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Initial–Boundary Value Problem for a Nonlinear Beam Vibration Equation

K. B. Sabitov[1,2*] **and A. A. Akimov**[1**]

[1] *Sterlitamak Branch of Bashkir State University, Sterlitamak, Bashkortostan, 453103 Russia*
[2] *Sterlitamak Branch, Institute for Strategic Studies, Sterlitamak, Bashkortostan, 453103 Russia*
*e-mail: *sabitov_fmf@mail.ru, **andakm@rambler.ru*

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Abstract—We consider an initial–boundary value problem for the beam vibration equation, which is a fourth-order nonlinear equation with two independent variables. It is shown that under certain conditions on the initial data this problem can be reduced to the Cauchy problem for a countable system of quasilinear ordinary differential equations. Using the method of energy inequalities, we prove that this Cauchy problem has a solution. Based on this, we establish the existence of a local solution of the original initial–boundary value problem and construct it in closed form. A theorem on the uniqueness of a global solution is proved.

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INTRODUCTION

Many problems concerning rod, beams, and plate vibrations have important applications in structural engineering, stability theory of rotating shafts, and vibration theory of ships and pipelines and lead to differential equations of orders higher than the second [1, pp. 264–326 of the Russian translation; 2; 3, pp. 322–430 of the Russian translation]. Consider a uniform beam of length π (for definiteness) pinned on two supports. The flexural rotational-transverse vibrations of the beam are described by the nonlinear fourth-order equation

$$w_{tt} + (C_0 + C_1 w_{xxx}^2) w_{xxxx} - C_2 w_{xxtt} = 0, \quad C_i = \text{const} > 0, \quad i = 0, 1, 2. \tag{1}$$

Note that the last term in Eq. (1) allows for the beam rotation inertia. This equation was considered in the books indicated above and in [4–6] for the linear case, i.e., for $C_1 = C_2 = 0$. The paper [7] was one of the first publications to lay foundation for the applications of the method of energy inequalities to studying nonlinear equations that model various physical processes, including rotational phenomena. In that paper, the method was used to prove the existence of a regular solution of the problem considered there. The paper [8] studied a nonlinear string vibration equation supplemented with a longitudinal string displacement term in the form of a mixed third derivative. In that paper, the method of energy inequalities was used to prove the existence of a regular solution, and its estimates were derived in terms of initial functions. A semilinear equation of the form (1) describing the vibrations of a beam with movable endpoints was considered in [9], where the existence of a generalized solution of this equation was proved. Similar results for nonlinear differential equations of the hyperbolic type were obtained in [10] and [11]. In [12–14], existence and uniqueness theorems were proved for the solutions of initial–boundary value problems for a nonlinear equation describing the vibrations of a beam in the case of its possible rotation and sufficient conditions for the oscillation of the solution were established.

In the present paper, we study nonlinear rotational-transverse vibrations of a beam with pinned endpoints. To this end, we consider Eq. (1) in the domain $D = \{(x,t) : 0 < x < \pi, 0 < t < T\}$, and pose the following problem.

Initial–boundary value problem. Find a solution $w(x,t)$ of Eq. (1) defined in the domain D and satisfying the conditions

$$w(x,t) \in C_{x,t}^{4,2}(\overline{D}), \tag{2}$$

$$w(x,0) = \varphi(x), \quad w_t(x,0) = \psi(x), \quad 0 \le x \le \pi, \tag{3}$$

$$w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \quad 0 \le t \le T, \tag{4}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

In the present paper, we prove the uniqueness of the solution of problem (1)–(4) for each $T > 0$ and establish the existence of a solution in the class of regular solutions, i.e., those satisfying conditions (1) and (2), for small T. The solution is constructed in the form of a series in an orthogonal sine system whose coefficients are determined as the solutions of the Cauchy problem for a countable nonlinear system of second-order ordinary differential equations. The uniqueness of the solution of this problem is proved based on integral identities and inequalities.

1. SOLVABILITY OF THE INITIAL VALUE PROBLEM FOR AN INFINITE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

In what follows, we assume that the functions specifying the initial conditions admit the series expansions

$$\varphi(x) = \sum_{j=1}^{\infty} \alpha_j \sin(jx), \quad \psi(x) = \sum_{j=1}^{\infty} \beta_j \sin(jx),$$
$$\alpha_j = \frac{2}{\pi}\int_0^{\pi} \varphi(x)\sin(jx)\,dx, \quad \beta_j = \frac{2}{\pi}\int_0^{\pi} \psi(x)\sin(jx)\,dx. \tag{5}$$

To prove the existence of a regular solution of Eq. (1) with conditions (2)–(4), it suffices to establish the existence of a solution of the infinite system of ordinary differential equations

$$\ddot{T}_j(t)(1 + C_2 j^2) + C_0 j^4 T_j(t) + C_1 j \int_0^{\pi} \left(\sum_{i=1}^{\infty} i^3 T_i(t)\cos(ix)\right)^3 \cos(jx)\,dx = 0, \quad j = 1,2,\ldots, \quad 0 < t < T, \tag{6}$$

where $C_i > 0$, $i = 0,1,2$, with the initial conditions

$$T_j(0) = \alpha_j, \quad \dot{T}_j(0) = \beta_j, \quad j \in \mathbb{N}, \tag{7}$$

and the additional condition

$$\sum_{j=1}^{\infty} j^{10} T_j^2(t) < \infty, \quad 0 \le t < T. \tag{8}$$

To this end, it suffices to note that if Eq. (1) has a solution $w(x,t)$ belonging to the class (2) and satisfying the boundary conditions (4), then this solution can be represented in the form of the series

$$w(x,t) = \sum_{j=1}^{\infty} T_j(t)\sin(jx), \tag{9}$$

where the $T_j(t)$, $j \in \mathbb{N}$, are twice continuously differentiable functions. Then system (6) can be obtained by formally substituting the series (9) into Eq. (1), multiplying the resulting relation by $\sin(jx)$, and integrating it over x from 0 to π. In this case, instead of the factor C_1 multiplying the integral in Eq. (6), we obtain $C_1/3$; we denote it again by C_1.

To prove the existence of a solution of system (6), it is convenient to start from considering the corresponding finite systems of equations. Define functions $T_{j,N}(t)$ as the solutions of the system

$$\ddot{T}_{j,N}(t)(1 + C_2 j^2) + C_0 j^4 T_{j,N}(t) + C_1 j \int_0^{\pi} \left(\sum_{i=1}^{N} i^3 T_{i,N}(t)\cos(ix)\right)^3 \cos(jx)\,dx = 0, \quad j = 1,\ldots,N, \tag{10}$$

with the initial conditions

$$T_{j,N}(0) = \alpha_j, \quad \dot{T}_{j,N}(0) = \beta_j, \quad j = 1, \dots, N. \tag{11}$$

In what follows, for uniformity, we assume that $T_{j,N}(t) \equiv 0$ for $j > N$.

If we set $Y_{j,N} = \dot{T}_{j,N}$ in the system of second-order equations (10), then problem (10), (11) is reduced to the Cauchy problem for the system of first-order differential equations

$$\dot{\overline{T}}_N = \overline{Y}_N, \quad \dot{\overline{Y}}_N = \overline{F}_N, \tag{12}$$

$$T_N(0) = (\alpha_1, \dots, \alpha_N), \quad Y_N(0) = (\beta_1, \dots, \beta_N), \tag{13}$$

where

$$\overline{T}_N = (T_{1,N}, T_{2,N}, \dots, T_{N,N}), \ \ \overline{Y}_N = (Y_{1,N}, Y_{2,N}, \dots, Y_{N,N}), \ \ \overline{F}_N = (F_{1,N}, F_{2,N}, \dots, F_{N,N}),$$

$$F_{j,N} = -\frac{1}{1 + C_2 j^2}\left[C_0 j^4 T_{j,N} + C_1 j \int\limits_0^{\pi} \left(\sum_{i=1}^{N} i^3 T_{i,N} \cos(ix)\right)^3 \cos(jx)\, dx\right]. \tag{14}$$

The existence of a solution of system (12) follows from the Picard theorem, because the right-hand sides of Eqs. (14) have continuous partial derivatives with respect to the variables $T_{j,N}$, $Y_{j,N}$, $j = 1, \dots, N$. Therefore, the Cauchy problem (13) for system (12) is locally solvable, and its solution is unique, while its extendability to all $t \ge 0$ follows from the fact that the solutions of this system satisfy the energy identity

$$\sum_{j=1}^{N} \dot{T}_{j,N}^2 (j^2 + C_2 j^4) + C_0 \sum_{j=1}^{N} j^6 T_{j,N}^2 + \frac{C_1}{2} \int\limits_0^{\pi} \left(\sum_{j=1}^{N} j^3 T_{j,N} \cos(jx)\right)^4 dx = h_N, \tag{15}$$

where we have denoted

$$h_N = \sum_{j=1}^{N} \beta_j^2 (j^2 + C_2 j^4) + C_0 \sum_{j=1}^{N} j^6 \alpha_j^2 + \frac{C_1}{2} \int\limits_0^{\pi} \left(\sum_{i=1}^{N} i^3 \alpha_i \cos(ix)\right)^4 dx, \quad N \in \mathbb{N}. \tag{16}$$

Identity (15), with allowance for (16), is obtained by multiplying the jth equation in system (10) by $j^2 \dot{T}_{j,N}$, integration over the interval $[0, t]$, and subsequent summation over $j = 1, \dots, N$ with allowance for the relations

$$\left(\left(\sum_{j=1}^{N} j^3 T_{j,N} \cos(jx)\right)^4\right)' = 4\left(\sum_{j=1}^{N} j^3 T_{j,N} \cos(jx)\right)^3 \left(\sum_{j=1}^{N} j^3 \dot{T}_{j,N} \cos(jx)\right),$$

$$(\dot{T}_{j,N}^2)' = 2\dot{T}_{j,N}\ddot{T}_{j,N}, \quad (T_{j,N}^2)' = 2\dot{T}_{j,N} T_{j,N}.$$

Thus, we must show that the solutions of the finite systems (10) converge to the solution of the infinite system (6) as $N \to \infty$. It follows from identity (15) that for each $N \in \mathbb{N}$ the quantities $|T_{j,N}(t)|$, $j = 1, \dots, N$, are uniformly bounded with respect to t, and hence for each N the solution of system (10) (or, equivalently, system (12)) is infinitely extendable, because its right-hand side is continuously differentiable.

If the sequence h_N defined by the relation (6) is convergent as $N \to \infty$, i.e., if there exists a finite limit $\lim\limits_{N \to \infty} h_N = h < \infty$, then identity (15) implies that the sequences $|T_{j,N}(t)|$, $N \in \mathbb{N}$, and $|\dot{T}_{j,N}(t)|$, $N \in \mathbb{N}$, are uniformly bounded on any interval of the real line. Then, according to the Arzelà–Ascoli lemma, for each interval $[0, t^*]$ ($t^* > 0$ is arbitrary) and any $j \in \mathbb{N}$ the sequence $\{T_{j,N}(t)\}$ contains a subsequence that uniformly converges on the interval $[0, t^*]$ to a function that is continuous on this interval. Moreover, by a standard diagonal argument one can readily show that there exists a sequence N_i, $i \in \mathbb{N}$, of positive integers such that for each $j \in \mathbb{N}$ the subsequence $\{T_{j,N_i}(t)\}$ of the sequence $\{T_{j,N}(t)\}$ converges uniformly on the interval $[0, t^*]$ to some function $T_j(t)$ that is continuous on this interval. To prove that the functions $T_j(t)$ thus-defined are solutions of (6), one has to derive more accurate estimates for the functions $T_{j,N}$ than the ones that readily follow from the energy identities (15).

The desired estimates can be obtained by multiplying the jth equation in system (10) by $j^6\dot{T}_{j,N}$ and by summing over all $j = 1, \dots, N$. The resulting expression can be written in the form

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{j=1}^{N}\dot{T}_{j,N}^2\left(j^6 + C_2 j^8\right) + C_0\sum_{j=1}^{\infty} j^{10}T_{j,N}^2\right) + C_1\int_0^\pi \left(w_{xxx}^{(N)}\right)^3 w_{xxxxxxxt}^{(N)}\,dx = 0, \tag{17}$$

where we have denoted

$$w^{(N)}(t,x) = \sum_{j=1}^{N} T_{j,N}(t)\sin(jx). \tag{18}$$

Twice integrating by parts in the last term in relation (17) with allowance for the fact that the even derivatives of the function $w^{(N)}$ with respect to x vanish at the points 0 and π, we obtain

$$\begin{aligned}\int_0^\pi \left(w_{xxx}^{(N)}\right)^3 w_{xxxxxxxt}^{(N)}\,dx &= \left(w_{xxx}^{(N)}\right)^3 w_{xxxxxxt}^{(N)}\Big|_0^\pi - 3\int_0^\pi w_{xxxx}^{(N)}\left(w_{xxx}^{(N)}\right)^2 w_{xxxxxxt}^{(N)}\,dx \\ &= -3w_{xxxx}^{(N)}\left(w_{xxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\Big|_0^\pi + 3\int_0^\pi w_{xxxxx}^{(N)}\left(w_{xxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\,dx + 6\int_0^\pi w_{xxx}^{(N)}\left(w_{xxxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\,dx \\ &= 3\int_0^\pi w_{xxxxx}^{(N)}\left(w_{xxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\,dx + 6\int_0^\pi w_{xxx}^{(N)}\left(w_{xxxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\,dx,\end{aligned}$$

or

$$\begin{aligned}\int_0^\pi \left(w_{xxx}^{(N)}\right)^3 w_{xxxxxxxt}^{(N)}\,dx = \frac{3}{2}\frac{d}{dt}&\int_0^\pi \left(w_{xxx}^{(N)}\right)^2\left(w_{xxxxx}^{(N)}\right)^2 dx \\ &- 3\int_0^\pi w_{xxxxx}^{(N)}\left(w_{xxxxx}^{(N)}\right)^2 w_{xxxt}^{(N)}\,dx + 6\int_0^\pi w_{xxx}^{(N)}\left(w_{xxxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\,dx.\end{aligned} \tag{19}$$

Relation (18) implies the identities

$$\int_0^\pi \left(w_{xxxt}^{(N)}\right)^2 dx = \frac{\pi}{2}\sum_{j=1}^{N} j^6\dot{T}_{j,N}^2,\quad \int_0^\pi \left(w_{xxxxt}^{(N)}\right)^2 dx = \frac{\pi}{2}\sum_{j=1}^{N} j^8\dot{T}_{j,N}^2,\quad \int_0^\pi \left(w_{xxxxx}^{(N)}\right)^2 dx = \frac{\pi}{2}\sum_{j=1}^{N} j^{10}T_{j,N}^2,$$

$$\int_0^\pi \left(w_{xxx}^{(N)}\right)^2\left(w_{xxxxx}^{(N)}\right)^2 dx = \int_0^\pi \left(\sum_{j=1}^{N} j^3 T_{j,N}\cos(jx)\right)^2\left(\sum_{j=1}^{N} j^5 T_{j,N}\cos(jx)\right)^2 dx.$$

Then relations (17) and (19) can be written as

$$\frac{d}{dt}E_N(t) = 6C_1\int_0^\pi w_{xxx}^{(N)} w_{xxxt}^{(N)}\left(w_{xxxxx}^{(N)}\right)^2 dx - 12C_1\int_0^\pi w_{xxx}^{(N)}\left(w_{xxxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\,dx, \tag{20}$$

where

$$
\begin{aligned}
E_N(t) &= \sum_{j=1}^{N}\left(j^6+C_2 j^8\right)\dot{T}_{j,N}^2 + C_0\sum_{j=1}^{N} j^{10}T_{j,N}^2 \\
&\quad + 3C_1\int_0^{\pi}\left(\sum_{j=1}^{N} j^3 T_{j,N}\cos(jx)\right)^2\left(\sum_{j=1}^{N} j^5 T_{j,N}\cos(jx)\right)^2 dx \\
&= \frac{2C_2}{\pi}\int_0^{\pi}\left(w_{xxxxt}^{(N)}\right)^2 dx + \frac{2}{\pi}\int_0^{\pi}\left(w_{xxxt}^{(N)}\right)^2 dx \\
&\quad + \frac{2C_0}{\pi}\int_0^{\pi}\left(w_{xxxxx}^{(N)}\right)^2 dx + 3C_1\int_0^{\pi}\left(w_{xxx}^{(N)}\right)^2\left(w_{xxxxx}^{(N)}\right)^2 dx.
\end{aligned}
\tag{21}
$$

In what follows, we estimate the right-hand side of relation (20) via the energy integral E_N. Since

$$w^{(N)}(0,t) = w^{(N)}(\pi,t) = w_{xx}^{(N)}(0,t) = w_{xx}^{(N)}(\pi,t) = 0,$$

it follows from the Rolle theorem that there exist points $\zeta = \zeta(t)$ and $\eta = \eta(t)$ such that $w_x^{(N)}(\zeta,t) = 0$ and $w_{xxx}^{(N)}(\eta,t) = 0$. Then

$$|w_{xxx}^{(N)}| \le \left|\int_{\eta}^{x} w_{xxxx}^{(N)}\,dx\right| \le \int_0^{\pi}|w_{xxxx}^{(N)}|\,dx \le \left(\pi\int_0^{\pi}(w_{xxxx}^{(N)})^2\,dx\right)^{1/2}. \tag{22}$$

Moreover, as follows from (18), we have $w_{xx}^{(N)}(0,t) = 0$ and $w_{xxxx}^{(N)}(0,t) = 0$, and therefore,

$$\left|w_{xxxx}^{(N)}\right| = \left|\int_0^{x} w_{xxxxx}^{(N)}\,dx\right| \le \int_0^{\pi}\left|w_{xxxxx}^{(N)}\right|dx \le \left(\pi\int_0^{\pi}\left(w_{xxxxx}^{(N)}\right)^2 dx\right)^{1/2}. \tag{23}$$

In a similar way, one can show that

$$\left|w_{xxxt}^{(N)}\right| = \left|\int_0^{x} w_{xxxxt}^{(N)}\,dx\right| \le \int_0^{\pi}\left|w_{xxxxt}^{(N)}\right|dx \le \left(\pi\int_0^{\pi}\left(w_{xxxxt}^{(N)}\right)^2 dx\right)^{1/2}. \tag{24}$$

Inequalities (22)–(24) imply an estimate for the derivatives $w_{xxx}^{(N)}$, $w_{xxxx}^{(N)}$ and $w_{xxxt}^{(N)}$ via the energy integral E_N given by formula (21). Thus, we have

$$
\begin{aligned}
&\left|w_{xxxx}^{(N)}\right| \le \left(\pi/\sqrt{2C_0}\right)E_N^{1/2}, \\
&\left|w_{xxx}^{(N)}\right| \le \left(\pi\int_0^{\pi}\left(w_{xxxx}^{(N)}\right)^2 dx\right)^{1/2} \le \left(\pi^2/\sqrt{2C_0}\right)E_N^{1/2},
\end{aligned}
\tag{25}
$$

$$\left|w_{xxxt}^{(N)}\right| \le \left(\pi/\sqrt{2C_2}\right)E_N^{1/2}. \tag{26}$$

Using inequalities (25) and (26), we estimate the first term on the right-hand side in relation (20) as

$$
\begin{aligned}
\left|\int_0^{\pi} w_{xxx}^{(N)} w_{xxxt}^{(N)}\left(w_{xxxxx}^{(N)}\right)^2 dx\right| &\le \int_0^{\pi}\left|w_{xxx}^{(N)} w_{xxxt}^{(N)}\left(w_{xxxxx}^{(N)}\right)^2\right| dx \\
&\le \frac{\pi^3}{2\sqrt{C_0C_2}}E_N\int_0^{\pi}\left(w_{xxxxx}^{(N)}\right)^2 dx \le \frac{\pi^4}{4\sqrt{C_0^3C_2}}E_N^2.
\end{aligned}
\tag{27}
$$

After integration by parts, for the second term on the right-hand side in relation (20) we obtain the estimate

$$\begin{aligned}
&\left|\int_0^\pi w_{xxx}^{(N)}\left(w_{xxxx}^{(N)}\right)^2 w_{xxxxxt}^{(N)}\,dx\right| \\
&\le \left|\int_0^\pi \left(w_{xxxx}^{(N)}\right)^3 w_{xxxxt}^{(N)}\,dx\right| + 2\left|\int_0^\pi w_{xxx}^{(N)} w_{xxxx}^{(N)} w_{xxxxx}^{(N)} w_{xxxxt}^{(N)}\,dx\right| \\
&\le \int_0^\pi \left|\left(w_{xxxx}^{(N)}\right)^3 w_{xxxxt}^{(N)}\right|\,dx + 2\int_0^\pi \left|w_{xxx}^{(N)} w_{xxxx}^{(N)} w_{xxxxx}^{(N)} w_{xxxxt}^{(N)}\right|\,dx \\
&\le \frac{\pi^3}{2\sqrt{2C_0^3}} E_N^{3/2}\left(\int_0^\pi \left(w_{xxxxt}^{(N)}\right)^2 dx\right)^{1/2} + \frac{\pi^3}{2C_0} E_N\left(\int_0^\pi \left(w_{xxxxx}^{(N)}\right)^2 dx \int_0^\pi \left(w_{xxxxt}^{(N)}\right)^2 dx\right)^{1/2} \\
&\le \frac{\pi^4}{4\sqrt{C_0^3 C_2}} E_N^2 + \frac{\pi^4}{4\sqrt{C_0^3 C_2}} E_N^2 = \frac{\pi^4}{2\sqrt{C_0^3 C_2}} E_N^2.
\end{aligned} \tag{28}$$

By virtue of the estimates (27) and (28), relation (20) implies the inequality

$$\frac{dE_N}{dt} \le \frac{15\pi^4 C_1}{2\sqrt{C_0^3 C_2}} E_N^2, \tag{29}$$

which gives the desired estimate for the function $E_N(t)$.

Lemma 1. *Assume that there exists a finite limit*

$$\begin{aligned}
\lim_{N\to\infty} E_N(0) = E_0 = \sum_{j=1}^{\infty}\left(j^6 + C_2 j^8\right)\beta_j^2 + C_0\sum_{j=1}^{\infty} j^{10}\alpha_j^2 \\
+ 3C_1\int_0^\pi \left(\sum_{j=1}^{\infty} j^3\alpha_j\cos(jx)\right)^2\left(\sum_{j=1}^{\infty} j^5\alpha_j\cos(jx)\right)^2 dx < \infty;
\end{aligned} \tag{30}$$

i.e., the sequence $E_N(0)$ converges as $N\to\infty$. Then the function sequence $E_N(t)$, $N\in\mathbb{N}$, is uniformly bounded on any interval $0\le t\le t^* < t_c$, where

$$t_c = \frac{2\sqrt{C_0^3 C_2}}{15\pi^4 C_1 E_0}.$$

Proof. From inequality (29), we obtain the estimate

$$E_N(t) \le \frac{2\sqrt{C_0^3 C_2}\,E_N(0)}{2\sqrt{C_0^3 C_2} - 15\pi^4 C_1 E_N(0) t} \tag{31}$$

for

$$0 \le t < t_N = \frac{2\sqrt{C_0^3 C_2}}{15\pi^4 C_1 E_N(0)}.$$

Then $t_c = \lim\limits_{N\to\infty} t_N$.

Let $t_{n_1}, \ldots, t_{n_k}$ be all of the t_N, $N\in\mathbb{N}$, that are not greater than t^*. As shown above, the functions $T_{j,N}(t)$ and $\dot{T}_{j,N}(t)$ are defined and continuous on any interval and, in particular, on the interval $[0,t^*]$. Consequently, the $E_{n_i}(t)$, $i=1,\dots,k$, are bounded on the interval $[0,t^*]$ by some constant. For the remaining t_N, i.e., for $N\notin\{n_1,\dots,n_k\}$, one has the inequality $t_N>t^*$, and therefore, by virtue of the estimate (31), for such N we have

$$E_N(t)\le\frac{2\sqrt{C_0^3C_2}E_N(0)}{2\sqrt{C_0^3C_2}-15\pi^2C_1E_N(0)t^*}\le\frac{2\sqrt{C_0^3C_2}m}{2\sqrt{C_0^3C_2}-15\pi^2C_1mt^*},\quad t\in[0,t^*],$$

where $m=\max\{E_N(0):N\in\mathbb{N}\setminus\{n_1,\dots,n_k\}\}$. The proof of the lemma is complete.

The uniform boundedness of the sequence $E_N(t)$, $N\in\mathbb{N}$, is instrumental in proving that the functions $T_j(t)$, i.e., the limits of the subsequence $T_{j,N_i}(t)$, are solutions of system (6). This result is a consequence of the following two lemmas.

Lemma 2. *If $E_0<\infty$ (see (30)), then the series $\sum_{j=1}^\infty j^{10}T_j^2(t)$ is convergent on any interval $[0,t^*]$, where $t^*<t_c$.*

Proof. The sequence of functions

$$S_n(t)=\sum_{j=1}^n j^{10}T_j^2(t)\tag{32}$$

is nondecreasing, because $S_{n+1}(t)\ge S_n(t)$. Consequently, to prove that the sequence (32) converges, it suffices to prove that it is uniformly bounded.

Since $T_{j,N_i}(t)\rightrightarrows T_j(t)$ on $[0,t^*]$ as $i\to\infty$, it follows that for each $n\in\mathbb{N}$ there exists a number $N_{i(n)}\ge n$ such that the inequality $|T_j^2(t)-T_{j,N_{i(n)}}^2(t)|\le n^{-12}$ holds for all $j=1,\dots,n$ and $t\in[0,t^*]$. Then

$$S_n\le\sum_{j=1}^n j^{10}\left|T_j^2(t)-T_{j,N_{i(n)}}^2(t)\right|+\sum_{j=1}^n j^{10}T_{j,N_{i(n)}}^2(t)\le\sum_{j=1}^n j^{10}n^{-12}+\frac{1}{C_0}E_{N_{i(n)}}(t)\le\frac{1}{n}+\frac{1}{C_0}E_{N_{i(n)}}(t).$$

Since the sequence $E_N(t)$ is uniformly bounded on the interval $[0,t^*]$ by Lemma 1, we see that this estimate implies the uniform boundedness of the sequence S_n, $n\in\mathbb{N}$, on that interval. The proof of the lemma is complete.

Lemma 3. *If $E_0<\infty$, then the functions $w^{(N_i)}$ and $w_{xxx}^{(N_i)}$ uniformly converge to w and w_{xxx}, respectively, as $N_i\to\infty$ on each interval $[0,t^*]$, where $t^*<t_c$.*

Proof. To prove that $w^{(N_i)}\rightrightarrows w$ and $w_{xxx}^{(N_i)}\rightrightarrows w_{xxx}$ as $i\to\infty$, note that Lemma 1 implies the following estimates: $T_j^2\le M/j^{10}$ and $T_{j,N_i}^2\le M/j^{10}$, where M is a constant bounding the sequence $E_N(t)$ on the interval $0\le t\le t^*$.

Fix an $\varepsilon>0$ and choose an $n\in\mathbb{N}$ such that $(1+2\sqrt{M})n^{-1}\le\varepsilon$. Let us estimate the modulus of the difference $w_{xxx}^{(N_i)}-w_{xxx}$ as follows (for $N_i\ge n$):

$$\begin{aligned}\left|w_{xxx}^{(N_i)}(t,x)-w_{xxx}(t,x)\right|&\le\left|\sum_{j=1}^n j^3(T_j(t)-T_{j,N_i}(t))\cos(jx)\right|\\&\quad+\sum_{j=n+1}^\infty j^3|T_j(t)|+\sum_{j=n+1}^{N_i}j^3|T_{j,N_i}(t)|\\&\le\sum_{j=1}^n j^3|T_j(t)-T_{j,N_i}(t)|+2\sqrt{M}\sum_{j=n+1}^\infty j^{-2}\\&=\sum_{j=1}^n j^3|T_j(t)-T_{j,N_i}(t)|+2\sqrt{M}n^{-1}.\end{aligned}\tag{33}$$

Since $T_{j,N_i}(t) \rightrightarrows T_j(t)$ on $[0,t^*]$ as $i\to\infty$, it follows that for each $n\in\mathbb{N}$ there exists a number $k(n)$ such that the inequality $|T_j(t)-T_{j,N_i}(t)|\le n^{-5}$ holds for all $i\ge k(n)$, $t\in[0,t^*]$, and $j=1,\dots,n$. Then, assuming that $i\ge k(n)$ in inequality (33), we obtain

$$\left|w_{xxx}^{(N_i)}(t,x)-w_{xxx}(t,x)\right|\le (1+2\sqrt{M})n^{-1}\le\varepsilon \quad \text{for all} \quad t\in[0,t^*] \text{ and } i\ge k(n).$$

The proof of the convergence $w^{(N_i)}\rightrightarrows w$ can be carried out in a similar way. The proof of the lemma is complete.

Theorem 1. *If* $E_0<\infty$, *then the functions* $T_j(t)$ *are solutions of system* (6) *and satisfy the initial conditions* (7) *and condition* (8) *in the interval* $0<t<t^*$.

Proof. The functions $T_{j,N_i}(t)$ satisfy the Volterra integral equation

$$\begin{aligned} T_{j,N_i}(t)&=\tilde{\alpha_j}+\tilde{\beta}_j t-\int_0^t (t-\tau)\left\{C_{0,j}T_{j,N_i}(\tau)+C_{1,j}\int_0^\pi \left(w_{xxx}^{(N_i)}(\tau,x)\right)^3\cos(jx)\,dx\right\}d\tau\\ &=\tilde{\alpha}_j+\tilde{\beta}_j t-G_j\left(w_{xxx}^{(N_i)},T_{j,N_i}\right),\quad j=1,\dots,N_i,\end{aligned}$$

where

$$\tilde{\alpha_j}=\frac{\alpha_j}{1+C_2j^2},\quad \tilde{\beta}_j=\frac{\beta_j}{1+C_2j^2},\quad C_{0,j}=\frac{j^4C_0}{1+C_2j^2},\quad C_{1,j}=\frac{jC_1}{1+C_2j^2},$$

$$G_j(w(t,x),T(t))=\int_0^t (t-\tau)\left\{C_{0,j}T(\tau)+C_{1,j}\int_0^\pi w^3(\tau,x)\cos(jx)\,dx\right\}d\tau.$$

Let us show that the functions $T_j(t)$ satisfy a similar equation. The following estimate holds:

$$\begin{aligned}\left|T_j-\tilde{\alpha}_j-\tilde{\beta}_j t+G_jw_{xxx}\right|&=\left|T_j-T_{j,N_i}-G_jw_{xxx}^{(N_i)}+G_jw_{xxx}\right|\\ &\le \|T_j-T_{j,N_i}\|_C+C_{0,j}t^*\|T_j-T_{j,N_i}\|_C+C_{1,j}t^*\left\|\int_0^\pi\left(\left(w_{xxx}^{(N_i)}\right)^3-w_{xxx}^3\right)\cos(jx)\,dx\right\|_C,\end{aligned}\tag{34}$$

where $\|f\|_C=\max\limits_{0\le t\le t^*}|f(t)|$. Since $T_{j,N_i}\rightrightarrows T_j$ and, as shown in Lemma 3, $w_{xxx}^{N_i}\rightrightarrows w_{xxx}$ on the interval $[0,t^*]$ as $i\to\infty$, it follows that the right-hand side of inequality (34) tends to zero as $i\to\infty$. Therefore, the function $T_j(t)$ can be represented in the form

$$T_j(t)=\tilde{\alpha}_j+\tilde{\beta}_j t-G_jw_{xxx.}\tag{35}$$

Since the right-hand side of identity (35) is differentiable with respect to t, it follows that so is its left-hand side. By differentiating this identity with respect to t, we obtain

$$\dot{T}_j(t)=\tilde{\beta}_j-\int_0^t\left\{C_{0,j}T_j(\tau)+C_{1,j}\int_0^\pi w_{xxx}^3(\tau,x)\cos(jx)\,dx\right\}d\tau.\tag{36}$$

By the same pattern, the right-hand side of identity (36) is differentiable with respect to t, and hence so is its left-hand side. By differentiating identity (36) with respect to t, we arrive at the desired assertion. The proof of the theorem is complete.

2. UNIQUENESS OF THE SOLUTION OF PROBLEM (1)–(4)

Let us prove that the solution of problem (1)–(4) is unique.

Lemma 4. *If there exists a solution of problem* (1)–(4), *then one has the relation*

$$\|w_{xt}\|^2_{L_2[0,\pi]} + C_0\|w_{xx}\|^2_{L_2[0,\pi]} + \frac{C_1}{6}\|w^2_{xxx}\|^2_{L_2[0,\pi]} + C_2\|w_{xxt}\|^2_{L_2[0,\pi]}$$
$$= \|\psi'(x)\|^2_{L_2[0,\pi]} + C_0\|\varphi'''(x)\|^2_{L_2[0,\pi]} + \frac{C_1}{6}\|(\varphi'''(x))^2\|^2_{L_2[0,\pi]} + C_2\|\psi''(x)\|^2_{L_2[0,\pi]}$$

and the estimate

$$\|w_{xxt}\|^2_{L_2[0,\pi]} + C_0\|w_{xxxx}\|^2_{L_2[0,\pi]} + C_1\|w_{xxxx}w_{xxx}\|^2_{L_2[0,\pi]} + C_2\|w_{xxxt}\|^2_{L_2[0,\pi]}$$
$$\le C(\|\psi''(x)\|^2_{L_2[0,\pi]} + C_0\|\varphi''''(x)\|^2_{L_2[0,\pi]} + C_1\|\varphi'''(x)\varphi''''(x)\|^2_{L_2[0,\pi]} + C_2\|\psi'''(x)\|^2_{L_2[0,\pi]}),$$

where C is a positive constant.

Proof. First, we prove the equality. To this end, we multiply Eq. (1) by w_{xxt} and integrate the resulting relation with respect to x over the interval $[0,\pi]$. Then we obtain

$$\int\limits_0^\pi w_{tt}w_{xxt}\,dx + \int\limits_0^\pi C_0 w_{xxxx}w_{xxt}\,dx + \int\limits_0^\pi C_1 w^2_{xxx}w_{xxxx}w_{xxt}\,dx - \int\limits_0^\pi C_2 w_{xxtt}w_{xxt}\,dx = 0. \qquad (37)$$

Let us calculate by parts the integrals in relation (37) while considering the boundary conditions (4). Taking into account the obvious relations

$$\int\limits_0^\pi w_{tt}w_{xxt}\,dx = w_{tt}w_{xt}\Big|_0^\pi - \int\limits_0^\pi w_{xt}w_{xtt}\,dx = -\frac{1}{2}\frac{d}{dt}\int\limits_0^\pi w^2_{xt}\,dx,$$

$$\int\limits_0^\pi w_{xxxx}w_{xxt}\,dx = w_{xxx}w_{xxt}\Big|_0^\pi - \int\limits_0^\pi w_{xxx}w_{xxxt}\,dx = -\frac{1}{2}\frac{d}{dt}\int\limits_0^\pi w^2_{xxx}\,dx,$$

$$\int\limits_0^\pi w^2_{xxx}w_{xxxx}w_{xxt}\,dx = \frac{1}{3}w^3_{xxx}w_{xxt}\Big|_0^\pi - \frac{1}{3}\int\limits_0^\pi w^3_{xxx}w_{xxxt}\,dx = -\frac{1}{12}\frac{d}{dt}\int\limits_0^\pi w^4_{xxx}\,dx,$$

$$\int\limits_0^\pi w_{xxtt}w_{xxt}\,dx = \frac{1}{2}\frac{d}{dt}\int\limits_0^\pi w^2_{xxt}\,dx,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\Bigg(\int\limits_0^\pi w^2_{xt}\,dx + C_0\int\limits_0^\pi w^2_{xxx}\,dx + \frac{C_1}{6}\int\limits_0^\pi w^4_{xxx}\,dx + C_2\int\limits_0^\pi w^2_{xxt}\,dx\Bigg) = 0.$$

Integrating the last relation over variable t on the interval $[0,t]$, we arrive at the assertion in the lemma,

$$\int\limits_0^\pi w^2_{xt}\,dx + C_0\int\limits_0^\pi w^2_{xxx}\,dx + \frac{C_1}{6}\int\limits_0^\pi w^4_{xxx}\,dx + C_2\int\limits_0^\pi w^2_{xxt}\,dx$$
$$= \int\limits_0^\pi (\psi'(x))^2\,dx + C_0\int\limits_0^\pi (\varphi'''(x))^2\,dx + \frac{C_1}{6}\int\limits_0^\pi (\varphi'''(x))^4\,dx + C_2\int\limits_0^\pi (\psi''(x))^2\,dx.$$

Let us proceed to proving the estimate. We multiply Eq. (1) by w_{xxxxt}. Integrating the resulting relation with respect to x over the interval $[0,\pi]$, we obtain

$$\int\limits_0^\pi w_{tt}w_{xxxxt}\,dx + \int\limits_0^\pi C_0 w_{xxxx}w_{xxxxt}\,dx + \int\limits_0^\pi C_1 w^2_{xxx}w_{xxxx}w_{xxxxt}\,dx - \int\limits_0^\pi C_2 w_{xxtt}w_{xxxxt}\,dx = 0. \quad (38)$$

Let us calculate by parts the integrals in relation (38) with allowance for the boundary conditions (4). Taking into account the obvious relations

$$\int_0^\pi w_{tt}w_{xxxxt}\,dx = w_{tt}w_{xxxt}\Big|_0^\pi - \int_0^\pi w_{xtt}w_{xxxt}\,dx$$
$$= -w_{xxtt}w_{xxt}\Big|_0^\pi + \int_0^\pi w_{xxtt}w_{xxt}\,dx = \frac{1}{2}\frac{d}{dt}\int_0^\pi w_{xxt}^2\,dx,$$
$$\int_0^\pi w_{xxxx}w_{xxxxt}\,dx = \frac{1}{2}\frac{d}{dt}\int_0^\pi w_{xxxx}^2\,dx,$$
$$\int_0^\pi w_{xxx}^2 w_{xxxx}w_{xxxxt}\,dx = \frac{1}{2}\frac{d}{dt}\int_0^\pi w_{xxx}^2 w_{xxxx}^2\,dx - \int_0^\pi w_{xxx}w_{xxxt}w_{xxxx}^2\,dx,$$
$$\int_0^\pi w_{xxtt}w_{xxxxt}\,dx = w_{xxtt}w_{xxxt}\Big|_0^\pi - \int_0^\pi w_{xxx}w_{xxxt}\,dx = -\frac{1}{2}\frac{d}{dt}\int_0^\pi w_{xxxt}^2\,dx,$$

we obtain

$$\frac{1}{2}\frac{dE}{dt} - C_1\int_0^\pi w_{xxx}w_{xxxt}w_{xxxx}^2\,dx = 0, \tag{39}$$

where we have denoted

$$E = \int_0^\pi w_{xxt}^2\,dx + C_0\int_0^\pi w_{xxxx}^2\,dx + C_1\int_0^\pi w_{xxx}^2 w_{xxxx}^2\,dx + C_2\int_0^\pi w_{xxxt}^2\,dx.$$

Let us estimate the second term in relation (39),

$$\left|\int_0^\pi w_{xxx}w_{xxxt}w_{xxxx}^2\,dx\right| \le \int_0^\pi |w_{xxx}||w_{xxxt}|w_{xxxx}^2\,dx$$
$$\le \frac{1}{2}\int_0^\pi \left(w_{xxx}^2 + w_{xxxt}^2\right) w_{xxxx}^2\,dx \le \frac{1}{2}\int_0^\pi w_{xxx}^2 w_{xxxx}^2\,dx + \int_0^\pi w_{xxxt}^2 w_{xxxx}^2\,dx$$
$$\le \frac{1}{2}\int_0^\pi w_{xxx}^2 w_{xxxx}^2\,dx + K\int_0^\pi w_{xxxx}^2\,dx \le \frac{1}{2}ME,$$

where a positive constant K is an upper bound for the continuous function w_{xxxt}^2 and $M = 1 + 2K$ is constant. Taking the last estimate into account in relation (39), we arrive at the inequality

$$\frac{dE}{dt} \le C_1 ME,$$

which, after the integration with respect to the variable t over the interval $[0, T]$, gives

$$\|w_{xxt}\|_{L_2[0,\pi]}^2 + C_0\|w_{xxxx}\|_{L_2[0,\pi]}^2 + C_1\|w_{xxxx}w_{xxx}\|_{L_2[0,\pi]}^2 + C_2\|w_{xxxt}\|_{L_2[0,\pi]}^2$$
$$\le C(\|\psi''(x)\|_{L_2[0,\pi]}^2 + C_0\|\varphi''''(x)\|_{L_2[0,\pi]}^2 + C_1\|\varphi'''(x)\varphi''''(x)\|_{L_2[0,\pi]}^2 + C_2\|\psi'''(x)\|_{L_2[0,\pi]}^2).$$

The proof of the lemma is complete.

Theorem 2. *If there exists a solution of problem* (1)–(4), *then it is unique.*

Proof. Assume that there exist two solutions w_1 and w_2 of problem (1)–(4). Then the function $w = w_1 - w_2$ satisfies the initial conditions (3) with $\varphi(x) = \psi(x) = 0$, the boundary conditions (4), and the equation

$$w_{tt} + C_0 w_{xxxx} + C_1 \left(w_{1xxx}^2 w_{1xxxx} - w_{2xxx}^2 w_{2xxxx}\right) - C_2 w_{xxtt} = 0, \quad C_i > 0, \quad i = 0, 1, 2. \tag{40}$$

Let us multiply Eq. (40) by

$$w_{xxt} = w_{1xxt} - w_{2xxt}.$$

By integrating the resulting relation with respect to x over the interval $[0, \pi]$, we obtain

$$\begin{gathered}\int\limits_0^\pi w_{tt} w_{xxt}\,dx + C_0 \int\limits_0^\pi w_{xxxx} w_{xxt}\,dx \\ + C_1 \int\limits_0^\pi \left(w_{1xxx}^2 w_{1xxxx} - w_{2xxx}^2 w_{2xxxx}\right)\left(w_{1xxt} - w_{2xxt}\right)\,dx - C_2 \int\limits_0^\pi w_{xxtt} w_{xxt}\,dx = 0.\end{gathered} \tag{41}$$

We integrate by parts in relation (41) with allowance for the boundary conditions (4) to obtain

$$\frac{dE}{dt} - \frac{C_1}{3}\int\limits_0^\pi w_{xxx}^2 (2w_{1xxx}w_{1xxxt} + w_{1xxxt}w_{2xxx} + w_{1xxx}w_{2xxxt} + 2w_{2xxx}w_{2xxxt})\,dx = 0, \tag{42}$$

where

$$E = \int\limits_0^\pi w_{xt}^2\,dx + C_0 \int\limits_0^\pi w_{xxx}^2\,dx + \frac{C_1}{3}\int\limits_0^\pi w_{xxx}^2 (w_{1xxx}^2 + w_{1xxx}w_{2xxx} + w_{2xxx}^2)\,dx + C_2 \int\limits_0^\pi w_{xxt}^2\,dx.$$

Let us estimate the second term in Eq. (42),

$$\begin{aligned} I &= \left|\int\limits_0^\pi w_{xxx}^2 (2w_{1xxx}w_{1xxxt} + w_{1xxxt}w_{2xxx} + w_{1xxx}w_{2xxxt} + 2w_{2xxx}w_{2xxxt})\,dx\right| \\ &\le 2\int\limits_0^\pi w_{xxx}^2 (w_{1xxx}^2 + w_{1xxxt}^2 + w_{2xxx}^2 + w_{2xxxt}^2)\,dx. \end{aligned}$$

Then, since the functions w_{ixxx}^2 and w_{ixxxt}^2, $i = 1, 2$, are bounded, one has the estimate

$$I \le M \int\limits_0^\pi w_{xxx}^2\,dx \le ME,$$

where M is a positive constant. Consequently, one has the inequality

$$\frac{dE}{dt} \le ME,$$

from which, taking into account the homogeneous boundary and initial conditions, we conclude that $w(x, t) \equiv 0$. The proof of the theorem is complete.

3. SOLVABILITY OF PROBLEM (1)–(4) FOR SMALL T

Based on the results in Sec. 1 and Sec. 2, we arrive at the following main result.

Theorem 3. *Under the conditions*

$$\varphi(x) \in C^6[0,\pi], \quad \varphi(0) = \varphi(\pi) = \varphi''(0) = \varphi''(\pi) = \varphi^{(4)}(0) = \varphi^{(4)}(\pi) = 0,$$
$$\psi(x) \in C^4[0,\pi], \quad \psi(0) = \psi(\pi) = \psi''(0) = \psi''(\pi) = 0,$$

there exists a unique solution of problem (1)–(4) *on the interval* $0 \le t \le t^*$.

Proof. By formally differentiating the series (9) term by term, we compose the series

$$w_{tt} = \sum_{j=1}^{\infty} \ddot{T}_j \sin(jx), \quad w_{ttxx} = -\sum_{j=1}^{\infty} j^2 \ddot{T}_j \sin(jx), \tag{43}$$

$$w_{xxx} = -\sum_{j=1}^{\infty} j^3 T_j \cos(jx), \quad w_{xxxx} = \sum_{j=1}^{\infty} j^4 T_j \sin(jx). \tag{44}$$

Let us establish estimates for each of these series.

According to Lemma 1, the sequence $E_N(t)$, $N \in \mathbb{N}$, is uniformly bounded on the interval $[0, t^*]$ provided that the quantity E_0 defined by relation (30) is finite.

To prove the finiteness of E_0, consider the coefficients in (5). For these coefficients, taking into account the assumptions of the theorem, one has the representations

$$\alpha_i = -\frac{2}{i^6\pi}\int_0^\pi \varphi^{(6)}(s)\sin(i\pi s)\,ds = -\frac{q_i}{i^6}, \quad \beta_i = \frac{2}{i^4\pi}\int_0^\pi \psi^{(4)}(s)\sin(i\pi s)\,ds = \frac{p_i}{i^4}.$$

Then, in view of these representations, we obtain

$$\begin{aligned} E_0 &= \sum_{j=1}^{\infty}(j^6 + C_2 j^8)\beta_j^2 + C_0\sum_{j=1}^{\infty} j^{10}\alpha_j^2 + 3C_1\int_0^\pi \left(\sum_{j=1}^{\infty} j^3\alpha_j\cos(jx)\right)^2 \left(\sum_{j=1}^{\infty} j^5\alpha_j\cos(jx)\right)^2 dx \\ &< \sum_{j=1}^{\infty}\left(\frac{1}{j^2} + C_2\right)p_j^2 + C_0\sum_{j=1}^{\infty}\frac{1}{j^2}q_j^2 + 3\pi C_1\left(\sum_{j=1}^{\infty} q_j^2\right)^2 \sum_{j=1}^{\infty}\frac{1}{j^6}\sum_{j=1}^{\infty}\frac{1}{j^2}. \end{aligned}$$

The right-hand side of this inequality is a sum of convergent series by virtue of the estimates

$$\sum_{j=1}^{\infty} p_j^2 \le \frac{2}{\pi}\int_0^\pi [\varphi^{(6)}(s)]^2\,ds \quad \text{and} \quad \sum_{j=1}^{\infty} q_j^2 \le \frac{2}{\pi}\int_0^\pi [\psi^{(4)}(s)]^2\,ds.$$

Since $E_0 < \infty$, it follows that the sequence $E_N(t)$, $N \in \mathbb{N}$, is uniformly bounded on the interval $[0, t^*]$, but then the first relation in (21) implies the uniform convergence of the series

$$\sum_{j=1}^{\infty} j^8 \dot{T}_j^2, \quad \sum_{j=1}^{\infty} j^{10} T_j^2, \quad \left(\sum_{j=1}^{\infty} j^3 T_j\cos(jx)\right)^2 \left(\sum_{j=1}^{\infty} j^5 T_j\cos(jx)\right)^2$$

on the interval $[0, t^*]$. Based on this, we arrive at the following estimates:

$$|w_{xxxx}| = \sum_{j=1}^{\infty} j^4 |T_j\sin(jx)| \le \sum_{j=1}^{\infty} j^{10} T_j^2 + \sum_{j=1}^{\infty}\frac{1}{j^2},$$
$$|w_{xxx}| = \sum_{j=1}^{\infty} j^3 |T_j\cos(jx)| \le \sum_{j=1}^{\infty} j^{10} T_j^2 + \sum_{j=1}^{\infty}\frac{1}{j^4}.$$

For further estimates, consider relation (6), having preliminarily integrated by parts twice in the integral appearing in this relation,

$$\ddot{T}_j(t)(1+C_2j^2)+C_0j^4T_j(t)+\frac{6C_1}{j}\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)\Big(\sum_{i=1}^\infty i^4T_i(t)\sin(ix)\Big)^2\cos(jx)\,dx$$
$$+\frac{6C_1}{j}\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)^2\Big(\sum_{i=1}^\infty i^5T_i(t)\cos(ix)\Big)\cos(jx)\,dx=0.$$

We find $\ddot{T}_j(t)$ from this relation, substitute it into the second series in (43), and obtain

$$w_{ttxx}=C_0\sum_{j=1}^\infty\frac{j^6T_j(t)}{1+C_2j^2}$$
$$+6C_1\sum_{j=1}^\infty\frac{j}{1+C_2j^2}\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)\Big(\sum_{i=1}^\infty i^4T_i(t)\sin(ix)\Big)^2\cos(jx)\,dx$$
$$+6C_1\sum_{j=1}^\infty\frac{j}{1+C_2j^2}\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)^2\Big(\sum_{i=1}^\infty i^5T_i(t)\cos(ix)\Big)\cos(jx)\,dx\equiv I_1+I_2+I_3.$$

Let us estimate the terms I_1, I_2, and I_3. We have

$$|I_1|=C_0\sum_{j=1}^\infty\frac{j^6}{1+C_2j^2}|T_j(t)|\le C_0\sum_{j=1}^\infty\frac{j^2}{(1+C_2j^2)^2}+C_0\sum_{j=1}^\infty j^{10}T_j^2(t),$$

$$|I_2|=6C_1\sum_{j=1}^\infty\frac{j}{1+C_2j^2}\bigg|\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)\Big(\sum_{i=1}^\infty i^4T_i(t)\sin(ix)\Big)^2\cos(jx)\,dx\bigg|$$
$$\le 6C_1\sum_{j=1}^\infty\frac{j^2}{(1+C_2j^2)^2}+6C_1\sum_{j=1}^\infty\bigg[\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3|T_i(t)\cos(ix)|\Big)$$
$$\times\Big(\sum_{i=1}^\infty i^4T_i(t)\sin(ix)\Big)^2\cos(jx)\,dx\bigg]^2,$$

$$|I_3|=6C_1\sum_{j=1}^\infty\frac{j}{1+C_2j^2}+\bigg|\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3|T_i(t)\cos(ix)|\Big)^2\Big(\sum_{i=1}^\infty i^5T_i(t)\cos(ix)\Big)\cos(jx)\,dx\bigg|$$
$$\le 6C_1\sum_{j=1}^\infty\frac{j^2}{(1+C_2j^2)^2}+6C_1\sum_{j=1}^\infty\bigg[\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)^2$$
$$\times\Big(\sum_{i=1}^\infty i^5T_i(t)\cos(ix)\Big)\cos(jx)\,dx\bigg]^2.$$

The finiteness of the terms I_2 and I_3 follows from the Parseval inequality and the estimates

$$\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)^2\Big(\sum_{i=1}^\infty i^4T_i(t)\sin(ix)\Big)^4dx\le 8\pi\Big(\sum_{i=1}^\infty i^{10}T_i^2(t)\Big)^2<\infty,$$
$$\int\limits_0^\pi\Big(\sum_{i=1}^\infty i^3T_i(t)\cos(ix)\Big)^4\Big(\sum_{i=1}^\infty i^5T_i(t)\cos(ix)\Big)^2dx\le 8\pi\Big(\sum_{i=1}^\infty i^{10}T_i^2(t)\Big)^2<\infty.$$

The estimates for the function w_{tt} readily follow from the estimate for the function w_{xxtt}. Then the series (43), (44) converge uniformly on the strip $0 \leq t \leq t^*$; consequently, the sum of the series (9) satisfies conditions (1)–(4). The proof of the theorem is complete.

REFERENCES

1. Strutt, J.V. (3rd Baron Rayleigh), *The Theory of Sound*, New York: Dover Publ., 1945. Translated under the title: *Teoriya zvuka*, Moscow: Gostekhizdat, 1955.
2. Krylov, A.N., *Vibratsiya sudov* (Ship Vibration), Moscow, 2012.
3. Timoshenko, S.P., *Vibration Problems in Engineering*, London: Constable and Co, 1937. Translated under the title: *Kolebaniya v inzhenernom dele*, Moscow: Nauka, 1967.
4. Sabitov, K.B., Vibrations of a beam with clamped ends, *Vestn. Samar. Gos. Tekh. Univ. Ser. Fiz.-Mat. Nauki*, 2015, vol. 19, no. 2, pp. 311–324.
5. Sabitov, K.B., A remark on the theory of initial–boundary value problems for the equation of rods and beams, *Differ. Equations*, 2017, vol. 53, no. 1, pp. 86–98.
6. Sabitov, K.B., Cauchy problem for the beam vibration equation, *Differ. Equations*, 2017, vol. 53, no. 5, pp. 658–664.
7. Dickey, R.W., Quasi-linear evolution equation and the method of Galerkin, *Proc. Am. Math. Soc.*, 1973, vol. 37, pp. 149–156.
8. Paul, L.D., A quasilinear hyperbolic and related third-order equations, *J. Anal. Appl.*, 1975, vol. 51, pp. 596–606.
9. Clark, H.R., Rincon, M.A., and Rodrigues, R.D., Beam equation with weak internal damping in domain with moving boundary, *Appl. Numer. Math.*, 2003, vol. 47, no. 2, pp. 139–157.
10. Pohozaev, S.I., On a class of quasilinear hyperbolic equations, *Sb. Math.*, 1975, vol. 96, no. 1, pp. 152–166.
11. Ball, J.M., Initial-boundary value problems for an extensible beam, *J. Math. Anal. Appl.*, 1973, vol. 42, pp. 61–90.
12. Andrey, A. and Gimaltdinova, A.A., The application of the method of energy inequalities for equation of beam, *Int. J. Pure Appl. Math.*, 2017, vol. 116, no. 4, pp. 1075–1080.
13. Andrey, A., Guzeliya, G., and Dinislam, G., On existence solution of equation of the beam with hinged ends, *Int. J. Pure Appl. Math.*, 2017, vol. 116, no. 4, pp. 1069–1073.
14. Gilmanov, S., Akimov, A., and Galiaskarova, G., On a nonlinear equation of damped oscillations beam, *Int. J. Pure Appl. Math.*, 2016, vol. 106, no. 2, pp. 625–629.