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PARTIAL DIFFERENTIAL EQUATIONS

Parabolic Problem with a Power-Law Boundary Layer

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Abstract—We construct a regularized asymptotics of the solution of the first boundary value problem for a singularly perturbed two-dimensional differential equation of the parabolic type for the case in which the limit equation has a regular singularity. There arise power-law and corner boundary layers along with parabolic ones in such problems.

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INTRODUCTION

Lomov's regularization method [1] for singularly perturbed problems was originally developed for equations whose order does not decrease as the small parameter tends to zero but exhibits some singularity [2]. The method allows one to construct a regularized asymptotics of the solution [1]. Subsequently, this method was generalized to many classes of singularly perturbed equations in various settings. (A bibliography of recent papers dealing with the construction of regularized asymptotics can be found in the monograph [3].) Problems with a power-law boundary layer were studied from various points of view in [2–7]. For example, the asymptotics of solutions of boundary and initial value problems was constructed in [4] for ordinary differential equations with a small parameter and with a power-law boundary layer. The same paper also gives examples of mixed boundary value problems for partial differential equations of parabolic and hyperbolic types which, when solved, give rise to the phenomenon of a power-law boundary layer. There is no small parameter multiplying the self-adjoint elliptic operator in the equations studied in [4]. The Fourier method was used there to reduce the original problem to an ordinary differential equation for which the asymptotics of the solution contains only a power-law boundary layer.

In contrast to [4], the parabolic equation studied in the present paper contains a small parameter multiplying part of the second spatial derivatives. The small parameter thus introduced into the equation results in the onset of an additional parabolic boundary layer described by the special function known as the complementary error function. Moreover, the asymptotics of the solution contains corner boundary layer functions, which are products of power-law and parabolic boundary layer functions. Fundamental results on power-law boundary layers for ordinary differential equations can be found in the monograph [3, pp. 379–401], where a regularized asymptotics is constructed using the regularization method for singularly perturbed problems. This asymptotics of the solution contains a polynomial in powers of $\ln(1+\tau)$, $\tau = t/\varepsilon$. By introducing regularizing functions in a different way, we manage to simplify the structure of the solution so that it does not contain a polynomial in powers of $\ln(1+\tau)$. For ordinary differential equations, such a result was published in [7]. An algebraic method was used in [5, 6] to study singularly perturbed initial and boundary value problems for systems of ordinary differential equations with singularities of various types, and asymptotics of the solution containing power-law boundary layers were constructed.

The method can be applied to problems in hydro- and aerodynamics. Singularly perturbed problems in fluid mechanics, explosion theory, and other applied fields are given in the paper [8], while the paper [9] describes such problems in radio engineering.

The present paper deals with the asymptotic solution of the first boundary value problem for a singularly perturbed two-dimensional differential equation of the parabolic type

$$\begin{gathered} L_\varepsilon u(x,y,t,\varepsilon) \equiv (\varepsilon+t)\partial_t u(x,y,t,\varepsilon) - \varepsilon^2 a(x)\partial_x^2 u(x,y,t,\varepsilon) - L(y,t)u(x,y,t,\varepsilon) = f(x,y,t), \\ (x,y)\in\Omega\equiv(0,1)\times(0,1), \quad (x,y,t)\in Q\equiv\Omega\times(0,T], \\ u(x,y,t,\varepsilon)|_{t=0} = u(x,y,t,\varepsilon)|_{\partial\Omega} = 0. \end{gathered} \tag{1}$$

Along with a parabolic boundary layer function, the asymptotics of the solution of this problem also contains the power-law boundary layer function

$$\Pi_\varepsilon(t) = \left(\frac{\varepsilon}{t+\varepsilon}\right)^\lambda, \quad \lambda > 0,$$

as well as their product, which describes a corner boundary layer [7].

The problem is solved under the following assumptions.

Assumption 1. *The function $a(x)$ belongs to the class $C^\infty[0,1]$ and is positive for all $x \in [0,1]$. The free term $f(x,y,t)$ belongs to the class $C^\infty(\overline{Q})$.*

Assumption 2. *For each $t \in [0,T]$, the self-adjoint operator $L(y,t)$ on the Hilbert space $L_2[0,1]$ has simple discrete spectrum $\{\lambda_k(t) : k \in \mathbb{N}\}$ (i.e., $L\psi_k(y,t) = \lambda_k(t)\psi_k(y,t)$, $\psi_k(y,t)|_{y=0} = \psi_k(y,t)_{y=1} = 0$) such that*

(a) *$\lambda_i(t) \neq \lambda_j(t)$ for any $i \neq j$ and $t \in [0,T]$.*

(b) *$\lambda_k(0) < 0$ for each $k \in \mathbb{N}$.*

1. REGULARIZATION OF THE PROBLEM

Along with the independent variables x and t, we introduce regularizing variables with the use of the relations

$$\mu_j = \lambda_j(0)\ln\left(\frac{t+\varepsilon}{\varepsilon}\right) \equiv K_j(t,\varepsilon), \quad \tau = \frac{1}{\varepsilon}\ln\left(\frac{t+\varepsilon}{\varepsilon}\right), \quad \zeta_l = \frac{\varphi_l(x)}{\varepsilon^{3/2}},$$
$$\varphi_l(x) = (-1)^{l-1}\int\limits_{l-1}^{x}\frac{ds}{\sqrt{a(s)}}, \quad l = 1,2, \tag{2}$$

and declare them to be independent variables of the extended function

$$\tilde u(M,\varepsilon)|_{\theta=\chi(x,t,\varepsilon)} \equiv u(x,y,t,\varepsilon),$$
$$M = (x,y,t,\theta), \quad \theta = (\zeta,\tau,\mu), \quad \mu = (\mu_1,\mu_2,\ldots), \quad \zeta = (\zeta_1,\zeta_2),$$
$$\chi(x,t,\varepsilon) = \left(\frac{\varphi(x)}{\varepsilon^{3/2}}, \frac{1}{\varepsilon}\ln\left(\frac{t+\varepsilon}{\varepsilon}\right), K_1(t,\varepsilon), K_2(t,\varepsilon),\ldots\right), \quad \varphi(x) = \big(\varphi_1(x),\varphi_2(x)\big). \tag{3}$$

In view of definition (2), from (3) we find the derivatives of the extended function,

$$\partial_t u(x,y,t,\varepsilon) \equiv \left(\partial_t\tilde u + \frac{1}{\varepsilon(t+\varepsilon)}\partial_\tau\tilde u + \sum_{j=1}^{\infty}\frac{\lambda_j(0)}{t+\varepsilon}\partial_{\mu_j}\tilde u\right)_{\theta=\chi(x,t,\varepsilon)},$$
$$\partial_x^2 u \equiv \left(\partial_x^2\tilde u + \sum_{l=1}^{2}\left[\left(\frac{\varphi_l'(x)}{\varepsilon^{3/2}}\right)^2\partial_{\zeta_l}^2\tilde u + \frac{1}{\varepsilon^{3/2}}\big(2\varphi_l'(x)\partial^2_{x,\zeta_l} + \varphi_l''(x)\partial_{\zeta_l}\big)\tilde u\right]\right)_{\theta=\chi(x,t,\varepsilon)}. \tag{4}$$

To simplify the notation, we omit the terms containing $\partial^2_{\zeta_1,\zeta_2}\tilde u(M)$, because the asymptotics does not contain functions depending on (ζ_1,ζ_2).

Based on (1) and (2)–(4), for the extended function $\tilde u(M,\varepsilon)$ we pose the problem

$$\tilde L_\varepsilon\tilde u(M,\varepsilon) \equiv \varepsilon\partial_t\tilde u + \frac{1}{\varepsilon}T_0\tilde u + T_1\tilde u - \sqrt{\varepsilon}L_\zeta\tilde u - \varepsilon^2 L_x\tilde u = f(x,y,t), \quad M \in B,$$
$$\tilde u(M,\varepsilon)|_{t=\tau=\mu=0} = 0, \quad \tilde u(M,\varepsilon)|_{\partial B} = 0, \tag{5}$$

where

$$B \equiv Q \times (0,\infty)^3, \quad T_0 \equiv \partial_\tau - \Delta_\zeta, \quad T_1 \equiv t\partial_t + \sum_{j=1}^{\infty} \lambda_j(0)\partial_{\mu_j} - L(y,t), \quad \Delta_\zeta \equiv \sum_{l=1}^{2} \partial^2_{\zeta_l},$$

$$L_\zeta \equiv a(x)\sum_{l=1}^{2} L_{\zeta,l}, \quad L_x \equiv a(x)\partial_x^2, \quad L_{\zeta,l} \equiv \partial_{\zeta_l} D_{x,l}, \quad D_{x,l} \equiv 2\varphi_l'(x)\partial_{x_l} + \varphi_l''(x).$$

Here one has the identity

$$\big(\tilde{L}_\varepsilon \tilde{u}(M,\varepsilon)\big)_{\theta=\chi(x,t,\varepsilon)} \equiv L_\varepsilon u(x,y,t,\varepsilon). \tag{6}$$

We seek a solution of problem (5) in the form of the series

$$\tilde{u}(M,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{k/2} u_k(M).$$

In a standard manner, for the coefficients of this series we obtain the iterative problems

$$\begin{aligned} &T_\nu u_0(M) = 0,\\ &T_0 u_2(M) = -T_1 u_0(M) + f(x,y,t),\\ &T_0 u_k(M) = -T_1 u_{k-2}(M) + L_\zeta u_{k-3}(M) - \partial_t u_{k-4}(M) + L_x u_{k-6}(M),\\ &\qquad u_k(M)|_{t=\tau=\mu=0} = 0, \quad u_k(M)|_{\partial B} = 0, \quad k \ge 0, \quad \nu = 0,1. \end{aligned} \tag{7}$$

2. SPACE OF RESONANCE-FREE SOLUTIONS

Let us define a function class in which each of problems (7) is uniquely solvable. To this end, we introduce the function spaces

$$G_0 = \Big\{ g_0(x,y,t) : g_0(x,y,t) = \big\langle \upsilon(x,t), \psi(y,t)\big\rangle, \quad \upsilon(x,t) \in C^\infty\big([0,1]\times[0,T]\big)\Big\},$$

$$G_1 = \Big\{ g_1(N^l) : g_1(N^l) = \sum_{l=1}^{2} \big\langle Y(N^l), \psi(y,t)\big\rangle, \quad \big\|Y(N^l)\big\| < c\exp\Big(-\frac{\zeta_l^2}{8\tau}\Big),$$
$$Y(N^l) = y(x,t)Y(\zeta_l,\tau), \quad y(x,t) \in C^\infty\big([0,1]\times[0,T]\big)\Big\},$$

$$G_2 = \Big\{ g_2(x,y,t,\mu) : g_2(x,y,t,\mu) = \Big\langle \big[C(x,t) + \Lambda\big(P(x)\big)\big]\exp(\mu), \psi(y,t)\Big\rangle,$$
$$C(x,t) \in C^\infty\big([0,1]\times[0,T]\big), \quad P(x) \in C^\infty\big([0,1]\big)\Big\},$$

$$G_3 = \Big\{ g_3(N^l) : g_3(N^l) = \sum_{l=1}^{2} \big\langle Z(N^l)\exp(\mu), \psi(y,t)\big\rangle, \quad \big\|Z(N^l)\big\| < c\exp\Big(-\frac{\zeta_l^2}{8\tau}\Big),$$
$$Z(N^l) = z(x,t)Z(\zeta_l,\tau), \quad z(x,t) \in C^\infty\big([0,1]\times[0,T]\big)\Big\},$$

$$N^l = (x,t,\zeta_l,\tau), \quad \mu = (\mu_1,\mu_2,\dots), \quad \upsilon(x,t) = \big(\upsilon_1(x,t),\upsilon_2(x,t),\dots\big), \quad Z(N^l) = \big(Z_{ij}(N^l)\big),$$
$$Y(N^l) = \big(Y_1(N^l),Y_2(N^l),\dots\big), \quad C(x,t) = \big(c_{ij}(x,t)\big), \quad \Lambda(P(x)) = \operatorname{diag}\big(P_1(x),P_2(x),\dots\big),$$
$$\exp(\mu) = \big(\exp(\mu_1),\exp(\mu_2),\dots\big), \quad \psi(y,t) = \big(\psi_1(y,t),\psi_2(y,t),\dots\big), \quad i,j = 1,2,\dots,$$
$$\big\langle \upsilon(x,t), \psi(y,t)\big\rangle = \sum_{i=1}^{\infty} \upsilon_i(x,t)\psi_i(y,t),$$

$$\Big\langle\Big[C(x,t)+\Lambda\big(P(x)\big)\Big]\exp(\mu),\psi(y,t)\Big\rangle=\sum_{ij=1}^{\infty}c_{ij}(x,t)\exp(\mu_j)\psi_i(y,t)+\sum_{i=1}^{\infty}P_i(x)\exp(\mu_i)\psi_i(y,t),$$

$$\big\langle Z(N^l)\exp(\mu),\psi(y,t)\big\rangle=\sum_{ij=1}^{\infty}Z_{ij}(N^l)\exp(\mu_j)\psi_i(y,t),\quad \big\langle Y(N^l),\psi(y,t)\big\rangle=\sum_{i=1}^{\infty}Y_i(N^l)\psi_i(y,t).$$

From these spaces we construct the new space defined as the direct sum of these spaces,

$$U=G_0\oplus G_1\oplus G_2\oplus G_3.$$

Following [1], we refer to this new space as the *space of resonance-free solutions*. An arbitrary element $u_k(M)$ of the space U has the form

$$\begin{aligned}u_k(M)&=\big\langle v_k(x,t),\psi(y,t)\big\rangle+\sum_{l=1}^{2}\big\langle Y^k(N^l),\psi(y,t)\big\rangle\\&\quad+\Big\langle\Big[C^k(x,t)+\Lambda\big(P^k(x)\big)\Big]\exp(\mu),\psi(y,t)\Big\rangle+\sum_{l=1}^{2}\big\langle Z^k(N^l)\exp(\mu),\psi(y,t)\big\rangle.\end{aligned}\tag{8}$$

Let us calculate the action of the operators T_0, T_1, and L_ζ on a function $u_k(M)\in U$. We have

$$\begin{aligned}T_0u_k(M)&=\sum_{l=1}^{2}\Big\langle\Big[\partial_\tau Y^k(N^l)-\partial_{\zeta_l}^2Y^k(N^l)+\big(\partial_\tau Z^k(N^l)-\partial_{\zeta_l}^2Z^k(N^l)\big)\exp(\mu)\Big],\psi(y,t)\Big\rangle,\\T_1u_k(M)&=\Big\langle\Big(D^1v_k(x,t)+\sum_{l=1}^{2}D^1Y^k(N^l)\Big),\psi(y,t)\Big\rangle\\&\quad+\Big\langle D^3\Big(C^k(x,t)+\Lambda\big(P^k(x)\big)\Big)\exp(\mu),\psi(y,t)\Big\rangle\\&\quad+\Big\langle\sum_{l=1}^{2}D^3Z^k(N^l)\exp(\mu),\psi(y,t)\Big\rangle,\\L_\zeta u_k(M)&=a(x)\sum_{l=1}^{2}\Big\langle\Big[\partial_{\zeta_l}\big(D_{x,l}\big(Y^k(N^l)\big)\big)+\partial_{\zeta_l}\big(D_{x,l}\big(Z^k(N^l)\big)\big)\exp(\mu)\Big],\psi(y,t)\Big\rangle,\\&D^1\equiv t\partial_t-\Lambda\big(\lambda(t)\big)+tA^{\mathrm T}(t),\quad D^3Z\equiv t\partial_tZ+tA^{\mathrm T}(t)Z+Z\Lambda(0)-\Lambda(t)Z,\\&\alpha_{ik}(t)=\big(\partial_t\psi_i(y,t),\psi_k(y,t)\big),\quad\Lambda(t)=\operatorname{diag}\big(\lambda_1(t),\lambda_2(t),\dots\big),\quad A(t)=\big(\alpha_{ik}(t)\big).\end{aligned}\tag{9}$$

3. SOLVABILITY OF THE ITERATIVE PROBLEMS

In the general case, the iterative equations (7) can be written in the form

$$T_0u_k(M)=h^k(M).\tag{10}$$

Theorem 1. *Let Assumptions 1 and 2 be satisfied, and let the function $h^k(M)$ lie in the space $G_1\oplus G_3$. Then Eq. (10) has a solution $u_k(M)$ in the space U.*

Proof. Let $h^k(M)\in G_1\oplus G_3$; i.e.,

$$h^k(M)=\Big\langle\sum_{l=1}^{2}\big[h^{k,1}(N^l)+h^{k,2}(N^l)\exp(\mu)\big],\psi(y,t)\Big\rangle,\quad\big\|h^{k,r}(N^l)\big\|<c\exp\Big(-\frac{\zeta_l^2}{8\tau}\Big),\quad r=1,2.$$

Let us substitute the representation (8) into Eq. (10). Then, based on the calculations in (9), for the functions $Y^k(N^l)$ and $Z^k(N^l)$ we obtain the equations

$$\partial_\tau Z^k_{ij}(N^l) - \partial^2_{\zeta_l} Z^k_{ij}(N^l) = h^{k,2}_{ij}(N^l), \quad \partial_\tau Y^k_i(N^l) - \partial^2_{\zeta_l} Y^k_i(N^l) = h^{k,1}_i(N^l).$$

These equations with the corresponding boundary conditions

$$Z^k_{ij}(N^l)|_{\tau=0} = 0, \quad Z^k_{ij}(N^l)|_{\zeta_l=0} = W^{k,l}_{ij}(x,t), \quad Y^k_i(N^l)|_{\tau=0} = 0, \quad Y^k_i(N^l)|_{\zeta_l=0} = d^{k,l}_i(x,t)$$

have solutions representable in the form

$$Z^k_{ij}(N^l) = W^{k,l}_{ij}(x,t)\,\mathrm{erfc}\left(\frac{\zeta_l}{2\sqrt{\tau}}\right) + h^{k,2}_{ij}(x,t) I_2(\zeta_l,\tau), \quad \mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}}\int\limits_x^\infty \exp(-t^2)\,dt,$$

$$Y^k_i(N_{1,l}) = d^{k,l}_i(x,t)\,\mathrm{erfc}\left(\frac{\zeta_l}{2\sqrt{\tau}}\right) + h^{k,1}_i(x,t) I_1(\zeta_l,\tau), \tag{11}$$

$$I_r(\zeta_l,\tau) = \frac{1}{2\sqrt{\pi}}\int\limits_0^\tau\int\limits_0^\infty \frac{h^{k,r}_1(\eta,s)}{\sqrt{\tau-s}}\left[\exp\left(-\frac{(\zeta_l-\eta)^2}{4(\tau-s)}\right) - \exp\left(-\frac{(\zeta_l+\eta)^2}{4(\tau-s)}\right)\right] d\eta\,ds, \quad r=1,2,$$

where $h^{k,r}_1(x,t)$ and $h^{k,r}_2(\eta,s)$ are known functions.

Let us estimate the integral $I_r(\eta_l,\tau)$ using the mean value theorem,

$$\begin{aligned}
\left|I_r(N^l)\right| &\le c\left|\frac{1}{2\sqrt{\pi}}\int\limits_0^\tau \frac{d\nu}{\sqrt{\tau-\nu}}\int\limits_0^\infty \left[\exp\left(-\frac{(\zeta_l-\eta)^2}{4(\tau-\nu)}\right) - \exp\left(-\frac{(\zeta_l+\eta)^2}{4(\tau-\nu)}\right)\right]\exp\left(-\frac{\eta^2}{8\nu}\right) d\eta\right| \\
&= \frac{c}{2\sqrt{\pi}}\left|\int\limits_0^\tau \frac{d\nu}{\sqrt{\tau-\nu}}\int\limits_0^\infty \exp\left[-\frac{(\zeta_l+\eta)^2}{4(\tau-\nu)} + \theta\left(-\frac{(\zeta_l-\eta)^2}{4(\tau-\nu)} + \frac{(\zeta_l+\eta)^2}{4(\tau-\nu)}\right)\right]\right. \\
&\qquad\left.\times\left(-\frac{(\zeta_l-\eta)^2}{4(\tau-\nu)} + \frac{(\zeta_l+\eta)^2}{4(\tau-\nu)}\right)\exp\left(-\frac{\eta^2}{8\nu}\right) d\eta\right| \\
&= \frac{c}{2\sqrt{\pi}}\left|\int\limits_0^\infty\int\limits_0^\tau \exp\left(-\frac{\eta^2}{8\nu}\right)\exp\left(-(1-\theta)\frac{(\zeta_l+\eta)^2}{4(\tau-\nu)} - \theta\frac{(\zeta_l-\eta)^2}{4(\tau-\nu)}\right)\frac{\zeta_l\eta}{\sqrt{(\tau-\nu)^3}}\,d\eta\,d\nu\right|.
\end{aligned}$$

Since

$$-\frac{1}{(\tau-\nu)} \le -\frac{1}{\tau}, \quad -\frac{1}{\nu} \le -\frac{1}{\tau}, \quad \left|\frac{\zeta_l\eta}{2\sqrt{\tau-\nu}}\exp\left(-\frac{\zeta_l\eta}{4(\tau-\nu)}\right)\right| < c,$$

we have, choosing $\theta = 1/4$,

$$\left|I_1(N_l)\right| \le c\left|\frac{1}{2\sqrt{\pi}}\int\limits_0^\infty \exp\left(-\frac{\eta_l^2}{4\tau}\right)\exp\left(-\frac{\eta^2}{8\tau}\right)\int\limits_0^\tau \exp\left(-\frac{\eta^2}{4(\tau-\nu)}\right)\frac{1}{(\tau-\nu)}\,d\nu\,d\eta\right|.$$

Let us make the change of variables $\tau-\nu = z$. Applying the mean value theorem, we obtain

$$\left|I_1(N_l)\right| \le c\left|\frac{1}{2\sqrt{\pi}}\int\limits_0^\infty \exp\left(-\frac{\eta_l^2}{4\tau}\right)\exp\left(-\frac{\eta^2}{8\tau}\right)\frac{1}{\tau}\int\limits_\theta^\tau \exp\left(-\frac{\eta^2}{4z}\right) dz\,d\eta\right|.$$

Sharpening the inequality and applying the mean value theorem one more time, we arrive at the inequality

$$\left|I_1(N_l)\right| \le c\left|\frac{1}{\tau}\exp\left(-\frac{\eta_l^2}{4\tau}\right)\int\limits_0^\infty \exp\left(-\frac{\eta^2}{8\tau}\right)\tau\exp\left(-\frac{\eta^2}{4\theta\tau}\right) d\eta\right|, \quad \theta\in(0,\tau).$$

Hence, using formula 3.321.3 in [10], we obtain the desired estimate. The proof of the theorem is complete.

In what follows, given a matrix C, by $\overline{C}$ (respectively, $\overline{\overline{C}}$) we denote the matrix with the same diagonal entries and zero off-diagonal entries (respectively, with the same off-diagonal entries and zero diagonal entries); in particular, $C = \overline{C} + \overline{\overline{C}}$.

Theorem 2. *Let Assumptions 1 and 2 be satisfied, and let* $\overline{h^{k,2}(x,t)}|_{t=0} = 0$ *(i.e.,* $h^{k,2}_{ii}(x,0) = 0$*). Then the problem*

$$
\begin{gathered}
D^3(C^k(x,t) + \Lambda(P^k(x))) = h^{k,2}(x,t),\\
\overline{C^k(x,t)}|_{t=0} = -\Big[\Lambda\big(\upsilon_k(x,0)\big) + \Lambda\big(\overline{\overline{C^k(x,t)}}\mathbf{1}\big) + \Lambda\big(P^k(x)\big)\Big]_{t=0}\\
\Big(i.e.,\quad c^k_{i,i}(xt)|_{t=0} = -\upsilon_{ki}(x,0) - P^k_i(x) - \sum_{i\neq j} c^k_{ij}(x,0)\Big),
\end{gathered}
\tag{12}
$$

where $\mathbf{1} = \operatorname{col}(1,1,\ldots)$, *is uniquely solvable.*

Proof. In Eq. (12), set

$$
\begin{gathered}
\overline{\overline{C^k(x,t)\Lambda(0) - \Lambda(t)C^k(x,t)}}|_{t=0} = \big[\,\overline{\overline{h^{k,2}(x,t)}}\,\big]|_{t=0}\\
\Big(\text{i.e.,}\quad c^k_{ij}(x,t)|_{t=0} = \frac{h^{k,2}_{i,j}(x,0)}{\lambda_j(0) - \lambda_i(0)},\quad i\neq j\Big).
\end{gathered}
\tag{13}
$$

Then, by virtue of the condition $\overline{h^{k,2}(x,t)}|_{t=0} = 0$, system (12) is nonsingular.

Under the corresponding initial conditions in (12) and (13), Eq. (12) unambiguously determines the function $C^k(x,t)$. The proof of the theorem is complete.

Remark. When solving the iterative equations, the condition $\overline{h^{k,2}(x,t)}|_{t=0} = 0$ is ensured by the choice of the vector function $P^k(x) = (P^k_1(x), P^k_2(x), \ldots)$.

Theorem 3. *Let Assumptions 1 and 2 be satisfied. Then Eq. (10) has a unique solution satisfying the conditions*

(a) $u_k(M)|_{t=\tau=\mu=0} = 0$, $u_k(M)|_{\partial B} = 0$.

(b) $T_1 u_k(M) + h^k(M) \in G_1 \oplus G_3$.

(c) $L_\varsigma u_k(M) = 0$.

Proof. By Theorem 1, there exists a solution of Eq. (10), which can be represented in the form (8). Let us subject the solution (8) to condition (b), which holds if the arbitrary functions $\upsilon_k(x,t)$ and $C^k(x,t)$ are chosen to be solutions of the equations

$$
D^1\upsilon_{k,i}(x,t) = -h^{k,1}_i(x,t),\quad D^3\big[c^k_{ij}(x,t) + P^k_i(x)\big] = -h^{k,2}_{ij}(x,t).
$$

Then, based on (9), the expression $T_1 u_k(M) + h^k(M)$ is written in the form

$$
\begin{aligned}
T_1 u_k(M) + h^k(M) = \sum_{l=1}^{2}\Big[&\big\langle D^1 Y^k(N^l) + h^{k,3}(N^l), \psi(y,t)\big\rangle\\
&+ \Big\langle \big(D^3 Z^k(N^l) + h^{k,4}(N^l)\big)\exp(\mu), \psi(y,t)\Big\rangle\Big] \in G_1 \oplus G_3,
\end{aligned}
$$

$$
h^k(M) = \sum_{l=1}^{2}\big\langle h^{k,3}(N^l) + h^{k,4}(N^l)\exp(\mu), \psi(y,t)\big\rangle.
$$

Subjecting the function (8) to the boundary conditions (a), we find

$$\begin{gathered} Y_i^k(N^l)|_{t=\tau=0}=0, \quad Y_i^k(N^l)|_{\zeta_l=0}=d_i^{k,l}(x,t), \quad d_i^{k,l}(x,t)|_{x=l-1}=-\upsilon_{k,i}(l-1,t), \\ \overline{C^k(x,t)}|_{t=0}=-\Lambda\big(\upsilon_k(x,0)\big)-\Lambda\big(P^k(x)\big)-\Lambda\big(\overline{\overline{C^k(x,0)}}\mathbf{1}\big) \\ \left(\text{i.e.,}\;\; c_{ii}^k(x,t)|_{t=0}=-\upsilon_{ki}(x,0)-P_i^k(x)-\sum_{i\neq j}c_{ij}^k(x,0)\right), \quad Z_{ij}^k(N^l)|_{t=\tau=0}=0, \\ Z_{ij}^k(N^l)|_{\zeta_l=0}=W_{ij}^{k,l}(x,t), \quad W_{ij}^{k,l}(x,t)|_{x=l-1}=-c_{ij}^k(l-1,t)-P_i^k(l-1). \end{gathered} \tag{14}$$

The matrix function $C^k(x,t)$ is determined unambiguously by Theorem 2.

Let us substitute the function $u_k(M)$ into condition (c). Then, taking into account the representations (11) as well as the relation $h^{k,r+3}(N^l)=h^{k,r+3}(x,t)I_r(\zeta_l,\tau)$, and noticing that the function $\operatorname{erfc}(\zeta_l/2\sqrt{\tau})$ satisfies the same estimate as the function $I_r(\zeta_l,\tau)$, $r=1,2$, according to (9) we obtain the equations

$$\begin{aligned} D_{x,l}\big[d_i^{k,l}(x,t)+h_i^{k,3}(x,t)\big]&=0, \\ D_{x,l}\big[W_{i,j}^{k,l}(x,t)+h_{i,j}^{k,4}(x,t)\big]&=0. \end{aligned}$$

Under the initial conditions in (14), from these equations we unambiguously determine the functions $d_i^{k,l}(x,t)$ and $W_{i,j}^{k,l}(x,t)$ and hence, by virtue of (11), uniquely find the functions $Y^k(N^l)$ and $Z^k(N^l)$.

The equation for $\upsilon_k(x,t)$ has a unique smooth solution (see [2, 3, 11, 12]) satisfying the condition $\|\upsilon_k(x,0)\|<\infty$.

Thus, the solution of Eq. (10) has been unambiguously determined. The proof of the theorem is complete.

4. SOLUTION OF THE ITERATIVE PROBLEMS

The iterative equation (7) is homogeneous for $k=0,1$; therefore, according to Theorem 1, these equations are solvable in the space U if the functions $Y^k(N^1)$ and $Z^k(N^l)$ are solutions of the equations

$$\begin{aligned} \partial_\tau Y_i^k(N^1)&=\partial_{\zeta_l}^2 Y_i^k(N^l), \\ \partial_\tau Z_{ij}^k(N^1)&=\partial_{\zeta_l}^2 Z_{ij}^k(N^l). \end{aligned}$$

Under the boundary conditions

$$Y_i^k(N^1)|_{\tau=0}=0, \quad Y_i^k(N^1)|_{\zeta_l=0}=d_i^{k,l}(x,t), \quad Z_{ij}^k(N^1)|_{\tau=0}=0, \quad Z_{ij}^k(N^1)|_{\zeta_l=0}=W_{ij}^{k,l}(x,t),$$

the solutions of these equations can be represented as

$$\begin{aligned} Y_i^k(N_1)&=d_i^{k,l}(x,t)\operatorname{erfc}\left(\frac{\zeta_l}{2\sqrt{\tau}}\right), \\ Z_{ij}^k(N_1)&=W_{ij}^{k,l}(x,t)\operatorname{erfc}\left(\frac{\zeta_l}{2\sqrt{\tau}}\right), \end{aligned} \tag{15}$$

where the arbitrary functions $d_i^{k,l}(x,t)$ and $W_{ij}^{k,l}(x,t)$ satisfy the conditions

$$\begin{aligned} d_i^{k,l}(x,t)|_{x=l-1}&=-\upsilon_{k,i}(l-1,t), \\ W_{ij}^{k,l}(x,t)|_{x=l-1}&=-c_{ij}^k(l-1,t)-P_i^k(l-1). \end{aligned}$$

Let us calculate the free term in Eq. (7) for $k=2$, having preliminarily expanded the free term $f(x,y,t)$ in the series

$$f(x,y,t)=\sum_{i=1}^{\infty}f_i(x,t)\psi_i(y,t).$$

As a result, we obtain

$$F_2(M) = -T_1 u_0(M) + f(x,y,t) = -\langle D^1 v_0(x,t) - f(x,t), \psi(y,t)\rangle$$
$$- \sum_{l=1}^{2} \langle D^1 Y^0(N^1), \psi(y,t)\rangle - \left\langle D^3 \left[C^0(x,t) + \Lambda\big(P^0(x)\big)\right] \exp(\mu), \psi(y,t)\right\rangle$$
$$- \sum_{l=1}^{2} \langle D^3 Z^0(N^l)\exp(\mu), \psi(y,t)\rangle,$$
$$f(x,t) = \big(f_1(x,t), f_2(x,t), \dots\big).$$

Set

$$D^1 v_{0i}(x,t) - f_i(x,t) = 0, \quad D^3\left[c^0_{ij}(x,t) + \Lambda\big(P^0_i(x)\big)\right] = 0; \tag{16}$$

then

$$F_2(M) = -\sum_{l=1}^{2} \left\langle \big[D^1 Y^0(N^1) + D^3 Z^0(N^l)\exp(\mu)\big], \psi(y,t)\right\rangle.$$

The equation with this right-hand side is solvable in the space U if the functions $Y^2_i(N^l)$ and $Z^2_{ij}(N^l)$ are the solutions of the equations

$$T_0 Y^2_i(N^l) = -D^1 Y^0_i(N^1),$$
$$T_0 Z^2_{ij}(N^l) = -D^3 Z^0_{ij}(N^l).$$

Consider Eqs. (16). The first equation has a solution satisfying the condition $\|v_0(x,0)\| < \infty$ (see [2, 3, 11, 12]).

Removing the degeneracy of the second system in (16), we set

$$\overline{\overline{C^0(x,t)\Lambda(0) - \Lambda(t)C^0(x,t)}}|_{t=0} = 0$$
$$\Big(\text{i.e.,}\quad \big(\lambda_i(0) - \lambda_j(t)\big)c^0_{i,j}(x,t)|_{t=0} = 0, \quad i \ne j\Big). \tag{17}$$

Moreover, from the initial condition (14) we find

$$\overline{C^0(x,t)}|_{t=0} = -\Big[\Lambda\big(v_0(x,t)\big) + \Lambda\big(\overline{\overline{C^0(x,t)}}\,1\big) + \Lambda\big(P^0(x)\big)\Big]_{t=0}; \tag{18}$$

in coordinate form, in view of (17), this relation can be written as

$$c^0_{ii}(x,t)|_{t=0} = -\big[v_{0,i}(x,0) + P^0_i(x)\big].$$

Relations (17) and (18) are used in the initial conditions of the second system in (16), which is uniquely solvable.

Let us proceed to the next iterative equation for $k = 3$. Based on the calculations in (9), the free term of this equation can be written in the form

$$F_3(M) = -T_1 u_1(M) + L_\zeta u_0(M) = -\left\langle D^1 v_1(x,t) + \sum_{l=1}^{2} D^1 Y^1(N^l), \psi(y,t)\right\rangle$$
$$- \left\langle D^3\Big(C^1(x,t) + \Lambda\big(P^1(x)\big)\Big)\exp(\mu), \psi(y,t)\right\rangle - \sum_{l=1}^{2}\langle D^3 Z^1(N^l)\exp(\mu), \psi(y,t)\rangle$$
$$+ a(x)\sum_{l=1}^{2}\left\langle \partial_{\zeta_l} D_{x,l}\big[Y^0(N^l) + Z^0(N^l)\exp(\mu)\big], \psi(y,t)\right\rangle.$$

To ensure the solvability of this equation, based on (15), we set

$$D^1 v_{1,i}(x,t) = 0, \quad D_{x,l} d_i^{0,l}(x,t) = 0, \quad D_{x,l} W_{i,j}^{0,l}(x,t) = 0, \quad D^3(c_{i,j}^1(x,t) + P_i^1(x)) = 0. \tag{19}$$

From the first equation in (19), we find $v_1(x,t) = 0$. Solving the second and third equations under the conditions $d_i^{0,l}(x,t)|_{t=0} = -v_{0,i}(x,0)$ and $W_{i,j}^{0,l}(x,t)|_{t=0} = -c_{i,j}^0(x,0)$, we determine $d_i^{0,l}(x,t)$ and $W_{i,j}^{0,l}(x,t)$. The fourth equation is solvable if

$$\big(\lambda_i(0) - \lambda_j(t)\big) c_{ij}^1(x,0)\big|_{t=0} = 0 \quad \text{for all} \quad i \neq j.$$

From the initial condition (14), we find

$$\overline{C^1(x,t)|_{t=0}} = -\Lambda\big(v_1(x,0)\big) - \Lambda\big(P^1(x)\big)$$
$$\big(c_{ii}^1(x,t)|_{t=0} = -v_{1,i}(x,0) - P_i^1(x)\big).$$

It will be shown below that $P_i^k(x) = 0$ for odd k. The equation for $c_{ij}^1(x,t)$ is homogeneous; therefore, $c_{ij}^1(x,t) = 0$. The free term of the iterative equation for $k = 3$ acquires the form

$$F_3(M) = -\sum_{l=1}^{2} \Big\langle \big[D^1 Y^1(N^l) + D^3 Z^1(N^l)\exp(\mu)\big], \psi(y,t)\Big\rangle.$$

By Theorem 1, this equation has a solution representable in the form (8) with $k = 3$.

At the next step ($k = 4$), the free term of the iterative equation is written in the form

$$\begin{aligned}
F_4(M) = -T_1 u_2(M) + L_\zeta u_1 - \partial_t u_0 = -\big\langle D^1 v_2(x,t) + \partial_t v_0(x,t) + A^{\mathrm{T}}(t) v_0(x,t), \psi(y,t)\big\rangle \\
- \sum_{l=1}^{2} \langle D^1 Y^2(N^l) + D^3 Z^2(N^l)\exp(\mu), \psi(y,t)\rangle \\
- \big\langle D^3[C^2(x,t) + \Lambda(P^2(x))]\exp(\mu), \psi(y,t)\big\rangle \\
+ a(x) \sum_{l=1}^{2} \Big\langle \partial_{\zeta_l} D_{x,l}\big[Y^1(N^l) + Z^1(N^l)\exp(\mu)\big], \psi(y,t)\Big\rangle \\
- \sum_{l=1}^{2} \big\langle \partial_t Y^0(N^l) + \partial_t Z^0(N^l)\exp(\mu) + A^{\mathrm{T}}(t) Y^0(N^l) + A^{\mathrm{T}}(t) Z^0(N^l)\exp(\mu), \psi(y,t)\big\rangle \\
- \Big\langle \Big[\partial_t C^0(x,t) + A^{\mathrm{T}}(t) C^0(x,t) + A^{\mathrm{T}}(t)\Lambda\big(P^0(x)\big)\Big]\exp(\mu), \psi(y,t)\Big\rangle.
\end{aligned}$$

To ensure the solvability of the iterative equation for $k = 4$, we set

$$\begin{gathered}
D^1 v_{2,i}(x,t) = -\Big[\partial_t v_{0,i} + \sum_{j=1}^{\infty} \alpha_{ji}(t) v_{0,i}(x,t)\Big], \\
D^3\big[c_{ij}^2(x,t) + P_i^2(x)\big] = -\big[\partial_t c_{ii}^0(x,t) + \alpha_{ii}(t) c_{ii}^0(x,t) + \alpha_{ii}(t) P_i^0(x)\big], \\
D_{x,l} d_i^{1,l}(x,t) = 0, \quad D_{x,l} W_{ij}^{1,l}(x,t) = 0.
\end{gathered} \tag{20}$$

The first equation permits one to determine the function $v_2(x,t)$. Removing the degeneracy of the second equation, we set

$$\big(\lambda_i(0) - \lambda_j(t)\big) c_{ij}^2(x,t)\big|_{t=0} = 0 \quad \text{for all} \quad i \neq j,$$
$$-\big(\partial_t c_{ii}^0 + \alpha_{ii}(t) c_{ii}^0(x,t) + \alpha_{ii}(t) P_i^0(x)\big)\big|_{t=0} = 0.$$

The last relation is ensured by the choice of the components of the vector $P^0(x) = (P_1^0(x), P_2^0(x), \dots)$,

$$P_i^0(x) = -\frac{\partial_t c_{ii}^0(x,t) + \alpha_{ii}(t) c_{ii}^0(x,t)}{\alpha_{ii}(t)}\bigg|_{t=0}.$$

The equations for the functions $d_i^{1,l}(x,t)$ and $W_{ij}^{1,l}(x,t)$ in (20) are solved under the zero initial conditions

$$d_i^{1,l}(x,t)|_{x=l-1} = -\upsilon_{1,i}(l-1,t) = 0,$$
$$W_{ij}^{1,l}(x,t)|_{x=l-1} = -c_{ij}^1(l-1,t) - P_i^1(l-1) = 0;$$

here we have taken into account the fact that $P_i^1(x) = 0$, hence $Y^1(N^l) = 0$ and $Z^1(N^l) = 0$, and consequently, $u_1(M) = 0$.

Based on (20), the free term $F_4(M)$ acquires the form

$$F_4(M) = -\sum_{l=1}^{2}\bigg\{\big\langle D^1 Y^2(N^l) + D^3 Z^2(N^l)\exp(\mu), \psi(y,t)\big\rangle + \Big\langle \partial_t Y^0(N^l) + A^T(t) Y^0(N^l) + \big[\partial_t Z^0(N^l) + A^T(t) Z^0(N^l)\big]\exp(\mu), \psi(y,t)\Big\rangle\bigg\} \in G_1 \oplus G_3;$$

the iterative equation for $k = 4$ is solvable in U by Theorem 1.

Consider one more iterative equation for $k = 5$. The free term of this equation is written in the form

$$\begin{aligned} F_5(M) = -T_1 u_3(M) + L_\zeta u_2(M) - \partial_t u_1 = &-\big\langle D^1 \upsilon_3(x,t) + \partial_t \upsilon_1(x,t) + A^{\mathrm{T}}(t)\upsilon_1(x,t), \psi(y,t)\big\rangle \\ &- \sum_{l=1}^{2}\big\langle D^1 Y^3(N^l) + D^3 Z^3(N^l)\exp(\mu), \psi(y,t)\big\rangle - \Big\langle D^3\big[C^3(x,t) + \Lambda(P^3(x))\big], \psi(y,t)\Big\rangle \\ &+ a(x)\sum_{l=1}^{2}\Big\langle \partial_{\zeta_l} D_{x,l}\big[Y^2(N^l) + Z^2(N^l)\exp(\mu)\big], \psi(y,t)\Big\rangle \\ &- \sum_{l=1}^{2}\Big\langle \partial_t Y^1(N^l) + \partial_t Z^1(N^l)\exp(\mu) + A^{\mathrm{T}}(t)\big[Y^1(N^l) + Z^1(N^l)\exp(\mu)\big], \psi(y,t)\Big\rangle \\ &- \Big\langle \partial_t C^1(x,t) + A^{\mathrm{T}}(t)\big[C^1(x,t) + \Lambda(P^1(x))\big], \psi(y,t)\Big\rangle. \end{aligned}$$

By Theorem 1, the iterative equation for $k = 5$ is solvable if

$$\begin{gathered} D^1 \upsilon_{3i}(x,t) = -\partial_t \upsilon_{1i}(x,t) - \sum_{k=1}^{\infty}\alpha_{ki}\upsilon_{1k}(x,t), \\ D^3\Big[C^3(x,t) + \Lambda\big(P^3(x)\big)\Big] = \partial_t C^1(x,t) + A^{\mathrm{T}}(t)\Big[C^1(x,t) + \Lambda\big(P^1(x)\big)\Big], \\ \partial_{\zeta_l} D_{x,l} Y^2(N^l) = 0, \quad \partial_{\zeta_l} D_{x,l} Z^2(N^l) = 0. \end{gathered} \tag{21}$$

Since $C^1(x,t) = 0$, we set $P^1(x) = 0$ to ensure the solvability of the second equation in (21). Further, in a similar way, we successively determine the coefficients of the partial sum

$$u_{n,\varepsilon}(M) = \sum_{k=0}^{n}\varepsilon^k u_{2k}(M).$$

5. REMAINDER ESTIMATE

We substitute the expression

$$\tilde{u}(M,\varepsilon)=\sum_{k=0}^{n+1}\varepsilon^k u_{2k}(M)-\varepsilon^{n+1}u_{2(n+1)}(M)+\varepsilon^{n+1}R_\varepsilon(M) \tag{22}$$

into the extended problem (5). Then, considering the iterative problems (7), for the remainder term we obtain the problem

$$\tilde{L}_\varepsilon R_\varepsilon(M)=g_{\varepsilon,n}(M),\quad R_\varepsilon(M)|_{t=\tau=\mu=0}(M)=R_\varepsilon(M)|_{x=l-1,\zeta_l=0}(M)=0, \tag{23}$$

where

$$g_{\varepsilon,n}(M)=-T_1u_{2n}(M)-\partial_t u_{2n-2}(M)+L_x u_{2n-4}(M)-\tilde{L}_\varepsilon u_{2(n+1)}(M). \tag{24}$$

In relations (23) and (24), we perform restriction by means of the regularizing functions $\theta=\chi(x,t,\varepsilon)$. Then, by virtue of identity (6), for $R_{\varepsilon,n}(x,t)\equiv R_\varepsilon(M)$ we obtain the problem

$$L_\varepsilon R_{\varepsilon,n}(x,t)=g_{\varepsilon,n}(x,t),\quad R_{\varepsilon,n}(x,t)|_{t=0}=R_{\varepsilon,n}(x,t)|_{x=l-1}=0. \tag{25}$$

The very construction of the functions $u_k(M)$ and identity (6) imply the boundedness of the right-hand side $g_{\varepsilon,n}(x,t)\equiv g_{\varepsilon,n}(M)|_{\theta=\chi(x,t,\varepsilon)}$ of the equation in problem (25). For sufficiently small $\varepsilon>0$, the operator L_ε satisfies all the conditions of the maximum principle [13, p. 22]; therefore, following [14], we obtain the estimate $\|R_{\varepsilon,n}(x,t)\|<c$. From (22), we have the estimate

$$\left\|\tilde{u}(M)-u_{n,\varepsilon}(M)\right\|_{\theta=\chi(x,t,\varepsilon)}<c\varepsilon^{n+1}, \tag{26}$$

where the constant c is independent of $\varepsilon>0$, $n=0,1,2,\ldots$

Theorem 4. *Let Assumptions 1 and 2 be satisfied. Then the partial sum (22) obtained by the above-described method with $\theta=\chi(x,t,\varepsilon)$ is an asymptotic solution of problem (1); i.e., for sufficiently small $\varepsilon>0$ and all $n=0,1,2,\ldots$ one has the estimate (26).*

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