval on which the second jump point resides. At the time τ_{i_1+N}, consider the system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f_{1k}\big(t, x(t), y(t)\big) + \tilde{u}(t), \quad t \in [\tau_{i_1+N}, \vartheta], \tag{67}$$

with the initial state

$$x(\tau_{i_1+N}) = x_0(\tau_{i_1-1}), \quad y(\tau_{i_1+N}+) = y_0(\tau_{i_1-1}) + \tilde{\nu}_{i_1+N} e_1 \tag{68}$$

and the control $\tilde{u}(\cdot) \in \{u(\cdot) \in L_2(T;\mathbb{R}^n) : u(t) \in E_1$ at a.a. $t \in [\tau_{i_1+N}, \vartheta]\}$. We solve the optimal control problem of bringing the state trajectory of system (67) with the initial state (68) at time ϑ into the minimum neighborhood of the set M. Let $u_1(\cdot)$ be an optimal control that solves this problem, and let M^{ε_1} be the corresponding closed ε_1-neighborhood of the set M. In particular, if the problem of bringing the trajectory to the set M is solvable, then we set $\varepsilon_1 = 0$. For the family of stable sets $W_1(t)$, $t \in [\tau_{i_1+N}, \vartheta]$, we take the solution of system (67) for $\tilde{u}(t) = u_1(t)$, $t \in T$. Just as above, we denote this solution by $\{x_0(t), y_0(t)\}$. Thus, $W_1(t) = \{x_0(t), y_0(t)\}$. On the intervals $\delta_i = [\tau_i, \tau_{i+1})$, $i \geq i_1+N$, we proceed as follows. At the times $t = \tau_i$, we set vectors u_i and v_i^h according to formulas (63), (64) in which B_0 and P_0 have been replaced by B_1 and P_1, respectively. Then we define controls $u(t)$ in system (1) and $v^h(t)$ in system (60) by formula (65). After forming the above-indicated controls, we calculate the trajectories $\{x_p(\cdot), y_p(\cdot)\}$ (of system (1)) and $w^h(\cdot)$ (of system (60)) on the interval $[\tau_i, \tau_{i+1}]$. Let inequality (66) be satisfied for the first time for some $i \in [i_1 + N + 1 : m_h^3 - 1]$; i.e., for all $j \leq i-1$, $\tau_{i_1-1}+d_0^* \leq \tau_j$ we have the inequalities $\tilde{\nu}_j \leq b^*/2$. Denote the time corresponding to this i by τ_{i_2+N}. Then the second jump point a_2^* lies on the half-interval $(\tau_{i_2-1}, \tau_{i_2}]$. Here the size b_2^* of the discontinuity satisfies the inequality

$$|b_2^* - \tilde{\nu}_{i_2+N}| \leq \chi(\alpha, \delta, h).$$

Similar actions are also performed at $t \in [\tau_{i_k+N}, \vartheta]$. Namely, at time τ_{i_k+N}, $k \geq 2$, consider the system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f_{1k}\big(t, x(t), y(t)\big) + \tilde{u}(t), \quad t \in [\tau_{i_k+N}, \vartheta], \tag{69}$$

with the initial state

$$x(\tau_{i_k+N}) = x_0(\tau_{i_k-1}), \quad y(\tau_{i_k+N}+) = y_0(\tau_{i_k-1}) + \tilde{\nu}_{i_k+N} e_k \tag{70}$$

and the control $\tilde{u}(\cdot) \in \{u(\cdot) \in L_2(T;\mathbb{R}^n) : u(t) \in E_k$ for almost all $t \in [\tau_{i_k+N}, \vartheta]\}$. We solve the optimal control problem of bringing the state trajectory of system (69) with the initial state (70) at time ϑ into the minimum neighborhood of the set M. Let $u_k(\cdot)$ be an optimal control that solves this problem, and let M^{ε_k} be the corresponding closed ε_k-neighborhood of the set M. In particular, if the problem on bringing the trajectory at time ϑ to the set M is solvable, then we set $\varepsilon_k = 0$. For the family of stable sets $W_k(t)$, $t \in [\tau_{i_k+N}, \vartheta]$, we take the solution of system (69) for $\tilde{u}(t) = u_k(t)$, $t \in [\tau_{i_k+N}, \vartheta]$. We denote this solution by $\{x_0(t), y_0(t)\}$. Thus, $W_k(t) = \{x_0(t), y_0(t)\}$. On the intervals $\delta_i = [\tau_i, \tau_{i+1})$, $i \geq i_k + N$, we proceed as follows. At the times $t = \tau_i$, we set vectors u_i and v_i^h according to formulas (63), (64) in which B_0 and P_0 are replaced by B_k and P_k, respectively. We set the controls $u(t)$ in system (1) and $v^h(t)$ in system (60) by formula (64). After forming the above-indicated controls, we calculate the trajectory $\{x_p(\cdot), y_p(\cdot)\}$ (of system (1)) and $w^h(\cdot)$ (of system (60)) on the interval $[\tau_i, \tau_{i+1}]$.

Let inequality (66) be satisfied for the first time for some $i \in [i_k + N + 1 : m_h^3 - 1]$; i.e., for all $j \leq i - 1$, $\tau_{i_k-1} + d_0^* \leq \tau_j$ one has the inequalities $\tilde{\nu}_j \leq b^*/2$. Denote the time corresponding to this i by $\tau_{i_{k+1}+N}$. Then the $(k+1)$st jump point a_{k+1}^* lies on the half-interval $(\tau_{i_{k+1}-1}, \tau_{i_{k+1}}]$. Here the size b_{k+1}^* of discontinuity is such that

$$|b_{k+1}^* - \tilde{\nu}_{i_{k+1}+N}| \leq \chi(\alpha, \delta, h).$$

Thus, in the course of algorithm operation, it is established that $a_k^* \in (\tau_{i_k-1}, \tau_{i_k}]$, $k \in [1 : r]$.

Thus, the ε-SGG is determined as a strategy of extremal aiming (see (63)) at a stable track of the form

$$W(t) = \begin{cases} W_0(t), & t \in [0, \tau_{i_1+N}) \\ W_k(t), & t \in [\tau_{i_k+N}, \tau_{i_{k+1}+N}), \quad k \in [1 : r-1] \\ W_r(t), & t \in [\tau_{i_r+N}, \vartheta]. \end{cases}$$

This fact follows from the Theorem below.

Let $\delta^{(k)} = [\tau_{i_k-1}, \tau_{i_k+N})$, $\Delta^{(r)} = \bigcup_{k=1}^{r}\delta^{(k)} \cup [0,\tau_1)$, and $\rho(h) = \vartheta(m_h^{-1} + m_h^{-3})$. Note that $\tau_{i_k+N} - \tau_{i_k-1} = \rho(h)$. Therefore, the Lebesgue measure of the set $\Delta^{(r)}$ is $r\rho(h) + \delta(h)$.

Theorem. *For each $\gamma_* > 0$ there exist numbers $h_* \in (0,1)$ and $\delta_* \in (0,1)$ such that for all $h \in (0,h_*)$ and $\delta \in (0,\delta_*)$ the inequality*

$$\varepsilon(\vartheta) \le \gamma_*$$

holds, where $\varepsilon(t) = |x_p(t) - x_0(t)|_n^2 + |y_p(t) - y_0(t)|_n^2$.

The **proof of the Theorem** follows from Lemma 9 below.

Let L_k be the Lipschitz constant of the function f_k, $L = \max_{k\in[0:r]} L_k$, and let $\omega_k(\delta)$, $k \in [0:r]$, be the modulus of continuity of the function $t \mapsto f_k(t,x,y,u,v)$ in the domain in which the solutions of system (1) and the stable track $W(t)$, $t \in T$, are confined. Denote also

$$\omega(\delta) = \max_{k\in[0:r]} \omega_k(\delta).$$

Note that all jump points are concentrated in the set $\Delta^{(r)}$.

Lemma 5. *Let $\delta_i \cap \Delta^{(r)} \ne \varnothing$. Then one has the inequality*

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + C_1\delta\varepsilon(\tau_i) + C_2\delta^2 + 4\omega^2(\delta)\delta + 2C_0h\delta,$$

where $C_0 = \sup\{|B_k u^{(1)} + C_k u^{(2)} + u^{(3)}|_n : u^{(1)} \in P_k,\ u^{(2)} \in Q_k,\ u^{(3)} \in E_k,\ k \in [0:r]\}$, $C_1 = 4(1+L)$, and $C_2 = 4L^2(F+d)^2 + 5F^2 + 4d^2$.

Proof. According to the statement in the lemma, there are no jump points on the interval $[\tau_i, \tau_{i+1}]$. Let

$$a_k^* < \tau_i, \quad \tau_{i+1} < a_{k+1}^*.$$

Then the trajectory $\{x_p(\cdot), y_p(\cdot)\}$ on the interval $[\tau_i, \tau_{i+1}]$ is a solution of the system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f_{1k}\big(t, x(t), y(t)\big) + B_k u_i - C_k v(t),$$

and the trajectory $\{x_0(\cdot), y_0(\cdot)\}$ on the same interval is a solution of the system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = f_{1k}\big(t, x(t), y(t)\big) + u_k(t),$$

where $u_k(\cdot)$ is the corresponding optimal control, $u_k(t) \in E_k$ for a.a. $t \in [\tau_i, \tau_{i+1}]$. In this case, we have the estimate

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + I_{1i} + I_{2i} + 4(d^2 + F^2)\delta^2, \tag{71}$$

where

$$I_{1i} = 2\left(x_p(\tau_i) - x_0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} \{y_p(s) - y_0(s)\}\,ds\right), \quad I_{2i} = 2\left(y_p(\tau_i) - y_0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} q_i(s)\,ds\right),$$

$$q_i(s) = f_*\big(s, x_p(s), y_p(s), u_i, v(s)\big) - f_*\big(s, x_0(s), y_0(s), u_k(s)\big),$$

$$f_*\big(s, x_p(s), y_p(s), u_i, v(s)\big) = f_{1k}\big(s, x_p(s), y_p(s)\big) + B_k u_i - C_k v(s),$$

$$f_*\big(s, x_0(s), y)(s), u_k(s)\big) = f_{1k}\big(s, x_0(s), y_0(s)\big) + u_k(s).$$

It can readily be seen that the inequality

$$\left|\int\limits_{\tau_i}^{\tau_{i+1}} \{y_p(s) - y_0(s)\}\,ds\right|_n = \left|\int\limits_{\tau_i}^{\tau_{i+1}} \left\{y_p(\tau_i) - y_0(\tau_i) + \left(\int\limits_{\tau_i}^{s} \{\dot{y}_p(\tau) - \dot{y}_0(\tau)\}\,d\tau\right)\right\} ds\right|_n$$

$$\le \delta\big|y_p(\tau_i) - y_0(\tau_i)\big|_n + 2F\delta^2$$

holds. Using this inequality, we obtain

$$I_{1i} \le 2\delta\big|x_p(\tau_i) - x_0(\tau_i)\big|_n\big|y_p(\tau_i) - y_0(\tau_i)\big|_n + 2F\delta^2\big|x_p(\tau_i) - x_0(\tau_i)\big|_n \le 2\delta\varepsilon(\tau_i) + F^2\delta^3. \tag{72}$$

Further, by virtue of the functions f_{1k} being Lipschitz in x, y and continuous in t, for $s \in \delta_i = [\tau_i, \tau_{i+1})$ we have the inequality

$$\big|q_i(s)\big|_n \le \Big|f_*\big(\tau_i, x_p(\tau_i), y_p(\tau_i), u_i, v(s)\big) - f_*\big(\tau_i, x_0(\tau_i), y_0(\tau_i), u_k(s)\big)\Big|_n + I_{3i}(s) + 2\omega(\delta). \tag{73}$$

Here $I_{3i}(s) = L\{|x_p(s) - x_p(\tau_i)|_n + |y_p(s) - y_p(\tau_i)|_n + |x_0(s) - x_0(\tau_i)|_n + |y_0(s) - y_0(\tau_i)|_n\} \le 2L\delta(f + d)$. Using the Lipschitz property of the functions f_{1k} one more time, from (73) we derive the inequality ($s \in \delta_i$)

$$\big|q_i(s)\big|_n \le \Big|f_*\big(\tau_i, x_0(\tau_i), y_0(\tau_i), u_i, v(s)\big) - f_*\big(\tau_i, x_0(\tau_i), y_0(\tau_i), u_k(s)\big)\Big|_n + q_{1i} + 2\omega(\delta) + 2L\delta(F + d),$$

where $q_{1i} = L\{|x_p(\tau_i) - x_0(\tau_i)|_n + |y_p(\tau_i) - y_0(\tau_i)|_n\}$. Obviously, $q_{1i} \le L\varepsilon^{1/2}(\tau_i)$. Therefore, $I_{2i} \le I_{4i} + I_{5i}$, where

$$I_{4i} = 2\delta\big|y_p(\tau_i) - y_0(\tau_i)\big|_n\big\{L\varepsilon^{1/2}(\tau_i) + 2\omega(\delta) + 2L\delta(F + d)\big\},$$

$$I_{5i} = 2\left(y_p(\tau_i) - y_0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} \Big\{f_*\big(\tau_i, x_0(\tau_i), y_0(\tau_i), u_i, v(s)\big) - f_*\big(\tau_i, x_0(\tau_i), y_0(\tau_i), u_k(s)\big)\Big\}\, ds\right).$$

It can readily be seen that

$$I_{5i} = 2\left(y_p(\tau_i) - y_0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} \big\{B_k u_i - C_k v(s) - u_k(s)\big\}\, ds\right).$$

Consequently,

$$I_{5i} \le 2\left(\psi_i^h - y_0(\tau_i), \int\limits_{\tau_i}^{\tau_{i+1}} \big\{B_k u_i - C_k v(s) - u_k(s)\big\}\, ds\right) + 2h\delta C_0.$$

Here $\psi_i^h \in \mathbb{R}^n$, $|\psi_i^h - y_p(\tau_i)|_n \le h$. In this case, taking into account Condition 1 as well as the rule of selection of vectors u_i (see (63), with B_0 and P_0 replaced in (63) by B_k and P_k, respectively), we conclude that the estimate $I_{5i} \le 2h\delta C_0$ holds. Hence

$$I_{2i} \le I_{4i} + I_{5i} \le 2(1 + L)\delta\varepsilon(\tau_i) + 4\omega^2(\delta)\delta + 4L^2(F + d)^2\delta^2 + 2h\delta C_0.$$

Based on this, by virtue of (71) and (72), we arrive at the assertion of the lemma. The proof of the lemma is complete.

Let $\tau_{i^{(k)}} = \max\{\tau_i : \tau_i < a^*_{k+1}\}$.

Lemma 6. *For all* $k \in [0 : r - 1]$ *one has the inequalities*

$$\varepsilon(a^*_{k+1}-) \le \nu_{k+1} = \Big[\varepsilon(\tau_{i_k+N}+) + (a^*_{k+1} - a^*_k)\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big] \exp C_1(a^*_{k+1} - a^*_k),$$

where $\varepsilon(a^*_{k+1}-) = \lim\limits_{t \to a^*_{k+1}-} \varepsilon(t)$.

Proof. By virtue of the Lemma in the paper [12] and Lemma 5 in the present paper, for $\tau_i \in [\tau_{i_k+N}, a^*_{k+1}]$ we have the estimates

$$\varepsilon(\tau_i) \le \Big[\varepsilon(\tau_{i_k+N}+) + (\tau_i - \tau_{i_k+N})\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big] \exp C_1(\tau_i - \tau_{i_k+N}). \tag{74}$$

Therefore,

$$\varepsilon(\tau_{i^{(k)}}) \le \Psi_k = \Big[\varepsilon(\tau_{i_k+N}+) + (\tau_{i^{(k)}} - \tau_{i_k})\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big] \exp C_1(\tau_{i^{(k)}} - \tau_{i_k}).$$

Denote $\Delta_k = a^*_{k+1} - \tau_{i^{(k)}}$. By analogy with Lemma 5, taking into account the last inequality, we obtain

$$\varepsilon(a^*_{k+1}-) \le (1 + C_1\Delta_k)\varepsilon(\tau_{i^{(k)}}) + \tilde{\rho}_k \le (1 + C_1\Delta_k)\Psi_k + \tilde{\rho}_k, \tag{75}$$

where $\tilde{\rho}_k = C_2\Delta_k^2 + 4\omega^2(\Delta_k)\Delta_k + 2hC_0\Delta_k$. It can readily be seen that the inequalities

$$(1 + C_1\Delta_k)\Psi_k \le \Big[\varepsilon(\tau_{i_k+N}+) + (\tau_{i^{(k)}} - \tau_{i_k+N})\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big] \exp C_1(a^*_{k+1} - a^*_k), \tag{76}$$

$$\tilde{\rho}_k \le (a^*_{k+1} - \tau_{i^{(k)}})\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big) \exp C_1(a^*_{k+1} - a^*_k) \tag{77}$$

hold. The assertion of the lemma follows from inequalities (75)–(77) and the inequality $a^*_k < \tau_{i_k+N}$. The proof of the lemma is complete.

We introduce the notation $\rho_1(h) = \rho(h) + \vartheta m_h^{-3}$.

Lemma 7. *One has the inequalities*

$$\varepsilon(\tau_{i_1+N}+) = \big|y_0(\tau_{i_1+N}+) - y_p(\tau_{i_1+N}+)\big|_n^2 + \big|x_0(\tau_{i_1+N}) - x_p(\tau_{i_1+N})\big|_n^2 \le 4\varepsilon(a^*_1-) + \phi_1(h,\delta), \tag{78}$$

$$\begin{aligned}\varepsilon(\tau_{i_k+N}+) &= \big|y_0(\tau_{i_k+N}+) - y_p(\tau_{i_k+N}+)\big|_n^2 + \big|x_0(\tau_{i_k+N}) - x_p(\tau_{i_k+N})\big|_n^2 \\ &\le 4\varepsilon(a^*_k-) + \phi(h,\delta) \quad \text{for} \quad k \in [2:r],\end{aligned} \tag{79}$$

where $\phi_1(h,\delta) = 4(\chi_1 + F\rho_1(h))^2 + 8d^2\rho^2(h)$, $\phi(h,\delta) = 4(\chi + F\rho_1(h))^2 + 8d^2\rho^2(h)$.

Proof. Let us verify inequality (78). By definition,

$$\varepsilon(a^*_k-) = \big|y_0(a^*_k-) - y_p(a^*_k-)\big|_n^2 + \big|x_0(a^*_k) - x_p(a^*_k)\big|_n^2.$$

At time τ_{i_1+N} we establish that $a^*_1 \in (\tau_{i_1-1}, \tau_{i_1+N}]$ and

$$|b^*_1 - \tilde{\nu}_{i_1+N}| \le \chi_1 = \chi_1(\alpha,\delta,h). \tag{80}$$

We set (see (68))

$$y_0(\tau_{i_1+N}+) = y_0(\tau_{i_1-1}) + \tilde{\nu}_{i_1+N}e_1. \tag{81}$$

By the statement of the problem, we have

$$y_p(a^*_1+) = y_p(a^*_1-) + b^*_1 e_1. \tag{82}$$

It can readily be seen that the inequality

$$\big|y_0(a^*_1-) - y_p(a^*_1-)\big|_n \le \varepsilon^{1/2}(a^*_1-) \tag{83}$$

holds. There are no jumps for $t \in (a^*_1, \tau_{i_1+N}]$. Moreover, $|F_k|_n \le F$, $k \in [0:r]$. In this case,

$$\big|y_p(\tau_{i_1+N}) - y_p(a^*_1+)\big|_n \le F\rho(h). \tag{84}$$

Since $a^*_1 \in (\tau_{i_1-1}, \tau_{i_1}]$, $\tau_{i_1} - \tau_{i_1-1} = \delta(h) = \vartheta m_h^{-3}$, and $|F_k|_n \le F$, one has the estimate

$$\big|y_0(\tau_{i_1-1}) - y_0(a^*_1-)\big|_n \le F\delta = F\vartheta m_h^{-3}. \tag{85}$$

Therefore, in view of relations (80)–(85), one has the chain of inequalities

$$\begin{aligned}\big|y_0(\tau_{i_1+N}+) - y_p(\tau_{i_1+N}+)\big|_n &= \big|y_0(\tau_{i_1-1}) + \tilde{\nu}_{i_1+N}e_1 - y_p(\tau_{i_1+N})\big|_n \\ &= \big|y_0(\tau_{i_1-1}) + \tilde{\nu}_{i_1+N}e_1 - y_p(\tau_{i_1+N}) + y_p(a_1^*+) - y_p(a_1^*+)\big|_n \\ &\le \big|y_0(\tau_{i_1-1}) + \tilde{\nu}_{i_1+N}e_1 - y_p(a_1^*+)|_n + |y_p(a_1^*+) - y_p(\tau_{i_1+N})\big|_n \\ &\le \big|y_0(\tau_{i_1-1}) - y_p(a_1^*-)\big|_n + \chi_1 + F\rho(h) \\ &\le \big|y_0(\tau_{i_1-1}) - y_0(a_1^*-)\big|_n + \big|y_0(a_1^*-) - y_p(a_1^*-)\big|_n + \chi_1 + F\rho(h) \\ &\le \big|y_0(a_1^*-) - y_p(a_1^*-)\big|_n + \chi_1 + F\rho_1(h) \\ &\le \varepsilon^{1/2}(a_1^*-) + \chi_1(\alpha,\delta,h) + F\rho_1(h).\end{aligned} \tag{86}$$

Since $a_1^* \in (\tau_{i_1-1}, \tau_{i_1}]$, $N\delta = \vartheta m_h^{-1}$, the function $x_p(\cdot)$ is continuous on T, and the function $x_0(\cdot)$ is continuous on $[a_1^*, \tau_{i_1+N}]$, we have the inequalities

$$\big|x_0(\tau_{i_1+N}) - x_0(a_1^*)\big|_n \le d(\tau_{i_1+N} - a_1^*) \le d\rho(h) \quad \text{and} \quad \big|x_p(\tau_{i_1+N}) - x_p(a_1^*)\big|_n \le d\rho(h).$$

Consequently,

$$\big|x_p(\tau_{i_1+N}) - x_0(\tau_{i_1+N})\big|_n \le \big|x_0(a_1^*) - x_p(a_1^*)\big|_n + 2d\rho(h) \le \varepsilon^{1/2}(a_1^*-) + 2d\rho(h). \tag{87}$$

Then (86) and (87) imply inequality (78).

Inequality (79) can be established in a similar way. The proof of the lemma is complete.

Let

$$\begin{aligned}\Psi_0(h,\delta) &= (1+4\delta^2)(h+2F\delta)^2 + h^2 + 4h\delta(h+2F\delta), \\ \Psi_1(h,\delta) &= \Big[\Psi_0(h,\delta) + a_1^*\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big]\exp C_1 a_1^*.\end{aligned}$$

Lemma 8. *For* $k \in [1 : r-1]$, *the inequalities*

$$\varepsilon(a_{k+1}^*-) \le \Psi_{k+1}(h,\delta) = \Big[4\Psi_k(h,\delta) + \phi_*(h,\delta) + \vartheta\Big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\Big)\Big]\exp C_1\vartheta$$

hold, where $\phi_*(h,\delta) = \phi_1(h,\delta)$ *if* $k=1$ *and* $\phi_*(h,\delta) = \phi(h,\delta)$ *if* $k \in [2 : r-1]$.

Proof. By Lemmas 6 and Lemma 7, one has the estimate

$$\varepsilon(a_{k+1}^*-) \le \Big[4\varepsilon(a_k^*-) + \phi_*(h,\delta) + (a_{k+1}^* - a_k^*)\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big]\exp C_1\vartheta.$$

In this case,

$$\varepsilon(a_{k+1}^*-) \le \Big[4\varepsilon(a_k^*-) + \phi_*(h,\delta) + \vartheta\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big]\exp C_1\vartheta. \tag{88}$$

By analogy with Lemma 6, we can establish the inequality

$$\varepsilon(a_1^*-) \le \Big[\varepsilon(\tau_1) + a_1^*\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big]\exp C_1 a_1^* \le \Psi_1(h,\delta). \tag{89}$$

Note that by virtue of (5), (62), and (59) one has the inequalities

$$\big|y_p(\tau_1) - y_0(\tau_1)\big|_n \le h + 2F\delta, \quad \big|x_p(\tau_1) - x_0(\tau_1)\big|_n \le h + 2\delta(h+2F\delta).$$

Therefore,

$$\varepsilon(\tau_1) \le (h+2F\delta)^2 + \big[h + 2\delta(h+2F\delta)\big]^2 \le \Psi_0(h,\delta).$$

The assertion of the lemma follows from (88), (89), and the last inequality. The proof of the lemma is complete.

Lemma 9. *For each $\gamma_0 > 0$, there exists an $h_* \in (0,1)$ and a $\delta_* \in (0,1)$ such that for all $\delta \le \delta_*$, $h \le h_*$, and $\tau_i \notin \Delta^{(r)}$ one has the inequalities*

$$\varepsilon(\tau_i) \le \Psi_r(h,\delta) \le \gamma_0.$$

Proof. The functions $\Psi_k(h,\delta)$ possess the following property:

$$\Psi_k(h,\delta) < \Psi_{k+1}(h,\delta) \quad \text{for} \quad k \in [0:r-1].$$

For each $\gamma_0 > 0$, there exists a $\delta_* = \delta_*(\gamma_0) > 0$ and an $h_* = h_*(\gamma_0) > 0$ such that the inequality $\Psi_r(h,\delta) \le \gamma_0$ holds for all $h \in (0,h_*)$ and $\delta \in (0,\delta_*)$. Therefore,the inequalities $\varepsilon(a_k^*-) \le \gamma_0$ hold for $\delta \in (0,\delta_*)$ and $h \in (0,h_*)$ for all $k \in [1:r]$. As was noted above, inequality (74) is satisfied. From this inequality and Lemma 7, for $\tau_i \in [\tau_{i_k+N}, a_{k+1}^*]$ we obtain

$$\begin{aligned}\varepsilon(\tau_i) &\le \Big[4\varepsilon(a_k^*-) + \phi_*(h,\delta) + (\tau_i - \tau_{i_k+N})\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big] \exp C_1(\tau_i - \tau_{i_k+N}) \\ &\le \Big[4\varepsilon(a_k^*-) + \phi_*(h,\delta) + \vartheta\big(C_2\delta + 2hC_0 + 4\omega^2(\delta)\big)\Big] \exp C_1\vartheta.\end{aligned}$$

Based on this and taking into account Lemma 8, we derive the estimate $\varepsilon(\tau_i) \le \Psi_{k+1}(h,\delta) \le \gamma_0$. The proof of the lemma is complete.

Remark 2. By virtue of Lemma 9 and the inequality $\vartheta - a_r^* > \rho(h)$, for $\delta \le \delta_*$ and $h \le h_*$ we have the inequality $\varepsilon(\vartheta) \le \gamma_0$. This implies the assertion of the Theorem.

Remark 3. Assume that at the initial time we have constructed a family of u-stable positional absorption sets ensuring the solution of the guaranteed guidance problem for system (3) with right-hand side $f = f_0$ from the initial state $\{x_0, y_0\}$ to the least neighborhood of the set M. Let it be the ε-neighborhood. Denote the constructed family by $\tilde{W}^\varepsilon(t)$, $t \in T$. An analysis of the above-described algorithm allows the conclusion that if the inclusions

$$\big\{x_p(a_k^*), y_p(a_k^*+)\big\} \in \tilde{W}^\varepsilon(a_k^*), \quad k \in [1:r]$$

are satisfied at the jump time a_k^*, then the SGG ensures bringing the state trajectory of system (3) into an arbitrarily small neighborhood of the set M^ε for sufficiently small h and δ.

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