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ORDINARY DIFFERENTIAL EQUATIONS

Lyapunov Vector Functions, Rotation of Vector Fields, Guiding Functions, and the Existence of Poisson Bounded Solutions

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Abstract—We use the Lyapunov vector function method and the guiding function method to obtain sufficient conditions for the existence of Poisson bounded and partially Poisson bounded solutions of systems of differential equations.

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The application of the Lyapunov function method [1] to the study of the boundedness of solutions of differential systems is given in [2], and its application to the study of the boundedness of solutions with respect to part of variables is given in the monograph [3, pp. 223–228]. A generalization of the Lyapunov function method—the Lyapunov vector function method—is presented in the monographs [4, 5]. The book [5] shows how the Lyapunov vector function method can be applied to the derivation of conditions ensuring the boundedness of all solutions of an arbitrary nonlinear system. Independently of the methods in the paper [2], the monograph [6] developed the guiding function method based on the technique of rotation of vector fields. The method was used in [6] to obtain sufficient conditions for the existence of solutions of an arbitrary nonlinear system bounded on the entire real line.

On the other hand, the present author (see, e.g., [7]) has recently begun a study of a new form of solution boundedness—the Poisson boundedness. The concept of Poisson boundedness of a solution on the half-line is that there exists a ball in the state space and a countable system of disjoint intervals on the time half-line with the sequence of their right endpoints tending to $+\infty$ such that the solution is contained in the ball at all times belonging to these intervals. Obviously, a bounded solution is Poisson bounded; the opposite, as is easily seen, is not true. In papers by the present author, conditions have been studied under which all solutions of a differential system are Poisson bounded.

To study conditions for the existence of Poisson bounded solutions, the present paper develops a method that is a synthesis of the Lyapunov vector function method and the guiding function method. Using this method, we obtain sufficient conditions for the existence of Poisson bounded solutions (Theorem 1) as well as partially (in part of the variables) Poisson bounded solutions (Theorem 2). Let us now proceed to rigorous definitions and statements.

Consider the system of differential equations

$$\frac{dx}{dt} = F(t,x), \quad F(t,x) = \big(F_1(t,x),\dots,F_n(t,x)\big)^{\mathrm{T}}, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0,+\infty), \quad n \ge 2, \tag{1}$$

where $F\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function satisfying the local Lipschitz condition in the variable $x \in \mathbb{R}^n$. In addition, we assume that all solutions of system (1) are extendible to the time half-line $\mathbb{R}_+$.

In what follows, by $\|\cdot\|$ we denote the Euclidean norm on $\mathbb{R}^n$. The solution $x = x(t)$ of system (1) with the initial condition $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ is denoted by $x(t, t_0, x_0)$. Any nonnegative increasing number sequence $(\tau_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to+\infty} \tau_i = +\infty$ will be called a *$\mathcal{P}$-sequence*.

Recall [2] that a solution $x = x(t, t_0, x_0)$ of system (1) is said to be *bounded* if there exists a number $\beta > 0$ such that $\|x(t, t_0, x_0)\| \le \beta$ for all $t \in \mathbb{R}_+$.

Definition 1. A solution $x = x(t, t_0, x_0)$ of system (1) is said to be *Poisson bounded* if there exists a $\mathcal{P}$-sequence $(\tau_i)_{i\in\mathbb{N}}$ and a number $\beta > 0$ such that $\|x(\tau_i, t_0, x_0)\| \le \beta$ for all $i \in \mathbb{N}$.

Obviously, if a solution of system (1) is bounded, then it is also Poisson bounded, because in this case one can take any $\mathcal{P}$-sequence for the required $\mathcal{P}$-sequence. It is also obvious that increasing the number β if necessary, we can assume that the inequality $\|x(t, t_0, x_0)\| \le \beta$ for a Poisson bounded solution is satisfied for all t belonging to some sequence of intervals whose right endpoints tend to $+\infty$, as was underlined above. Therefore, the above definition is equivalent to the definition of a Poisson bounded solution given in the paper [8].

Following [4, pp. 46–48], let us recall some facts needed in what follows about Lyapunov vector functions. Given a continuously differentiable vector function

$$V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^k, \quad V(t,x) = \big(V_1(t,x), \dots, V_k(t,x)\big)^{\mathrm{T}}, \quad k \ge 1,$$

the *derivative according to system* (1) of this vector function is defined by the relation

$$\dot V(t,x) = \big(\dot V_1(t,x), \dots, \dot V_k(t,x)\big)^{\mathrm{T}},$$

where $\dot V_i(t,x)$ is the derivative according to system (1) of the function $V_i(t,x)$, $1 \le i \le k$. In what follows, the notation $\xi \le \eta$ for vectors $\xi = (\xi_1, \dots, \xi_k)^{\mathrm{T}}$ and $\eta = (\eta_1, \dots, \eta_k)^{\mathrm{T}} \in \mathbb{R}^k$ means that $\xi_i \le \eta_i$ for each $1 \le i \le k$. We say [3, p. 235] that a continuous vector function

$$f\colon \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}^k, \quad f(t,\xi) = \big(f_1(t,\xi), \dots, f_k(t,\xi)\big)^{\mathrm{T}}, \quad k \ge 1,$$

satisfies the *Ważewski condition* if for each $1 \le s \le k$ the function f_s is nondecreasing with respect to the variables $\xi_1, \dots, \xi_{s-1}, \xi_{s+1}, \dots, \xi_k$, i.e., if it follows from the relations $\xi_i \le \eta_i$, $1 \le i \le k$, $i \ne s$, and $\xi_s = \eta_s$ that $f_s(t,\xi) \le f_s(t,\eta)$. If f satisfies the Ważewski condition, then we write $f \in W$. Note that the condition $f \in W$ degenerates for $k = 1$; therefore, we adopt the convention that $f \in W$ for each continuous function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

A continuously differentiable vector function $V\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^k$ satisfying the condition $V(t,x) \ge 0$ for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, where 0 is the zero vector in $\mathbb{R}^k$, and a system

$$\frac{d\xi}{dt} = f(t,\xi), \quad f \in W, \tag{2}$$

are called a *Lyapunov vector function* and a *comparison system*, respectively, for system (1) if the following condition is satisfied: $\dot V(t,x) \le f(t, V(t,x))$. In what follows, we always assume that the right-hand side of system (2) satisfies the local Lipschitz condition with respect to ξ and, in addition, the solutions of this system are extendible to the entire half-line $\mathbb{R}_+$. Since we have the uniqueness of solution of the Cauchy problem for system (2), it follows from the Ważewski theorem (see, e.g., [3, p. 236]) that for each point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the solution $x(t, t_0, x_0)$ of system (1), the Lyapunov vector function $V\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^k$, and the solution $\xi(t, t_0, V(t_0, x_0))$ of the comparison system (2) for system (1) are related for all $t \ge t_0$ by the inequality

$$V\big(t, x(t, t_0, x_0)\big) \le \xi(t, t_0, V(t_0, x_0)). \tag{3}$$

Now let us recall necessary notions and constructions related to the rotation of vector fields and to operators of shift along trajectories [6] (see also [9]). Let Ω be an arbitrary compact subset of $\mathbb{R}^n$ with boundary $\partial\Omega$. Following [6], we define a continuous vector field, or, for brevity, a vector field Q, on Ω as an arbitrary continuous mapping $Q\colon \Omega \to \mathbb{R}^n$. For a vector field Q on Ω, consider its restriction to $\partial\Omega$, i.e., the vector field $Q|_{\partial\Omega}\colon \partial\Omega \subset \Omega \to \mathbb{R}^n$. A vector field Q on Ω is said to be *nondegenerate* on $\partial\Omega$ if $Q(x) \ne 0 \in \mathbb{R}^n$ for all $x \in \partial\Omega$. It can readily be seen that any vector field Q nondegenerate on $\partial\Omega$ defines a continuous mapping

$$T\colon \partial\Omega \to S^{n-1} = \big\{a \in \mathbb{R}^n : \|a\| = 1\big\}, \quad T(x) = Q(x)/\big\|Q(x)\big\|, \quad x \in \partial\Omega.$$

The *rotation* $\gamma(Q, \partial\Omega)$ of a vector field Q nondegenerate on $\partial\Omega$ is the degree $\deg(T) \in \mathbb{Z}$ of the mapping $T : \partial\Omega \to S^{n-1}$. In the case where the compact subset Ω in $\mathbb{R}^n$ is an n-dimensional smooth orientable manifold with boundary $\partial\Omega$ (see, e.g., [10]), the integer $\deg(T)$ is easy to define, for example, by using the functor $H_{n-1}(-;\mathbb{Z})$ of singular $(n-1)$-dimensional homology of topological spaces with integer coefficients [11]. Indeed, the continuous mapping $T:\partial\Omega\to S^{n-1}$ induces a homomorphism $H_{n-1}(T;\mathbb{Z})\colon H_{n-1}(\partial\Omega;\mathbb{Z})\to H_{n-1}(S^{n-1};\mathbb{Z})$ of the singular homology groups. It is well known (see, e.g., [11]) that the groups $H_{n-1}(\partial\Omega;\mathbb{Z})$ and $H_{n-1}(S^{n-1};\mathbb{Z})$ are isomorphic to the group $\mathbb{Z}$ and the generators of these groups are the fundamental classes $[\partial\Omega]$ and $[S^{n-1}]$ of the manifolds $\partial\Omega$ and S^{n-1}, respectively. Using the group homomorphism $H_{n-1}(T;\mathbb{Z})$ and the fundamental classes $[\partial\Omega]$ and $[S^{n-1}]$, the degree of the mapping $\deg(T)\in\mathbb{Z}$ is defined by the rule

$$H_{n-1}(T;\mathbb{Z})\big([\partial\Omega]\big)=\deg(T)[S^{n-1}].$$

In the general case, where Ω is an arbitrary compact subset in $\mathbb{R}^n$, the definition of the degree $\deg(T)\in\mathbb{Z}$ of the mapping $T:\partial\Omega\to S^{n-1}$ is described in detail in the monograph [9, p. 9–29].

We use the following terminology. The subset

$$\operatorname{Tr}(x_0)=\big\{x\in\mathbb{R}^n : x=x(t,0,x_0),\ \ t\geq 0\big\}\subset\mathbb{R}^n,$$

where $x(t,0,x_0)$ is the solution of system (1) and x_0 is an arbitrary point in $\mathbb{R}^n$, will be called the *trajectory* of system (1) *issuing from the point* x_0. For each $\tau>0$, consider the continuous mapping

$$U(\tau)\colon\Omega\to\mathbb{R}^n,\quad U(\tau)(x_0)=x(\tau,0,x_0),$$

where $x(t,0,x_0)$ is the solution of system (1) and x_0 is an arbitrary point in Ω. The mapping $U(\tau)$ is called [6, pp. 11–12] the *operator of shift along the trajectories* of system (1) for time $0\leq t\leq\tau$. A τ*-nonrecurrent* point of a trajectory of system (1) is a point $x_0\in\mathbb{R}^n$ such that the solution $x(t,0,x_0)$ of system (1) satisfies the condition $x(t,0,x_0)\neq x_0$ for all $0<t\leq\tau$ [6, p. 101].

Consider the vector field

$$S_0\colon\Omega\to\mathbb{R}^n,\quad S_0(x)=-F(0,x),$$

where $F(t,x)$ is the right-hand side of system (1). The rotation $\gamma(S_0,\partial\Omega)$ of this vector field is closely related to the problem on the existence of fixed points of the operator $U(\tau)$ of shift along the trajectories of system (1). Indeed, it was shown in [6, p. 102–104] that if a vector field $S_0:\Omega\to\mathbb{R}^n$ nondegenerate on $\partial\Omega$ has rotation $\gamma(S_0,\partial\Omega)\neq 0$ and all points of the boundary $\partial\Omega$ are τ-nonrecurrent points of the trajectories of system (1), then the operator $U(\tau)$ of shift along the trajectories of system (1) has at least one fixed point inside Ω, i.e., a point $x\in\Omega\setminus\partial\Omega$ such that $U(\tau)(x)=x$. We will use the following terminology. The subsets

$$\operatorname{Tr}^+(x_0,t_0)=\big\{x\in\mathbb{R}^n : x=x(t,t_0,x_0),\ \ t>t_0\big\}\subset\mathbb{R}^n,$$
$$\operatorname{Tr}^-(x_0,t_0)=\big\{x\in\mathbb{R}^n : x=x(t,t_0,x_0),\ \ 0\leq t\leq t_0\big\}\subset\mathbb{R}^n,$$

where $x(t,t_0,x_0)$ is the solution of system (1) and (t_0,x_0) is any point in $\mathbb{R}_+\times\mathbb{R}^n$, will be called the *right part* and the *left part*, respectively, of the trajectory $\operatorname{Tr}(x(0,t_0,x_0))$ of system (1).

We state and prove the following sufficient condition for the existence of Poisson bounded solutions of system (1) in terms of Lyapunov vector functions and rotations of vector fields.

Proposition 1. *Assume that for system* (1) *there exists a* $\mathcal{P}$*-sequence* $(\tau_i)_{i\in\mathbb{N}}$, *a Lyapunov vector function* $V\colon\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^k$ *with a comparison system* (2), *and a nondecreasing function* $b\colon\mathbb{R}_+\to\mathbb{R}_+$ *with the property* $b(r)\to+\infty$ *as* $r\to+\infty$ *such that the inequality*

$$b(\|x\|)\leq\sum_{q=1}^{k}V_q(\tau_i,x)\tag{4}$$

holds for any $x\in\mathbb{R}^n$ *and* $i\in\mathbb{N}$. *Moreover, let the following conditions be satisfied for the comparison system* (2):

1. *There exists a subset* $\Omega\subset\operatorname{Im}(V\colon\{0\}\times\mathbb{R}^n\to\mathbb{R}^k)$ *compact in* $\mathbb{R}^k$ *such that the vector field* $S_0:\Omega\to\mathbb{R}^k$, $S_0(\xi)=-f(0,\xi)$, *where* $f(t,\xi)$ *is the right-hand side of the comparison system* (2), *is nondegenerate on* $\partial\Omega$ *and* $\gamma(S_0,\partial\Omega)\neq 0$.

2. *For each* $\xi_0 \in \partial\Omega$, *the right part* $\mathrm{Tr}^+(\xi_0, t_0)$ *of the trajectory* $\mathrm{Tr}\,(\xi(0, t_0, \xi_0))$ *of the comparison system* (2) *does not have any points in* Ω *in common with the left part* $\mathrm{Tr}^-(\xi_0, t_0)$ *of the same trajectory.*

Then system (1) *has at least one Poisson bounded solution.*

Proof. For each $m \in \mathbb{N}$, consider the operator $U(m)\colon \Omega \to \mathbb{R}^k$ of shift along the trajectories of the comparison system (2) for system (1) for time $0 \le t \le m$. Since, by assumption, the intersection $\mathrm{Tr}^+(t_0, \xi_0) \cap \mathrm{Tr}^-(t_0, \xi_0) \cap \Omega$ is empty for each $(t_0, \xi_0) \in \mathbb{R}_+ \times \partial\Omega$, we conclude that all points of the boundary $\partial\Omega$ are m-nonrecurrent points of the trajectories of system (2) for each $m \in \mathbb{N}$. As was indicated above, it follows that for each $m \in \mathbb{N}$ the operator $U(m)$ has a fixed point $\vartheta_m \in \Omega \setminus \partial\Omega$.

Consider the family of solutions $\{\xi(t, 0, \vartheta_m)\}_{m\in\mathbb{N}}$ of system (2). It follows from the assumptions of the theorem to be proved that $\xi(t, 0, \vartheta_m) \in \Omega \setminus \partial\Omega$ for each $0 \le t \le m$. Indeed, if the opposite were true, then for some point $\xi_0 = x(t_0, 0, \vartheta_m) \in \mathrm{Tr}\,(\vartheta_m)$, where $0 < t_0 < m$ and $\xi_0 \in \partial\Omega$, we would have

$$\mathrm{Tr}^+(t_0, \xi_0) \cap \mathrm{Tr}^-(t_0, \xi_0) \cap \Omega = \{\vartheta_m\} \ne \varnothing,$$

which contradicts the assumptions of the theorem.

Consider the sequence of points $(\vartheta_m)_{m\in\mathbb{N}}$ in $\Omega \setminus \partial\Omega$. Using the fact that the set Ω is compact, from the sequence $(\vartheta_m)_{m\in\mathbb{N}}$ we select a subsequence $(\vartheta_{m_i})_{i\in\mathbb{N}}$ converging to some point $\mu \in \Omega$. Let us show that the solution $\xi(t, 0, \mu)$ of system (2) satisfies the condition $\xi(t, 0, \mu) \in \Omega$ for all $t \ge 0$. Assume the contrary: there exists a number $\eta \ge 0$ such that $\xi(\eta, 0, \mu) \notin \Omega$. Since system (2) satisfies the assumptions of the theorem on the continuous dependence on the initial conditions (see, e.g., [12]), it follows that $\xi(\eta, 0, \vartheta_{m_i}) \notin \Omega$, where $\eta \le m_i$, for sufficiently large i. We have arrived at a contradiction with the inclusion

$$\xi(t, 0, \vartheta_{m_i}) \in \Omega \setminus \partial\Omega \subset \Omega \quad \text{for all} \quad 0 \le t \le m_i.$$

Thus, we have shown that $\xi(t, 0, \mu) \in \Omega$ for any $t \ge 0$. Since the set Ω is compact in $\mathbb{R}^k$, we conclude that in $\mathbb{R}^k$ there exists a ball of radius $\alpha > 0$ centered at the origin such that Ω is contained in this ball, and hence $\|\xi(t, 0, \mu)\| \le \alpha$ for all $t \ge 0$.

Now let us show that system (1) has a Poisson bounded solution $x(t, 0, x_0)$ for some $x_0 \in \mathbb{R}^n$. Since, by assumption, $\Omega \subset \mathrm{Im}\,(V : \{0\} \times \mathbb{R}^n \to \mathbb{R}^k)$, it follows that there exists a point $(0, x_0) \in \{0\} \times \mathbb{R}^n$ such that $V(0, x_0) = \mu$. We use condition (4) and inequality (3) to obtain the inequalities

$$b\Big(\big\|x(\tau_i, 0, x_0)\big\|\Big) \le \sum_{q=1}^{k} V_q\big(\tau_i, x(\tau_i, 0, x_0)\big) \le \sum_{q=1}^{k} \xi_q\big(\tau_i, 0, V(0, x_0)\big),$$

which hold for all $i \in \mathbb{N}$, for the solution $x(t, 0, x_0)$ of system (1) and the solution $\xi(t, 0, V(0, x_0))$ of the comparison system (2). Moreover, we have the obvious inequalities

$$\sum_{i=1}^{k} \xi_i\big(t, 0, V(0, x_0)\big) \le \sum_{i=1}^{k} \Big|\xi_i\big(t, 0, V(0, x_0)\big)\Big| \le k\Big\|\xi\big(t, 0, V(0, x_0)\big)\Big\|$$

for each $t \ge 0$. Since $V(0, x_0) = \mu$, we have $\|\xi(t, 0, V(0, x_0))\| \le \alpha$ for all $t \ge 0$. This fact, as well as the above-indicated inequalities, implies that $b(\|x(\tau_i, 0, x_0)\|) \le k\alpha$ for all $i \in \mathbb{N}$. Using the condition $b(r) \to +\infty$ as $r \to +\infty$ and the fact that the number $k\alpha$ is fixed, we select a number $\beta > 0$ such that $k\alpha \le b(\beta)$. It follows that $b(\|x(\tau_i, 0, x_0)\|) \le b(\beta)$ for all $i \in \mathbb{N}$. Since the function $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, it follows from the last inequality that $\|x(\tau_i, 0, x_0)\| \le \beta$ for all $i \in \mathbb{N}$. Thus, we have shown that the solution $x(t, 0, x_0)$ of system (1) is Poisson bounded. The proof of the proposition is complete.

Given arbitrary positive integers $n \ge 2$ and $1 \le m < n$, for each fixed element $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ and the corresponding solution $x(t, t_0, x_0) = (x_1(t, t_0, x_0), \ldots, x_n(t, t_0, x_0))^{\mathrm{T}}$ of system (1) consider the mapping

$$y\colon \mathbb{R}_+ \times \big\{(t_0, x_0)\big\} \to \mathbb{R}^m, \quad y(t, t_0, x_0) = \big(x_1(t, t_0, x_0), \ldots, x_m(t, t_0, x_0)\big)^{\mathrm{T}}.$$

Recall [3] that a solution $x(t, t_0, x_0)$ of system (1) is said to be *y-bounded* if there exists a number $\beta > 0$ such that $\|y(t, t_0, x_0)\| \leq \beta$ for all $t \in \mathbb{R}_+$.

Definition 2. A solution $x = x(t, t_0, x_0)$ of system (1) is said to be *y-bounded in the sense of Poisson* if there exists a $\mathcal{P}$-sequence $(\tau_i)_{i\in\mathbb{N}}$ and a number $\beta > 0$ such that $\|y(\tau_i, t_0, x_0)\| \leq \beta$ for all $i \in \mathbb{N}$.

It is obvious that if a solution of system (1) is y-bounded, then it is also y-bounded in the sense of Poisson, because in this case we can take any $\mathcal{P}$-sequence for the required $\mathcal{P}$-sequence. It can readily be seen that the above definition is equivalent to the definition of solution y-bounded in the sense of Poisson in the paper [8].

The following assertion is a sufficient condition for system (1) to have solutions y-bounded in the sense of Poisson.

Proposition 2. *Let all the assumptions in Proposition 1 with inequality (4) replaced by the inequality*

$$b(\|y\|) \leq \sum_{q=1}^{k} V_q(\tau_i, x) \tag{5}$$

be satisfied. Then system (1) has at least one solution that is y-bounded in the sense of Poisson.

Proof. By reproducing the argument in the proof of Proposition 1 word for word and by replacing inequality (4) with inequality (5), we obtain the inequality

$$b(\|y(\tau_i, 0, x_0)\|) \leq \sum_{q=1}^{k} V_q(\tau_i, x(\tau_i, 0, x_0)) \leq \sum_{q=1}^{k} \xi_q(\tau_i, 0, V(0, x_0)), \quad i \in \mathbb{N}.$$

Now, by reproducing the argument in the proof of Proposition 1 word for word with $x(t, 0, x_0)$ replaced by $y(t, 0, x_0)$, we obtain the desired inequality $\|y(\tau_i, 0, x_0)\| \leq \beta$ for all $i \in \mathbb{N}$. Thus, we have shown that the solution $x(t, 0, x_0)$ of system (1) is y-bounded in the sense of Poisson. The proof of the proposition is complete.

Now let us recall necessary facts about guiding functions and their indices [6]. A continuously differentiable function $G\colon \mathbb{R}^n \to \mathbb{R}$ is called a *guiding function*, or, more precisely, an r_0*-guiding function*, for system (1) if the following condition is satisfied:

$$\big(\operatorname{grad} G(x), F(t, x)\big) > 0, \quad t \geq 0, \quad \|x\| \geq r_0. \tag{6}$$

Consider the vector field $\operatorname{grad} G\colon B^n(r_0) \to \mathbb{R}^n$, where $B^n(r_0) = \{x \in \mathbb{R}^n : \|x\| \leq r_0\}$. It can be seen from condition (6) that the vector field $\operatorname{grad} G\colon B^n(r_0) \to \mathbb{R}^n$ is nondegenerate on $\partial B^n(r_0)$, and consequently, the rotation $\gamma(\operatorname{grad} G, \partial B^n(r_0))$ of this vector field is well defined. It was shown in the monograph [6, p. 90–91] that if for each $r > r_0$ we consider the corresponding vector field $\operatorname{grad} G\colon B^n(r) \to \mathbb{R}^n$, which is obviously nondegenerate on $\partial B^n(r)$, then one has the equality of rotations $\gamma(\operatorname{grad} G, \partial B^n(r)) = \gamma(\operatorname{grad} G, \partial B^n(r_0))$. The *index* of an r_0-guiding function G for system (1) is the integer $\operatorname{ind}(G)$ defined by the formula

$$\operatorname{ind}(G) = \gamma\big(\operatorname{grad} G, \partial B^n(r_0)\big) = \gamma\big(\operatorname{grad} G, \partial B^n(r)\big), \quad r > r_0.$$

It was shown in [6, p. 110–111] that if system (1) has an r_0-guiding function G, then for each $r \geq r_0$ the rotation of the vector field $S_0\colon B(r) \to \mathbb{R}^n$, $S_0(x) = -F(0, x)$, where $F(t, x)$ is the right-hand side of system (1), and the index of the r_0-guiding function G are related by the formula

$$\gamma\big(S_0, \partial B^n(r)\big) = (-1)^n \operatorname{ind} G.$$

An *unbounded* r_0-guiding function for system (1) is any r_0-guiding function G for this system satisfying the condition $G(x) \to +\infty$ as $\|x\| \to +\infty$. It was shown in [6, p. 111–113] that $\operatorname{ind}(G) = 1$ for each unbounded r_0-guiding function G for system (1). It follows that if system (1) has an unbounded r_0-guiding function, then the rotation of the above-indicated vector field $S_0\colon B^n(r) \to \mathbb{R}^n$ is calculated by the formula $\gamma(S_0, \partial B^n(r)) = (-1)^n$.

Let us state and prove a sufficient condition in terms of Lyapunov vector functions and guiding functions for system (1) to have Poisson bounded solutions.

Theorem 1. *Assume that for system (1) there exists a $\mathcal{P}$-sequence $(\tau_i)_{i\in\mathbb{N}}$, a nondecreasing function $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ for which $b(r) \to +\infty$ as $r \to +\infty$, and a Lyapunov vector function $V\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^k$ with the comparison system (2) such that inequality (4) is satisfied for any $x \in \mathbb{R}^n$ and $i \in \mathbb{N}$. Moreover, assume that there exist numbers $r_1 > r_0$ and an unbounded r_0-guiding function G for the system*

$$\frac{d\varrho}{dt} = g(t,\varrho), \quad (t,\varrho) \in \mathbb{R}^+ \times \mathbb{R}^k, \quad g(t,\varrho) = f(t,\varrho+\overline{r}_1), \quad \overline{r}_1 = (r_1,\dots,r_1)^{\mathrm{T}} \in \mathbb{R}^k, \tag{7}$$

where $f(t,\xi)$ is the right-hand side of system (2), with the number r_1 satisfying the conditions

1. *$G(\varrho) \geq M_0$ for all $\varrho \in \mathbb{R}^k$, $\|\varrho\| = r_1$, where $M_0 = \max\limits_{\|\varrho\|\leq r_0} G(\varrho)$.*
2. *$B^k_{\overline{r}_1}(r_1) = \{\xi \in \mathbb{R}^k \,|\, \|\xi - \overline{r}_1\| \leq r_1\} \subset \operatorname{Im}(V\colon \{0\}\times\mathbb{R}^n \to \mathbb{R}^k)$.*

Then system (1) has at least one Poisson bounded solution.

Proof. Consider the vector field $L_0\colon B^k(r_1) \to \mathbb{R}^k$ defined by the formula $L_0(\varrho) = -g(0,\varrho)$, where $g(t,\varrho)$ is the right-hand side of system (7). By the preceding, $\gamma(L_0, \partial B^k(r_1)) = (-1)^k$. Thus, $\gamma(L_0, \partial B^k(r_1)) \neq 0$.

Let us show that whatever the point $\varrho_0 \in \partial B^k(r_1)$ is, the right part $\mathrm{Tr}^+(\varrho_0,t_0)$ of the trajectory $\mathrm{Tr}\,(\varrho(0,t_0,\varrho_0))$ of system (7) does not have points in $B^k(r_1)$ in common with the left part $\mathrm{Tr}^-(\varrho_0,t_0)$ of the same trajectory. Consider the function $\varphi(t) = G(\varrho(t,t_0,\varrho_0))$, $t \geq 0$, and its derivative

$$\varphi'(t) = \frac{d\Big(G\big(\varrho(t,t_0,\varrho_0)\big)\Big)}{dt} = \Big(\operatorname{grad} G\big(\varrho(t,t_0,\varrho_0)\big), g\big(t,\varrho(t,t_0,\varrho_0)\big)\Big), \quad t \geq 0.$$

Since G is an r_0-guiding function for system (7), it follows that $\varphi'(t) > 0$ for $t \geq 0$ such that $\|\varrho(t,t_0,\varrho_0)\| \geq r_0$. It is obvious that $\varphi(t_0) = G(\varrho_0) \geq M_0$ and $\varphi(t) \leq M_0$ for $t \geq 0$ such that $\|\varrho(t,t_0,\varrho_0)\| \leq r_0$. Moreover, it is obvious that $\varphi(t)$ is an increasing function for $t \geq 0$ such that $\|\varrho(t,t_0,\varrho_0)\| \geq r_0$. It follows that $\varphi(t) \leq \varphi(t_0)$ for each point $\varrho(t,t_0,\varrho_0) \in \mathrm{Tr}^-(\varrho_0,t_0)$.

Now let us show that if $\varrho(t,t_0,\varrho_0) \in \mathrm{Tr}^+(\varrho_0,t_0)$, then

$$\big\|\varrho(t,t_0,\varrho_0)\big\| > r_0.$$

Assume the contrary: $\|\varrho(t_1,t_0,\varrho_0)\| \leq r_0$ for some $t_1 > t_0$ and hence $\varphi(t_1) \leq M_0$. Since $\varphi(t_0) \geq M_0$ and $\varphi'(t_0) > 0$, it follows that there exists a $t_0 < t' < t_1$ such that $\varphi(t') > M_0$ and hence $\|\varrho(t',t_0,\varrho_0)\| > r_0$. Since the function $\|\varrho(t,t_0,\varrho_0)\|$ is continuous, it follows from the preceding that there exists a $t' < t_2 \leq t_1$ such that $\|\varrho(t_2,t_0,\varrho_0)\| = r_0$ and $\|\varrho(t,t_0,\varrho_0)\| \geq r_0$ for $t' < t \leq t_2$. It is obvious that $\varphi(t_2) \leq M_0$, because $\|\varrho(t_2,t_0,\varrho_0)\| = r_0$. Since $\|\varrho(t,t_0,\varrho_0)\| \geq r_0$ for $t' \leq t \leq t_2$, we conclude that $\varphi'(t) > 0$ for $t' \leq t \leq t_2$ and hence $\varphi(t') < \varphi(t_2)$. Based on this, we obtain $\varphi(t_2) > M_0$; this contradicts the above-indicated inequality $\varphi(t_2) \leq M_0$. Thus, $\|\varrho(t,t_0,\varrho_0)\| > r_0$ for each point $\varrho(t,t_0,\varrho_0) \in \mathrm{Tr}^+(\varrho_0,t_0)$.

Based on the preceding, we conclude that $\varphi'(t) > 0$ for $t > t_0$ and hence $\varphi(t) > \varphi(t_0)$ for $t > t_0$. Thus, $\varphi(t) \leq \varphi(t_0)$ for $0 \leq t \leq t_0$ and $\varphi(t) > \varphi(t_0)$ for $t > t_0$. It follows that the right part $\mathrm{Tr}^+(\varrho_0,t_0)$ of the trajectory $\mathrm{Tr}\,(\varrho(0,t_0,\varrho_0))$ of system (7) does not have points in $B^k(r_1)$ in common with the left part $\mathrm{Tr}^-(\varrho_0,t_0)$ of the same trajectory. Arguing by analogy with the proof of Theorem 1, we obtain a solution $\varrho(t,0,\mu)$ (where $\mu \in B^k(r_1)$) of system (7) for which the inclusion $\varrho(t,0,\mu) \in B^k(r_1)$ is satisfied for all $t \geq 0$.

Since system (7) has been obtained from system (2) by the change of variables $\xi = \varrho + \overline{r}_1$, we see that the solution $\xi(t,0,\mu+\overline{r}_1) = \varrho(t,0,\mu) + \overline{r}_1$ of system (2) satisfies $\xi(t,0,\mu+\overline{r}_1) \in B^k_{\overline{r}_1}(r_1)$ for all $t \geq 0$. One obviously has the inclusion $\mu + \overline{r}_1 \in B^k_{\overline{r}_1}(r_1)$. It follows from condition 2 in the theorem that there exists a point $(0,x_0) \in \{0\}\times\mathbb{R}^n$ such that $V(0,x_0) = \mu + \overline{r}_1$. Arguing by analogy with the proof of Proposition 1, we obtain the Poisson boundedness of the solution $x(t,0,x_0)$ of system (1). The proof of the theorem is complete.

The next assertion is a sufficient condition for system (1) to have solutions y-bounded in the sense of Poisson.

Theorem 2. *Let all assumptions of Theorem 1 with inequality (4) replaced by (5) be satisfied. Then system (1) has at least one solution y-bounded in the sense of Poisson.*

Proof. Let us repeat word for word the argument in the proof of Theorem 1 up to the point of considering the solutions $\xi(t,0,\mu+\bar{r}_1)=\varrho(t,0,\mu)+\bar{r}_1$ of system (2) for which the inclusion $\xi(t,0,\mu+\bar{r}_1)\in B^k_{\bar{r}_1}(r_1)$ holds for all $t\ge 0$, where $V(0,x_0)=\mu+\bar{r}_1\in B^k_{\bar{r}_1}(r_1)$. After this, arguing by analogy with the proof of Proposition 2, we conclude that the solution $x(t,0,x_0)$ of system (1) is y-bounded in the sense of Poisson. The proof of the theorem is complete.

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