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NUMERICAL METHODS

Locally One-Dimensional Difference Scheme for a Nonlocal Boundary Value Problem for a Parabolic Equation in a Multidimensional Domain

Z. V. Beshtokova

Institute of Applied Mathematics and Automation, Kabardino-Balkarian Scientific Center, Russian Academy of Sciences, Nalchik, Kabardino-Balkar Republic, 360000 Russia
e-mail: zarabaeva@yandex.ru

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Abstract—We study a nonlocal boundary value problem for a parabolic equation in the multidimensional case. A locally one-dimensional difference scheme is constructed to solve this problem numerically. A priori estimates are derived by the method of energy inequalities in the differential and difference settings. The uniform convergence of the locally one-dimensional scheme is proved.

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INTRODUCTION

Boundary value problems with integral conditions are of special interest in the theory of differential equations. Note that from the physical viewpoint, such conditions are natural and occur in mathematical modeling of those cases where it is unfeasible to obtain information about the process developing at the boundary of the domain where it occurs by direct measurement or where only certain averaged (integral) characteristics of the variable concerned can be measured (see, e.g., [1]). For example, problems with integral conditions can serve as mathematical models of physical phenomena related, say, to problems encountered in plasma physics. In his survey article [2], Samarskii pointed out problems of the kind as qualitatively new and arising when solving contemporary problems in physics and exemplified this type of problems with the statement of the problem with an integral condition for the heat equation.

For the equation

$$L_\nu(u) \equiv u_t - u_{xx} - \nu u_{xxt} + c(x,t)u = q(x,t),$$

the paper [3] considered the nonlocal boundary value problem with the boundary conditions

$$u(0,t) = \alpha(t)u(1,t) + \int_0^t h(t,\tau)u(1,\tau)\,d\tau, \quad 0 < t < T, \tag{0.1}$$

$$u_x(1,t) = 0, \quad 0 < t < T,$$
$$u(x,0) = u_0(x), \quad 0 < x < 1.$$

The nonlocal problem considered in the present paper contains the nonlocal boundary condition of the integral form (0.1).

Various classes of nonlocal problems for partial differential equations were studied in [4–10].

In the present paper, we propose a locally one-dimensional (economical) difference scheme for numerically solving a nonlocal boundary value problem for a partial differential equation of the parabolic type in the multidimensional case. The main idea behind the scheme is to reduce the layer-to-layer transition to the successive solution of a number of one-dimensional problems in each of the coordinate spatial directions. In this case, for each intermediate problems we construct an unconditionally stable scheme that is solved with the number of operators proportional to the number of grid nodes on each time layer. Using the method of energy inequalities, we derive a priori estimates in the differential and difference settings. The uniform convergence of the locally one-dimensional scheme is proved.

1. STATEMENT OF THE PROBLEM AND THE A PRIORI ESTIMATE IN DIFFERENTIAL FORM

In the cylinder $\overline{Q}_T = \overline{G} \times [0 \le t \le T]$ with the base in the form of the p-dimensional rectangular parallelepiped $\overline{G} = \{x = (x_1, \dots, x_p) : 0 \le x_\alpha \le l_\alpha, \alpha = 1, \dots, p\}$ with boundary Γ, consider the nonlocal problem

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad (x,t) \in Q_T, \tag{1.1}$$

$$k_\alpha(0,x',t)u_{x_\alpha}(0,x',t) = \beta_{-\alpha}(0,x',t)u(l_\alpha,x',t) + \int\limits_0^t \rho(t,\tau)u(l_\alpha,x',\tau)\,d\tau - \mu_{-\alpha}(0,x',t), \tag{1.2}$$

$$-k_\alpha(l_\alpha,x',t)u_{x_\alpha}(l_\alpha,x',t) = \beta_{+\alpha}(l_\alpha,x',t)u(l_\alpha,x',t) - \mu_{+\alpha}(l_\alpha,x',t), \quad x_\alpha = l_\alpha, \quad t \in [0,T], \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \overline{G}; \tag{1.4}$$

here $Q_T = G \times [0 < t \le T]$, $G = \overline{G} \setminus \Gamma$,

$$Lu = \sum_{\alpha=1}^p L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) - q_\alpha(x,t)u,$$

$$0 < c_0 \le k_\alpha(x,t) \le c_1, \quad |q_\alpha(x,t)|, |\beta_{\pm\alpha}(x,t)|, |\rho(t,\tau)| \le c_2, \quad 0 \le \tau \le t,$$

where c_0, c_1, and c_2 are positive constants and $\alpha = 1, \dots, p$.

In what follows, by M_i, $i \in \mathbb{N}$, we denote positive constants depending only on the input data of the problem under consideration.

Assuming that there exists a regular solution of the differential problem (1.1)–(1.4) in the cylinder $\overline{Q}_T$, we obtain an a priori estimate for this solution using the method of energy inequalities. Multiplying Eq. (1.1) in the sense of the inner product by u, we derive the energy identity

$$\left(\frac{\partial u}{\partial t}, u\right) = \left(\sum_{\alpha=1}^p \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right), u\right) - \left(\sum_{\alpha=1}^p q_\alpha(x,t)u, u\right) + (f(x,t), u). \tag{1.5}$$

Let us transform the integrals occurring in identity (1.5) as follows:

$$\left(\frac{\partial u}{\partial t}, u\right) = \frac{1}{2}\frac{\partial}{\partial t}\|u\|_0^2, \tag{1.6}$$

$$\left(\sum_{\alpha=1}^p \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right), u\right) = \sum_{\alpha=1}^p \int\limits_{G'} k_\alpha(x,t)u\frac{\partial u}{\partial x_\alpha}\Big|_0^{l_\alpha} dx' - \sum_{\alpha=1}^p \int\limits_G k_\alpha(x,t)\left(\frac{\partial u}{\partial x_\alpha}\right)^2 dx, \tag{1.7}$$

where $G' = \{x' = (x_1, x_2, \dots, x_{\alpha-1}, x_{\alpha+1}, \dots, x_p) : 0 < x_k < l_k,\ k = 1, 2, \dots, \alpha-1, \alpha+1, \dots, p\}$ and $dx' = dx_1\,dx_2 \cdots dx_{\alpha-1}\,dx_{\alpha+1} \cdots dx_p$. Further, to estimate the terms on the right-hand side, we apply the Cauchy ε-inequality

$$-\left(\sum_{\alpha=1}^p q_\alpha(x,t)u, u\right) \le c_2 \sum_{\alpha=1}^p \int\limits_G u^2\,dx, \tag{1.8}$$

$$(f(x,t), u) \le \frac{1}{2}\|f\|_0^2 + \frac{1}{2}\|u\|_0^2. \tag{1.9}$$

In view of the transformations (1.6), (1.7) and the estimates (1.8), (1.9), we obtain the following inequality from identity (1.5):

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_0^2+\sum_{\alpha=1}^{p}\int\limits_G k_\alpha(x,t)\left(\frac{\partial u}{\partial x_\alpha}\right)^2 dx\le\sum_{\alpha=1}^{p}\int\limits_{G'} uk_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\bigg|_0^{l_\alpha} dx'+M_1\|u\|_0^2+\frac{1}{2}\|f\|_0^2. \quad (1.10)$$

Considering conditions (1.2) and (1.3), we write the first term on the right-hand side in (1.10) as

$$\sum_{\alpha=1}^{p}\int\limits_{G'} uk_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\bigg|_0^{l_\alpha} dx'=\sum_{\alpha=1}^{p}\int\limits_{G'}(k_\alpha(l_\alpha,x',t)u(l_\alpha,x',t)-k_\alpha(0,x',t)u(0,x',t))\,dx'$$
$$=\sum_{\alpha=1}^{p}\int\limits_{G'}\Big(\mu_{+\alpha}(l_\alpha,x',t)u(l_\alpha,x',t)-\beta_{+\alpha}(l_\alpha,x',t)u^2(l_\alpha,x',t)-\beta_{-\alpha}(0,x't)u(l_\alpha,x',t) \quad (1.11)$$
$$\times u(0,x',t)-u(0,x',t)\int\limits_0^t\rho(t,\tau)u(l_\alpha,x',\tau)\,d\tau-u(0,x',t)\mu_{-\alpha}(0,x',t)\Big)\,dx'.$$

The following assertion holds.

Theorem 1 [11; p. 73]. *Let Ω be a domain with smooth boundary $\partial\Omega$. For elements $u(x)$ in $W_2^1(\Omega)$, on the domains Π lying on smooth hypersurfaces and belonging to the domain Ω, the traces are defined as elements in $L_2(\Pi)$, with these traces changing continuously under a continuous shift of Π. For these traces, one has the inequalities*

$$\int\limits_\Pi[u(x+le_1)-u(x)]^2\,ds\equiv\|u(x+le_1)-u(x)\|_{2,\Pi}^2\le cl\int\limits_{Q_l(\Pi)}u_x^2\,dx,\quad 0\le l\le\delta,$$

and

$$\|u(x)\|_{2,\Pi}^2\le c[\delta^{-1}\|u(x)\|_{2,Q_\delta(\Pi)}^2+\delta\|u_x(x)\|_{2,Q_\delta(\Pi)}^2],$$

where e_1 is the unit vector of the normal to Π at a point x; $Q_l(\Pi)$ is the curvilinear cylinder formed by segments of these normals of length l (δ is the greatest of those lengths l for which $Q_l(\Pi)\subset\Omega$); and c is a constant independent of the function $u(x)$.

For all elements $v(x)$ in $W_2^1(\Omega)$ with piecewise smooth boundary $\partial\Omega$, one has the estimate

$$\int\limits_{\partial\Omega}v^2\,ds\le\bar c_1\int\limits_\Omega(|v|\cdot|v_x|+v^2)\,dx\le\bar c_1\int\limits_\Omega\left[\frac{\varepsilon}{\bar c_1}v_x^2+\left(\frac{\bar c_1}{4\varepsilon}+1\right)v^2\right]dx\equiv\int\limits_\Omega(\varepsilon v_x^2+c_\varepsilon v^2)\,dx,\quad\varepsilon>0.$$

Using the representation (1.11), Theorem 1, and the Cauchy ε-inequality, we obtain the estimate

$$\sum_{\alpha=1}^{p}\int\limits_{G'} uk_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\bigg|_0^{l_\alpha} dx'\le\varepsilon M_2\|u_x\|_0^2+M_3(\varepsilon)\|u\|_0^2$$
$$+\varepsilon\int\limits_0^t\|u_x\|_0^2\,d\tau+M_4(\varepsilon)\int\limits_0^t\|u\|_0^2\,d\tau+\frac{1}{2}\sum_{\alpha=1}^{p}\int\limits_{G'}(\mu_{-\alpha}^2+\mu_{+\alpha}^2)\,dx',$$

which, by virtue of inequality (1.10), implies that

$$\frac{\partial}{\partial t}\|u\|_0^2+\sum_{\alpha=1}^{p}\int\limits_G k_\alpha(x,t)\left(\frac{\partial u}{\partial x_\alpha}\right)^2 dx\le\varepsilon M_5\|u_x\|_0^2+M_6(\varepsilon)\|u\|_0^2$$
$$+\varepsilon\int\limits_0^t\|u_x\|_0^2\,d\tau+M_7(\varepsilon)\int\limits_0^t\|u\|_0^2\,d\tau+\sum_{\alpha=1}^{p}\int\limits_{G'}(\mu_{-\alpha}^2+\mu_{+\alpha}^2)\,dx'+\|f\|_0^2. \quad (1.12)$$

We denote the integration variable τ in inequality (1.12) by τ_1 and the variable t by τ, then integrate the result over τ between 0 and t, and obtain

$$\begin{aligned}\|u\|_0^2 + \int_0^t \|u_x\|_0^2\,d\tau &\le \varepsilon M_8 \int_0^t \|u_x\|_0^2\,d\tau + M_9(\varepsilon)\int_0^t \|u\|_0^2\,d\tau + \varepsilon M_{10}\int_0^t\int_0^\tau \|u_x\|_0^2\,d\tau_1\,d\tau \\ &+ M_{11}(\varepsilon)\int_0^t\int_0^\tau \|u\|_0^2\,d\tau_1\,d\tau + M_{12}\left(\int_0^t\left(\|f\|_0^2 + \int_{G'}(\mu_{-\alpha}^2+\mu_{+\alpha}^2)\,dx'\right)d\tau + \|u_0(x)\|_0^2\right).\end{aligned} \tag{1.13}$$

Let us estimate the third and fourth terms on the right-hand side in inequality (1.13) as follows:

$$\begin{aligned}\int_0^t\int_0^\tau \|u_x\|_0^2\,d\tau_1\,d\tau &\le T\int_0^t \|u_x\|_0^2\,d\tau,\\ \int_0^t\int_0^\tau \|u\|_0^2\,d\tau_1\,d\tau &\le T\int_0^t \|u\|_0^2\,d\tau.\end{aligned} \tag{1.14}$$

Taking the estimates (1.14) into account in inequality (1.13), we obtain

$$\begin{aligned}\|u\|_0^2 + \int_0^t \|u_x\|_0^2\,d\tau &\le M_{13}(\varepsilon)\int_0^t \|u\|_0^2\,d\tau + \varepsilon(T+1)\int_0^t \|u_x\|_0^2\,d\tau \\ &+ M_{14}\left(\int_0^t\left(\|f\|_0^2 + \int_{G'}(\mu_{-\alpha}^2+\mu_{+\alpha}^2)\,dx'\right)d\tau + \|u_0(x)\|_0^2\right).\end{aligned} \tag{1.15}$$

We take $\varepsilon = 1/(2T+2)$ in inequality (1.15) and obtain

$$\begin{aligned}\|u\|_0^2 + \int_0^t \|u_x\|_0^2\,d\tau &\le M_{15}\int_0^t \|u\|_0^2\,d\tau \\ &+ M_{16}\left(\int_0^t\left(\|f\|_0^2 + \int_{G'}(\mu_{-\alpha}^2+\mu_{+\alpha}^2)\,dx'\right)d\tau + \|u_0(x)\|_0^2\right).\end{aligned} \tag{1.16}$$

Dropping the second terms on the left-hand side in inequality (1.16) and applying Gronwall's lemma [11, p. 152; 12] to the resulting inequality, we obtain an upper bound for the integral $\int_0^t \|u_x\|_0^2\,d\tau$. Substituting this bound for the integral on the right-hand side in inequality (1.16), we arrive at the desired a priori estimate for the solution,

$$\|u\|_0^2 + \|u_x\|_{2,Q_t}^2 \le M(t)\left(\int_0^t\left(\|f\|_0^2 + \int_{G'}(\mu_{-\alpha}^2+\mu_{+\alpha}^2)\,dx'\right)d\tau + \|u_0(x)\|_0^2\right), \tag{1.17}$$

where $\|u_x\|_{2,Q_t}^2 = \int_0^t \|u_x\|_0^2\,d\tau$ and the function $M(t)$ depends only on the input data of problem (1.1)–(1.4).

The a priori estimate (1.17) implies the uniqueness of solution to the original problem (1.1)–(1.4), as well as the continuous, in the norm $\|u\|_1^2 = \|u\|_0^2 + \|u_x\|_{2,Q_t}^2$, dependence of the solution on the input data on each time layer.

2. LOCALLY ONE-DIMENSIONAL SCHEME

We select the spatial mesh uniform with respect to each direction Ox_α with step $h_\alpha = l_\alpha/N_\alpha$, $\alpha = 1, \dots, p$:

$$\overline{\omega}_h = \prod_{\alpha=1}^{p} \overline{\omega}_{h_\alpha}, \quad \overline{\omega}_{h_\alpha} = \{x_\alpha^{(i_\alpha)} = i_\alpha \hbar_\alpha : i_\alpha = 0, \dots, N_\alpha, \quad \alpha = 1, \dots, p\}$$

$$\hbar_\alpha = \begin{cases} h_\alpha, & i_\alpha = 1, \dots, N_\alpha - 1, \\ h_\alpha/2, & i_\alpha = 0, N_\alpha. \end{cases}$$

On the closed interval $[0, T]$, we also introduce the uniform mesh $\overline{\omega}_\tau = \{t_j = j\tau : j = 0, \dots, j_0\}$ with step $\tau = T/j_0$. We partition each of the closed intervals $[t_j, t_{j+1}]$ into p parts by introducing the points $t_{j+\alpha/p} = t_j + \tau\alpha/p$, $\alpha = 1, \dots, p-1$, and denote the half-interval $(t_{j+(\alpha-1)/p}, t_{j+\alpha/p}]$ by Δ_α, where $\alpha = 1, \dots, p$.

We write Eq. (1.1) in the form

$$\operatorname{Re} u \equiv \frac{\partial u}{\partial t} - Lu - f = 0,$$

or

$$\sum_{\alpha=1}^{p} \operatorname{Re}_\alpha u = 0, \quad \operatorname{Re}_\alpha u \equiv \frac{1}{p}\frac{\partial u}{\partial t} - L_\alpha u - f_\alpha,$$

where $f_\alpha(x, t)$, $\alpha = 1, \dots, p$, are arbitrary functions possessing the same smoothness as the function $f(x, t)$ and satisfying the normalization condition $\sum_{\alpha=1}^{p} f_\alpha = f$.

On each half-interval Δ_α, $\alpha = 1, \dots, p$, we successively solve the problems

$$\operatorname{Re}_\alpha \vartheta_\alpha \equiv \frac{1}{p}\frac{\partial \vartheta_{(\alpha)}}{\partial t} - L_\alpha \vartheta_{(\alpha)} - f_\alpha = 0, \quad x \in G, \quad t \in \Delta_\alpha, \quad \alpha = 1, \dots, p, \tag{2.1}$$

$$k_\alpha(0, x', t)(\vartheta_{(\alpha)})_{x_\alpha}(0, x', t) = \beta_{-\alpha}(0, x', t)\vartheta_{(\alpha)}(l_\alpha, x', t) + \int_0^t \rho(t, \tau)\vartheta_{(\alpha)}(l_\alpha, x', \tau)\, d\tau - \mu_{-\alpha}(0, x', t)$$

$$-k_\alpha(l_\alpha, x', t)(\vartheta_{(\alpha)})_{x_\alpha}(l_\alpha, x', t) = \beta_{+\alpha}(l_\alpha, x', t)\vartheta_{(\alpha)}(l_\alpha, x', t) - \mu_{+\alpha}(l_\alpha, x', t), \quad 0 \le t \le T,$$

while assuming that [13, p. 522]

$$\vartheta_{(1)}(x, 0) = u_0(x), \quad \vartheta_{(\alpha)}(x, t_{j+(\alpha-1)/p}) = \vartheta_{(\alpha-1)}(x, t_{j+(\alpha-1)/p}), \quad \alpha = 2, \dots, p,$$
$$\vartheta_{(1)}(x, t_j) = \vartheta_{(p)}(x, t_j).$$

Let us approximate Eq. (2.1) on the half-interval Δ_α by an implicit two-layer scheme to produce a chain of p one-dimensional difference equations:

$$\begin{gathered} \frac{y^{j+\alpha/p} - y^{j+(\alpha-1)/p}}{\tau} = \Lambda_\alpha y^{j+\alpha/p} + \varphi_\alpha^{j+\alpha/p}, \quad \alpha = 1, \dots, p, \\ \Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha}^{j+\alpha/p})_{x_\alpha} - d_\alpha y^{j+\alpha/p}, \end{gathered} \tag{2.2}$$

where $a^{(1_\alpha)} = a_{i_\alpha+1}$, $a_i = k_{i-0.5}(\bar{t})$, $\bar{t} = t_{j+0.5}$, $\varphi_\alpha^{j+\alpha/p} = f_\alpha(x, t_{j+0.5})$, $d_\alpha = q_\alpha$.

Equation (2.2) must be equipped with the boundary and initial conditions. We write a difference analog for the boundary conditions (1.2), (1.3):

$$a_\alpha^{(+1_\alpha)} y_{x_\alpha,0}^{j+\alpha/p} = \beta_{-\alpha}\, y_{N_\alpha}^{j+\alpha/p} + \sum_{s=0}^{j} \rho_{s,j} y_{N_\alpha}^{s} \tau - \mu_{-\alpha}, \quad x_\alpha = 0,$$

$$-a_\alpha^{(N_\alpha)} y_{\bar{x}_\alpha,N_\alpha}^{j+\alpha/p} = \beta_{+\alpha} y_{N_\alpha}^{j+\alpha/p} - \mu_{+\alpha}, \quad x_\alpha = l_\alpha.$$

Conditions (1.2) and (1.3) have the approximation order $O(h_\alpha)$. Let us increase the approximation order to $O(h_\alpha^2)$ on the solutions of Eq. (2.1). The following relations hold:

$$\begin{aligned}
a_\alpha^{(1\alpha)}\vartheta_{x_\alpha,0}^{j+\alpha/p} &= \beta_{-\alpha}\vartheta_{(\alpha),0}^{j+\alpha/p} - \mu_{-\alpha} + O(h_\alpha),\\
a_\alpha^{(1\alpha)} &= k_{1/2}^{(\alpha)} = k_0 + k_0'\frac{h_\alpha}{2} + k_0''\frac{h_\alpha^2}{8} + O(h_\alpha^3),\\
\frac{\vartheta_{(\alpha)}^1 - \vartheta_{(\alpha)}^0}{h_\alpha} &= \vartheta_{(\alpha)x_\alpha,0} = \vartheta_{(\alpha)}' + \vartheta_{(\alpha)}''\frac{h_\alpha}{2} + O(h_\alpha^2),\\
a_\alpha^{(1_\alpha)}\vartheta_{(\alpha)x_\alpha,0}^{j+\alpha/p} &= k^{(\alpha)}\vartheta_{(\alpha),0}' + (k^{(\alpha)}\vartheta_{(\alpha)}')'\frac{h_\alpha}{2} + O(h_\alpha^2),\\
k^{(\alpha)}\vartheta_{(\alpha),0}' &= a_\alpha^{(1_\alpha)}\vartheta_{(\alpha)x_\alpha,0}^{j+\alpha/p} - 0.5h_\alpha(k^{(\alpha)}\vartheta_{(\alpha)}')' + O(h_\alpha^2)\\
&= a_\alpha^{(1_\alpha)}\vartheta_{(\alpha)x_\alpha,0}^{j+\alpha/p} - 0.5h_\alpha\left(\frac{1}{p}\frac{\partial\vartheta_{(\alpha)}^{j+\alpha/p}}{\partial t} + q_\alpha\vartheta_{(\alpha)} - f_\alpha\right) + O(h_\alpha^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&a_\alpha^{(1_\alpha)}\vartheta_{(\alpha)x_\alpha,0}^{j+\alpha/p} - 0.5h_\alpha(\vartheta_{(\alpha)\bar t}^{j+\alpha/p} + d_\alpha\vartheta_{(\alpha)} - f_\alpha)\\
&\qquad = \beta_{-\alpha}\vartheta_{(\alpha),N_\alpha}^{j+\alpha/p} + \sum_{s=0}^{j}\rho_{s,j}\vartheta_{(\alpha),N_\alpha}^{s}\tau - \mu_{-\alpha} + O(h_\alpha^2) + O(h_\alpha\tau).
\end{aligned}\tag{2.3}$$

In relation (2.3), we drop quantities of the orders $O(h_\alpha^2)$ and $O(h_\alpha\tau)$ and replace $\vartheta_{(\alpha)}$ with $y_{(\alpha)} = y^{j+\alpha/p}$. Then (2.3) acquires the form

$$a_\alpha^{(1_\alpha)}y_{x_\alpha,0}^{j+\alpha/p} - 0.5h_\alpha y_{\bar t}^{j+\alpha/p} = 0.5h_\alpha d_\alpha^{(0)}y_0^{j+\alpha/p} + \beta_{-\alpha}y_{N_\alpha}^{j+\alpha/p} + \sum_{s=0}^{j}\rho_{s,j}y_{N_\alpha}^{s}\tau - \mu_{-\alpha} - 0.5h_\alpha f_{\alpha,0}, \quad x_\alpha = 0,$$

or

$$\begin{aligned}
y_{\bar t,0}^{j+\alpha/p} &= \frac{1}{0.5h_\alpha}\left(a_\alpha^{(1_\alpha)}y_{x_\alpha,0}^{j+\alpha/p} - 0.5h_\alpha d_\alpha^{(0)}y_0^{j+\alpha/p} - \beta_{-\alpha}y_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j}\rho_{s,j}y_{N_\alpha}^{s}\tau\right) - \overline{\mu}_{-\alpha}, \quad x_\alpha = 0,\\
y_{\bar t,N_\alpha}^{j+\alpha/p} &= -\frac{1}{0.5h_\alpha}(a_\alpha^{(N_\alpha)}y_{\bar x_\alpha,N_\alpha}^{j+\alpha/p} + \overline{\beta}_{+\alpha}y_{N_\alpha}^{j+\alpha/p}) - \overline{\mu}_{+\alpha}, \quad x_\alpha = l_\alpha,
\end{aligned}$$

where

$$\overline{\beta}_{+\alpha} = \beta_{+\alpha} + 0.5h_\alpha d_\alpha^{(N_\alpha)}, \quad \overline{\mu}_{-\alpha} = \frac{\mu_{-\alpha}}{0.5h_\alpha} + f_{\alpha,0}, \quad \overline{\mu}_{+\alpha} = \frac{\mu_{+\alpha}}{0.5h_\alpha} + f_{\alpha,N_\alpha}.$$

Thus, we arrive at the chain of one-dimensional schemes

$$\begin{gathered}
y_{\bar t}^{(\alpha)} = \overline{\Lambda}_\alpha y^{(\alpha)} + \Phi_\alpha^{j+\alpha/p}, \quad \alpha = 1,\ldots,p, \quad x \in \overline{\omega}_{h_\alpha},\\
y(x,0) = u_0(x), \quad y_{\bar t} = \frac{y^{j+\alpha/p} - y^{j+(\alpha-1)/p}}{\tau},
\end{gathered}\tag{2.4}$$

where

$$\overline{\Lambda}_\alpha y^{(\alpha)} = \begin{cases}
\Lambda_\alpha y^{(\alpha)} = (a_\alpha y_{x_\alpha}^{(\alpha)})_{x_\alpha} - d_\alpha y^{(\alpha)},\\
\Lambda_\alpha^- y^{(\alpha)} = \dfrac{1}{0.5h_\alpha}\left(a_\alpha^{(1_\alpha)}y_{x_\alpha,0}^{(\alpha)} - 0.5h_\alpha d_\alpha^{(0)}y_0^{j+\alpha/p} - \beta_{-\alpha}y_{N_\alpha}^{j+\alpha/p} - \displaystyle\sum_{s=0}^{j}\rho_{s,j}y_{N_\alpha}^{s}\tau\right), & x_\alpha = 0,\\
\Lambda_\alpha^+ y^{(\alpha)} = -\dfrac{a_\alpha^{(N_\alpha)}y_{\bar x_\alpha,N_\alpha}^{(\alpha)} + \overline{\beta}_{+\alpha}y_{N_\alpha}^{(\alpha)}}{0.5h_\alpha}, & x_\alpha = l_\alpha,
\end{cases}$$

$$\Phi_\alpha = \begin{cases} \varphi_\alpha, \ x_\alpha \in \omega_{h_\alpha}, \\ \overline{\mu}_{-_\alpha}, \ x_\alpha = 0, \\ \overline{\mu}_{+_\alpha}, \ x_\alpha = l_\alpha. \end{cases} \tag{2.5}$$

3. APPROXIMATION ERROR OF THE LOCALLY ONE-DIMENSIONAL SCHEME

The accuracy of the solution by the locally one-dimensional scheme is characterized by the difference $z^{j+\alpha/p} = y^{j+\alpha/p} - u^{j+\alpha/p}$, where $u^{j+\alpha/p}$ is the solution of the original problem (1.1)–(1.4). Substituting $y^{j+\alpha/p} = z^{j+\alpha/p} + u^{j+\alpha/p}$ into the difference equation (2.2), we obtain the following problem for the error $z^{j+\alpha/p}$:

$$\frac{z^{j+\alpha/p} - z^{j+(\alpha-1)/p}}{\tau} = \Lambda_\alpha z^{j+\alpha/p} + \psi_\alpha^{j+\alpha/p}, \tag{3.1}$$

where $\psi_\alpha^{j+\alpha/p} = \Lambda_\alpha u^{j+\alpha/p} + \varphi_\alpha^{j+\alpha/p} - \frac{u^{j+\alpha/p} - u^{j+(\alpha-1)/p}}{\tau}$.

Denoting

$$\overset{\circ}{\psi}_\alpha = \left(L_\alpha u + f_\alpha - \frac{1}{p}\frac{\partial u}{\partial t} \right)^{j+1/2}$$

and noting that $\sum_{\alpha=1}^{p} \overset{\circ}{\psi}_\alpha = 0$ if $\sum_{\alpha=1}^{p} f_\alpha = f$, we represent the error $\psi_\alpha^{j+\alpha/p}$ as the sum

$$\begin{aligned} \psi_\alpha^{j+\alpha/p} &= \Lambda_\alpha u^{j+\alpha/p} + \varphi_\alpha^{j+\alpha/p} - \frac{u^{j+\alpha/p} - u^{j+(\alpha-1)/p}}{\tau} + \overset{\circ}{\psi}_\alpha - \overset{\circ}{\psi}_\alpha \\ &= (\Lambda_\alpha u^{j+\alpha/p} - L_\alpha u^{j+1/2}) + (\varphi_\alpha^{j+\alpha/p} - f_\alpha^{j+1/2}) - \left(\frac{u^{j+\alpha/p} - u^{j+(\alpha-1)/p}}{\tau} - \frac{1}{p}\left(\frac{\partial u}{\partial t}\right)^{j+1/2} \right) + \overset{\circ}{\psi}_\alpha \\ &= \psi_\alpha^* + \overset{\circ}{\psi}_\alpha. \end{aligned}$$

It is obvious that

$$\psi_\alpha^* = O(h_\alpha^2 + \tau), \quad \overset{\circ}{\psi}_\alpha = O(1), \quad \sum_{\alpha=1}^{p} \psi_\alpha^{j+\alpha/p} = \sum_{\alpha=1}^{p} \overset{\circ}{\psi}_\alpha + \sum_{\alpha=1}^{p} \psi_\alpha^* = O(|h|^2 + \tau), \quad |h|^2 = h_1^2 + h_2^2 + \ldots + h_p^2.$$

Write the boundary condition $x_\alpha = 0$ as follows:

$$0.5 h_\alpha y_{\bar{t}}^{j+\alpha/p} = a_\alpha^{(1_\alpha)} y_{x_\alpha,0}^{j+\alpha/p} - 0.5 h_\alpha d_\alpha^{(0)} y_0^{j+\alpha/p} - \beta_{-\alpha} y_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} y_{N_\alpha}^s \tau + 0.5 h_\alpha f_{\alpha,0} + \mu_{-\alpha}.$$

We substitute $y^{j+\alpha/p} = z^{j+\alpha/p} + u^{j+\alpha/p}$ into (3.1). Then we obtain

$$\begin{aligned} 0.5 h_\alpha z_{\bar{t}}^{j+\alpha/p} &= a_\alpha^{(1_\alpha)} z_{x_\alpha,0}^{j+\alpha/p} - 0.5 h_\alpha d_\alpha^{(0)} z_0^{j+\alpha/p} - \beta_{-\alpha} z_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} z_{N_\alpha}^s \tau \\ &+ a_\alpha^{(1_\alpha)} u_{x_\alpha,0}^{j+\alpha/p} - 0.5 h_\alpha d_\alpha^{(0)} u_0^{j+\alpha/p} - \beta_{-\alpha} u_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} u_{N_\alpha}^s \tau - 0.5 h_\alpha u_{\bar{t}}^{j+\alpha/p} + 0.5 h_\alpha f_{\alpha,0} + \mu_{-\alpha}. \end{aligned}$$

We add and subtract the quantity

$$0.5 h_\alpha \overset{\circ}{\psi}_\alpha = 0.5 h_\alpha \left[\frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha} \right) - q_\alpha u + f_\alpha - \frac{1}{p}\frac{\partial u}{\partial t} \right]_{x_\alpha=0}^{j+1/2}$$

to and from the right-hand side of the resulting relation. Then

$$\psi_{-\alpha} = 0.5h_\alpha(f_\alpha - u_{\bar t}^{j+\alpha/p}) + a_\alpha^{(1_\alpha)} u_{x_\alpha,0}^{j+\alpha/p} - \beta_{-\alpha} u_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} u_{N_\alpha}^s \tau - 0.5h_\alpha d_{\alpha,0} u_0^{j+\alpha/p} + \mu_{-\alpha}$$
$$- 0.5h_\alpha \left[\frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right) - q_\alpha u + f_\alpha - \frac{1}{p}\frac{\partial u}{\partial t}\right]_{x_\alpha=0}^{j+1/2} + 0.5h_\alpha \overset{\circ}{\psi}_\alpha$$
$$= 0.5h_\alpha(f_\alpha - u_{\bar t}^{j+\alpha/p}) + a_\alpha^{(1_\alpha)} u_{x_\alpha,0}^{j+\alpha/p} - \beta_{-\alpha} u_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} u_{N_\alpha}^s \tau - 0.5h_\alpha d_{\alpha,0} u_0^{j+\alpha/p} + \mu_{-\alpha}$$
$$- 0.5h_\alpha \left[\frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right)\right]^{j+1/2} - 0.5h_\alpha(f_\alpha - u_{\bar t}^{j+\alpha/p}) + 0.5h_\alpha d_{\alpha,0} u_0^{j+\alpha/p} + 0.5h_\alpha \overset{\circ}{\psi}_\alpha + O(h_\alpha \tau)$$
$$= a_\alpha^{(1_\alpha)} u_{x_\alpha,0}^{j+\alpha/p} - \beta_{-\alpha} u_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} u_{N_\alpha}^s \tau + \mu_{-\alpha}$$
$$+ 0.5h_\alpha \left[\frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right)\right]^{j+1/2} + 0.5h_\alpha \overset{\circ}{\psi}_\alpha + O(h_\alpha^2) + O(h_\alpha \tau)$$
$$= k_\alpha \frac{\partial u^{j+\alpha/p}}{\partial x_\alpha} + 0.5h_\alpha \left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right)_0^{j+\alpha/p} - \beta_{-\alpha} u_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} u_{N_\alpha}^s \tau$$
$$- 0.5h_\alpha \left[\frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right)\right]^{j+\alpha/p} + \mu_{-\alpha} + O(h_\alpha^2) + O(h_\alpha \tau)$$
$$= \left(k_\alpha \frac{\partial u^{j+\alpha/p}}{\partial x_\alpha} - \beta_{-\alpha} u_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} u_{N_\alpha}^s \tau + \mu_{-\alpha}\right)_{x_\alpha=0} + 0.5h_\alpha \overset{\circ}{\psi}_\alpha + O(h_\alpha^2) + O(h_\alpha \tau).$$

In view of the boundary conditions (1.2) and (1.3), the bracketed expression is zero. Therefore,

$$\psi_{-\alpha} = 0.5h_\alpha \overset{\circ}{\psi}_{-\alpha} + \psi_{-\alpha}^*, \quad \psi_{-\alpha}^* = O(h_\alpha^2 + \tau) + O(h_\alpha \tau).$$

We have

$$0.5h_\alpha z_{t,0}^{j+\alpha/p} = a_\alpha^{(1_\alpha)} z_{x_\alpha,0}^{j+\alpha/p} - 0.5h_\alpha d_\alpha^{(0)} z_0^{j+\alpha/p} - \beta_{-\alpha} z_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} z_{N_\alpha}^s \tau + 0.5h_\alpha \overset{\circ}{\psi}_{-\alpha} + \psi_{-\alpha}^*,$$

or

$$z_{t,0}^{j+\alpha/p} = \frac{1}{0.5h_\alpha}\left(a_\alpha^{(1_\alpha)} z_{x_\alpha,0}^{j+\alpha/p} - 0.5h_\alpha d_\alpha^{(0)} z_0^{j+\alpha/p} - \beta_{-\alpha} z_{N_\alpha}^{j+\alpha/p} - \sum_{s=0}^{j} \rho_{s,j} z_{N_\alpha}^s \tau\right) + \overset{\circ}{\psi}_{-\alpha} + \frac{\psi_{-\alpha}^*}{0.5h_\alpha},$$
$$z_{\bar t,N_\alpha}^{j+\alpha/p} = -\frac{a_\alpha^{(N_\alpha)} z_{\bar x_\alpha,N_\alpha}^{j+\alpha/p} + \bar\beta_{+\alpha} z_{N_\alpha}^{j+\alpha/p}}{0.5h_\alpha} + \overset{\circ}{\psi}_{+\alpha} + \frac{\psi_{+\alpha}^*}{0.5h_\alpha}.$$

We thus write the problem for the error $z^{j+\alpha/p}$ in the form

$$z_{\bar t}^{(\alpha)} = \overline{\Lambda}_\alpha z^{(\alpha)} + \psi_\alpha^{j+\alpha/p},$$
$$z(x,0) = 0,$$

where

$$\overline{\Lambda}_\alpha = \begin{cases} \Lambda_\alpha, & x_\alpha \in \omega_{h_\alpha}, \\ \Lambda_\alpha^-, & x_\alpha = 0, \\ \Lambda_\alpha^+, & x_\alpha = l_\alpha, \end{cases} \qquad \psi_\alpha = \begin{cases} \psi_\alpha, & x_\alpha \in \omega_{h_\alpha}, \\ \psi_\alpha^-, & x_\alpha = 0, \\ \psi_\alpha^+, & x_\alpha = l_\alpha, \end{cases}$$

$$\psi_\alpha = \overset{\circ}{\psi}_\alpha + \psi^*_\alpha, \quad \overset{\circ}{\psi}_\alpha = O(1), \quad \psi^*_\alpha = O(h^2_\alpha + \tau), \quad \psi_{-\alpha} = \overset{\circ}{\psi}_{-\alpha} + \frac{\psi^*_{-\alpha}}{0.5h_\alpha}, \quad \psi_{+\alpha} = \overset{\circ}{\psi}_{+\alpha} + \frac{\psi^*_{+\alpha}}{0.5h_\alpha},$$

$$\psi_{\pm\alpha} = O(h^2_\alpha + \tau), \quad \overset{\circ}{\psi}_{\pm\alpha} = O(1), \quad \sum_{\alpha=1}^{p} \overset{\circ}{\psi}_{\pm\alpha} = 0.$$

4. STABILITY OF THE LOCALLY ONE-DIMENSIONAL SCHEME

We multiply Eq. (2.4) in the sense of the inner product by $y^{(\alpha)} = y^{j+\alpha/p}$ to obtain

$$[y^{(\alpha)}_{\bar t}, y^{(\alpha)}]_\alpha - [\overline{\Lambda}_\alpha y^{(\alpha)}, y^{(\alpha)}]_\alpha = [\Phi^{(\alpha)}, y^{(\alpha)}]_\alpha, \tag{4.1}$$

where

$$[u,v] = \sum_{x\in\bar\omega_h} uvH, \quad H = \prod_{\alpha=1}^{N_\alpha} \hbar_\alpha, \quad [u,v]_\alpha = \sum_{i_\alpha=0}^{N_\alpha} u_{i_\alpha} v_{i_\alpha} \hbar_\alpha, \quad \hbar_\alpha = \begin{cases} h_\alpha, & i_\alpha = 1,\dots,N_\alpha - 1, \\ h_\alpha/2, & i_\alpha = 0, N_\alpha. \end{cases}$$

Let us transform each term in identity (4.1). For the first term, we have

$$[y^{(\alpha)}_{\bar t}, y^{(\alpha)}]_\alpha = \frac{1}{2}(\|y^{(\alpha)}\|^2_{L_2(\alpha)})_{\bar t} + \frac{\tau}{2}\|y_{\bar t}\|^2_{L_2(\alpha)},$$

where $\|\cdot\|_{L_2(\alpha)}$ means that the norm is taken with respect to the variable x_α for fixed values of the other variables. Further,

$$\begin{aligned}
[\overline{\Lambda}_\alpha y^{(\alpha)}, y^{(\alpha)}]_\alpha &= (\Lambda_\alpha y^{(\alpha)}, y^{(\alpha)})_\alpha + \Lambda^-_\alpha y^{(\alpha)} y^{(\alpha)}_0 \hbar_\alpha + \Lambda^+_\alpha y^{(\alpha)} y^{(\alpha)}_{N_\alpha} \hbar_\alpha \\
&= ((a_\alpha y^{(\alpha)}_{\bar x_\alpha})_{x_\alpha}, y^{(\alpha)}) - (d_\alpha y^{(\alpha)}, y^{(\alpha)})_\alpha - \left(a^{(1_\alpha)}_\alpha y^{(\alpha)}_{x_\alpha,0} - 0.5h_\alpha d^{(0)}_\alpha y^\alpha_0 - \beta_{-\alpha} y^\alpha_{N_\alpha} - \sum_{s=0}^{j} \rho_{s,j} y^s_{N_\alpha}\tau\right) y^{(\alpha)}_0 \\
&\quad - (a^{(N_\alpha)}_\alpha y^{(\alpha)}_{\bar x_\alpha, N_\alpha} + \overline{\beta}_{+\alpha} y^{(\alpha)}_{N_\alpha}) y^{(\alpha)}_{N_\alpha},
\end{aligned}$$

where $a^{(1_\alpha)} = a_{i_\alpha+1}$, $a_i = k_{i-1/2}(\bar t)$, and $\bar t = t_{j+1/2}$. Then the last expression can be written in the form

$$\begin{aligned}
[\overline{\Lambda}_\alpha y^{(\alpha)}, y^{(\alpha)}]_\alpha &= -(a_\alpha, y^2_{\bar x_\alpha}]_\alpha - (d_\alpha y^{(\alpha)}, y^{(\alpha)}) \\
&\quad - 0.5h_\alpha d^{(0)}_\alpha y^2_0 - \beta_{-\alpha} y_{N_\alpha} y_0 - y_0 \sum_{s=0}^{j} \rho_{s,j} y^s_{N_\alpha}\tau - \bar\beta_{+\alpha} y^2_{N_\alpha}.
\end{aligned} \tag{4.2}$$

For the right-hand side of identity (4.1), we have

$$\begin{aligned}
[\Phi^{(\alpha)}, y^{(\alpha)}]_\alpha &= (\varphi^{(\alpha)}, y^{(\alpha)}) + \overline{\mu}_{-\alpha} y^{(\alpha)}_0 \hbar_\alpha + \overline{\mu}_{+\alpha} y^{(\alpha)}_{N_\alpha} \hbar_\alpha = (\varphi^{(\alpha)}, y^{(\alpha)}) + \left(\frac{\mu_{-\alpha}}{0.5h_\alpha} + f_{\alpha,0}\right) y^{(\alpha)}_0 \hbar_\alpha \\
&\quad + \left(\frac{\mu_{+\alpha}}{0.5h_\alpha} + f_{\alpha,N_\alpha}\right) y^{(\alpha)}_{N_\alpha} \hbar_\alpha = [\varphi^{(\alpha)}, y^{(\alpha)}]_\alpha + \mu_{-\alpha} y^{(\alpha)}_0 + \mu_{+\alpha} y^{(\alpha)}_{N_\alpha}.
\end{aligned}$$

Using Lemma 1 in [14], we find the following estimates for the terms occurring on the right-hand side in relation (4.2):

$$(d, (y^{(\alpha)})^2) \le c_2 \|y^{(\alpha)}\|^2_{L_2(\omega_h)},$$

$$-0.5h_\alpha d^{(0)}_\alpha y^2_0 - \beta_{-\alpha} y_{N_\alpha} y_0 - \bar\beta_{+\alpha} y^2_{N_\alpha} \le M_1(y^2_0 + y^2_N) \le M_1(\varepsilon \|y_{\bar x_\alpha}]|^2_{L_2(\alpha)} + c(\varepsilon)\|y\|^2_{L_2(\alpha)}),$$

$$\begin{aligned}
-y_0 \sum_{s=0}^{j} \rho_{s,j} y^s_{N_\alpha}\tau &\le \frac{1}{2} y^2_0 + \frac{1}{2}\left(\sum_{s=0}^{j} \rho_{s,j} y^s_{N_\alpha}\tau\right)^2 \le M_2(\varepsilon \|y_{\bar x_\alpha}]|^2_{L_2(\alpha)} + c(\varepsilon)\|y\|^2_{L_2(\alpha)}) \\
&\quad + M_3 \sum_{s=0}^{j} (\varepsilon \|y^s_{\bar x_\alpha}]|^2_{L_2(\alpha)} + c(\varepsilon)\|y^s\|^2_{L_2(\alpha)})\tau,
\end{aligned}$$

$$[\varphi^{(\alpha)}, y^{(\alpha)}]_\alpha \le \frac{1}{2}\|\varphi^{(\alpha)}\|^2_{L_2(\alpha)} + \frac{1}{2}\|y^{(\alpha)}\|^2_{L_2(\alpha)},$$

$$\mu_{-\alpha} y_0^{(\alpha)} + \mu_{+\alpha} y_{N_\alpha}^{(\alpha)} \le \frac{\mu_{-\alpha}^2}{2} + \frac{\mu_{+\alpha}^2}{2} + \frac{1}{2}[(y_0^{(\alpha)})^2 + (y_{N_\alpha}^{(\alpha)})^2] \le \frac{1}{2}(\mu_{-\alpha}^2 + \mu_{+\alpha}^2) + \varepsilon\|y_{\bar{x}_\alpha}^{(\alpha)}]|^2_{L_2(\alpha)} + c(\varepsilon)\|y^{(\alpha)}\|^2_{L_2(\alpha)},$$

where $\varepsilon > 0$ and $c(\varepsilon) = 1/l_\alpha + 1/\varepsilon$.

Substituting the resulting estimates into identity (4.1), after summation over $i_\beta \ne i_\alpha$, $\beta = 1, \dots, p$, we find that

$$\begin{aligned} &\frac{1}{2}\|y^{j+\alpha/p}\|^2_{L_2(\bar{\omega}_h)} + \frac{c_0}{2}\tau\|y_{\bar{x}_\alpha}^{(\alpha)}]|^2_{L_2(\bar{\omega}_h)} \\ &\quad \le M_4\tau\|y^{(\alpha)}\|^2_{L_2(\bar{\omega}_h)} + M_5\tau\sum_{s=0}^{j}(\varepsilon\|y_{\bar{x}_\alpha}^s]|^2_{L_2(\bar{\omega}_h)} \\ &\qquad + c(\varepsilon)\|y^s\|^2_{L_2(\bar{\omega}_h)})\tau + \frac{1}{2}\tau\|\varphi^{j+\alpha/p}\|_{L_2(\bar{\omega}_h)} + \frac{1}{2}\tau\sum_{i_\beta\ne i_\alpha}(\mu_{-\alpha}^2(t_j) \\ &\qquad + \mu_{+\alpha}^2(t_j))\frac{H}{\hbar_\alpha} + \frac{1}{2}\|y^{j+(\alpha-1)/p}\|^2_{L_2(\bar{\omega}_h)}. \end{aligned} \tag{4.3}$$

We sum the inequalities in (4.3) first over $\alpha = 1, \dots, p$,

$$\begin{aligned} \|y^{j+1}\|^2_{L_2(\bar{\omega}_h)} + \frac{c_0}{2}\tau\sum_{\alpha=1}^{p}\|y_{\bar{x}_\alpha}^{j+\alpha/p}\|^2_{L_2(\bar{\omega}_h)} \le \sum_{\alpha=1}^{p}\tau\|\varphi^{j+\alpha/p}\|^2_{L_2(\bar{\omega}_h)} + \|y^j\|^2_{L_2(\bar{\omega}_h)} + M_4\tau\sum_{\alpha=1}^{p}\|y^{(\alpha)}\|^2_{L_2(\bar{\omega}_h)} \\ + M_5\tau\sum_{\alpha=1}^{p}\sum_{s=0}^{j}(\varepsilon\|y_{\bar{x}_\alpha}^s]|^2_{L_2(\bar{\omega}_h)} + c(\varepsilon)\|y^s\|^2_{L_2(\bar{\omega}_h)})\tau + \tau\sum_{\alpha=1}^{p}\sum_{i_\beta\ne i_\alpha}(\mu_{-\alpha}^2(t_j) + \mu_{+\alpha}^2(t_j))H/\hbar_\alpha, \end{aligned}$$

and then over j' from 0 to j,

$$\begin{aligned} \|y^{j+1}\|^2_{L_2(\bar{\omega}_h)} + \frac{c_0}{2}\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y_{\bar{x}_\alpha}^{j'+\alpha/p}]|^2_{L_2(\overline{\omega}_h)} \le \sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|\varphi^{j'+\alpha/p}\|^2_{L_2(\bar{\omega}_h)} + \|y^0\|^2_{L_2(\overline{\omega}_h)} \\ + M_4\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y^{j'+\alpha/p}\|^2_{L_2(\bar{\omega}_h)} + M_5\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{s=0}^{j'}(\varepsilon\|y_{\bar{x}_\alpha}^s]|^2_{L_2(\bar{\omega}_h)} + c(\varepsilon)\|y^s\|^2_{L_2(\bar{\omega}_h)})\tau \\ + \sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{i_\beta\ne i_\alpha}(\mu_{-\alpha}^2(t_j) + \mu_{+\alpha}^2(t_j))H/\hbar_\alpha. \end{aligned} \tag{4.4}$$

By estimating the fourth expression on the right-hand side in (4.4), we obtain

$$\begin{aligned} &\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{s=0}^{j'}(\varepsilon\|y_{\bar{x}_\alpha}^s]|^2_{L_2(\bar{\omega}_h)} + c(\varepsilon)\|y^s\|^2_{L_2(\bar{\omega}_h)})\tau \\ &\quad \le \varepsilon\sum_{j'=0}^{j}\sum_{s=0}^{j'}\tau\sum_{\alpha=1}^{p}\|y_{\bar{x}_\alpha}^s]|^2_{L_2(\bar{\omega}_h)}\tau + c(\varepsilon)\sum_{j'=0}^{j}\sum_{s=0}^{j'}\tau\sum_{\alpha=1}^{p}\|y^s\|^2_{L_2(\bar{\omega}_h)}\tau \\ &\quad \le \varepsilon T\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y_{\bar{x}_\alpha}^{j'+\alpha/p}]|^2_{L_2(\bar{\omega}_h)} + c(\varepsilon)T\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y^{j'+\alpha/p}\|^2_{L_2(\bar{\omega}_h)}. \end{aligned} \tag{4.5}$$

We take $\varepsilon = c_0/(4T)$ in inequality (4.5) and find from (4.4) that

$$\|y^{j+1}\|^2_{L_2(\bar\omega_h)} + M_6 \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|y^{j'+\alpha/p}_{\bar x_\alpha}]|^2_{L_2(\overline\omega_h)} \le \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|\varphi^{j'+\alpha/p}\|^2_{L_2(\bar\omega_h)} + \|y^0\|^2_{L_2(\overline\omega_h)}$$
$$+ M_7 \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|y^{j'+\alpha/p}\|^2_{L_2(\bar\omega_h)} + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \sum_{i_\beta \ne i_\alpha} (\mu^2_{-\alpha}(t_j) + \mu^2_{+\alpha}(t_j)) H/\hbar_\alpha.$$

The paper [15] showed that the following inequality holds:

$$\max_{1\le\alpha\le p} \|y^{j+\alpha/p}\|^2_{L_2(\bar\omega_h)} \le \nu_1 \sum_{j'=0}^{j} \tau \max_{1\le\alpha\le p} \|y^{j'+\alpha/p}\|^2_{L_2(\bar\omega_h)} + \nu_2 F^j, \tag{4.6}$$

where

$$F^j = \|y^0\|^2_{L_2(\overline\omega_h)} + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|\varphi^{j'+\alpha/p}\|^2_{L_2(\overline\omega_h)} + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \sum_{i_\beta \ne i_\alpha} (\mu^2_{-\alpha}(t_j) + \mu^2_{+\alpha}(t_j)) H/\hbar_\alpha.$$

Introducing the notation $g_{j+1} = \max\limits_{1\le\alpha\le p} \|y^{j+\alpha/p}\|^2_{L_2(\omega_h)}$, we write inequality (4.6) in the form

$$g_{j+1} \le \nu_1 \sum_{k=1}^{j} \tau g_k + \nu_2 F^j, \tag{4.7}$$

where ν_1 and ν_2 are known positive constants.

Using inequality (4.7), based on Lemma 4 in [12], we arrive at the a priori estimate

$$\begin{aligned} \|y^{j+1}\|^2_{L_2(\overline\omega_h)} + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|y^{j'+\alpha/p}_{\bar x_\alpha}]|^2_{L_2(\overline\omega_h)} \le M(t) \Bigg[\sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|\varphi^{j'+\alpha/p}\|^2_{L_2(\overline\omega_h)} + \|y^0\|^2_{L_2(\overline\omega_h)} \\ + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \sum_{i_\beta \ne i_\alpha} (\mu^2_{-\alpha}(t_{j'}) + \mu^2_{+\alpha}(t_{j'})) H/\hbar_\alpha \Bigg], \end{aligned} \tag{4.8}$$

where $M(t) > 0$ is independent of h_α and τ.

Thus, the following assertion holds.

Theorem 2. *The locally one-dimensional scheme* (2.4), (2.5) *is stable with respect to the initial data and the right-hand side*, *the estimate* (4.8) *holding true for the solution of problem* (2.4), (2.5).

5. UNIFORM CONVERGENCE OF THE LOCALLY ONE-DIMENSIONAL SCHEME

By analogy with [13, p. 528], we represent the solution of problem (3.1) as the sum $z_{(\alpha)} = \upsilon_{(\alpha)} + \eta_{(\alpha)}$, $z_{(\alpha)} = z^{j+\alpha/p}$, where the quantity $\eta_{(\alpha)}$ is determined by the conditions

$$\begin{gathered} (\eta_{(\alpha)} - \eta_{(\alpha-1)})/\tau = \overset{\circ}{\psi}_\alpha, \quad x \in \omega_{h_\alpha} + \gamma_{h,\alpha}, \quad \alpha = 1, \dots, p, \\ \eta(x, 0) = 0; \end{gathered} \tag{5.1}$$

here

$$\overset{\circ}{\psi}_\alpha = \begin{cases} \overset{\circ}{\psi}_\alpha, & x_\alpha \in \omega_{h_\alpha}, \\ \overset{\circ}{\psi}_{-\alpha}, & x_\alpha = 0, \\ \overset{\circ}{\psi}_{+\alpha}, & x_\alpha = l_\alpha. \end{cases}$$

It follows from (5.1) that $\eta^{j+1} = \eta_{(p)} = \eta^j + \tau(\mathring{\psi}_1 + \mathring{\psi}_2 + \ldots + \mathring{\psi}_p) = \eta^j = \ldots = \eta^0 = 0$ and $\eta_{(\alpha)} = \tau(\mathring{\psi}_1 + \mathring{\psi}_2 + \ldots + \mathring{\psi}_\alpha) = -\tau(\mathring{\psi}_{\alpha+1} + \ldots + \mathring{\psi}_p) = O(\tau)$.

The function $\upsilon_{(\alpha)}$ is determined by the conditions

$$(\upsilon_{(\alpha)} - \upsilon_{(\alpha-1)})/\tau = \Lambda_\alpha \upsilon_{(\alpha)} + \tilde{\psi}_\alpha, \quad \tilde{\psi}_\alpha = \Lambda_\alpha \eta_{(\alpha)} + \psi^*_\alpha, \quad x \in \omega_{h_\alpha}, \tag{5.2}$$

$$(\upsilon_{(\alpha)} - \upsilon_{(\alpha-1)})/\tau = \Lambda^-_\alpha \upsilon_{(\alpha)} + \Lambda^-_\alpha \eta_{(\alpha)} + 2\psi^*_{-\alpha}/h_\alpha, \quad x_\alpha = 0, \tag{5.3}$$

$$(\upsilon_{(\alpha)} - \upsilon_{(\alpha-1)})/\tau = \Lambda^+_\alpha \upsilon_{(\alpha)} + \Lambda^+_\alpha \eta_{(\alpha)} + 2\psi^*_{+\alpha}/h_\alpha, \quad x_\alpha = l_\alpha, \tag{5.4}$$

$$\upsilon(x, 0) = 0. \tag{5.5}$$

If there exist derivatives $\partial^4 u/\partial x_\alpha^2 \partial x_\beta^2$, $\alpha \neq \beta$, continuous in the closed domain $\overline{Q}_T$, then $\Lambda_\alpha \eta_{(\alpha)} = -\tau \Lambda_\alpha(\mathring{\psi}_{\alpha+1} + \ldots + \mathring{\psi}_p) = O(\tau)$.

Estimating the solution of problem (5.2)–(5.5) with the use of Theorem 2, we obtain

$$\begin{aligned} \|\upsilon^{j+1}\|^2_{L_2(\overline{\omega}_h)} + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|\upsilon^{j+\alpha/p}_{\bar{x}_\alpha}]|^2_{L_2(\overline{\omega}_h)} \le M(t)\Bigg[\sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|\tilde{\psi}^{j'+\alpha/p}_\alpha\|^2_{L_2(\overline{\omega}_h)} \\ + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \sum_{i_\beta \neq i_\alpha} (\tilde{\psi}^2_{-\alpha}(t_{j'}) + \tilde{\psi}^2_{+\alpha}(t_{j'}))H/\hbar_\alpha\Bigg]. \end{aligned} \tag{5.6}$$

Since $\eta^j = 0$, $\eta_{(\alpha)} = O(\tau)$ and $\|z^j\| \le \|\upsilon^j\|$, we see that the estimate (5.6) implies the following assertion.

Theorem 3. *Let problem* (1.1)–(1.4) *have a unique solution* $u(x,t)$ *continuous in* $\overline{Q}_T$, *and assume that there exist derivatives* $\partial^2 u/\partial t^2$, $\partial^4 u/\partial x_\alpha^2 \partial x_\beta^2$, $\partial^3 u/\partial x_\alpha^2 \partial t$, *and* $\partial^2 f/\partial x_\alpha^2$, $1 \le \alpha, \beta \le p$, *continuous in the domain* $\overline{Q}_T$. *Then the locally one-dimensional scheme* (2.4), (2.5) *converges at the rate* $O(|h|^2 + \tau)$, *so that*

$$\|y^{j+1} - u^{j+1}\|_1 \le M(|h|^2 + \tau), \quad |h|^2 = h_1^2 + h_2^2 + \ldots + h_p^2,$$
$$\|z^{j+1}\|_1 = \left(\|z^{j+1}\|^2_{L_2(\overline{\omega}_h)} + \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \|z_{\bar{x}_\alpha}]|^2_{L_2(\overline{\omega}_h)}\right)^{1/2}.$$

6. ALGORITHM FOR THE NUMERICAL SOLUTION OF THE NONLOCAL BOUNDARY VALUE PROBLEM

Let us write the Robin boundary value problem (1.1)–(1.4) for $0 \le x_\alpha \le l_\alpha$, $\alpha = 2$, $p = 2$. Then we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1(x_1, x_2, t)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2(x_1, x_2, t)\frac{\partial u}{\partial x_2}\right) - q_1(x_1, x_2, t)u - q_2(x_1, x_2, t)u + f(x_1, x_2, t),$$

$$k_\alpha(0, x', t)u_{x_\alpha}(0, x', t) = \beta_{-\alpha}(0, x', t)u(l_\alpha, x', t) + \int_0^t \rho(t, \tau)u(l_\alpha, x', \tau)\, d\tau - \mu_{-\alpha}(0, x', t),$$
$$-k_\alpha(l_\alpha, x', t)u_{x_\alpha}(l_\alpha, x', t) = \beta_{+\alpha}(l_\alpha, x', t)u(l_\alpha, x', t) - \mu_{+\alpha}(l_\alpha, x', t), \quad 0 \le t \le T,$$
$$u(x_1, x_2, 0) = u_0(x_1, x_2).$$

Consider the mesh $x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha$, $\alpha = 1, 2$, $t_j = j\tau$, where $i_\alpha = 0, \ldots, N_\alpha$, $h_\alpha = l_\alpha/N_\alpha$, $j = 0, \ldots, m$, $\tau = T/m$. We introduce one fractional step $t_{j+1/2} = t_j + 0.5\tau$. Consider the mesh function $y^{j+\alpha/p}_{i_1,i_2} = y^{j+\alpha/p} = y(i_1 h_1, i_2 h_2, (j + 0.5\alpha)\tau)$, $\alpha = 1, 2$.

We write the locally one-dimensional scheme

$$(y^{j+1/2}-y^j)/\tau=\Lambda_1 y^{j+1/2}+\varphi_1,\quad (y^{j+1}-y^{j+1/2})/\tau=\Lambda_2 y^{j+1}+\varphi_2, \tag{6.1}$$

$$y^{j+1/2}_{0,i_2}=\varkappa_{11}(i_2h_2,t_{j+1/2})y^{j+1/2}_{1,i_2}+\widetilde{\varkappa}_{11}(i_2h_2,t_{j+1/2})y^{j+1/2}_{N_1,i_2}+\mu_{11}(i_2h_2,t_{j+1/2}),$$

$$y^{j+1/2}_{N_1,i_2}=\varkappa_{12}(i_2h_2,t_{j+1/2})y^{j+1/2}_{N_1-1,i_2}+\mu_{12}(i_2h_2,t_{j+1/2}),$$

$$y^{j+1}_{i_1,0}=\varkappa_{21}(i_1h_1,t_{j+1})y^{j+1}_{i_1,1}+\widetilde{\varkappa}_{21}(i_1h_1,t_{j+1})y^{j+1}_{i_1,N_2}+\mu_{21}(i_1h_1,t_{j+1}),$$

$$y^{j+1}_{i_1,N_2}=\varkappa_{22}(i_1h_1,t_{j+1})y^{j+1}_{i_1,N_2-1}+\mu_{22}(i_1h_1,t_{j+1}), \tag{6.2}$$

$$y^0_{i_1,i_2}=u_0(i_1h_1,i_2,h_2), \tag{6.3}$$

$$\Lambda_\alpha y^{j+\alpha/p}=(a_\alpha y^{j+\alpha/p}_{\bar{x}_\alpha})_{x_\alpha}-d_\alpha y^{j+\alpha/p},\quad \alpha=1,2,$$

$$\varphi_\alpha=\frac{1}{2}f(x_1,x_2,t_{j+0.5\alpha})\quad\text{or}\quad \varphi_1=0,\quad \varphi_2=f(x_1,x_2,t_{j+1}).$$

Let us provide design formulas for the solution of problem (6.1)–(6.3).

At the first stage, we find the solution $y^{j+1/2}_{i_1,i_2}$. To this end, the following problem is solved for various $i_2=1,\dots,N_2-1$:

$$A_{1(i_1,i_2)}y^{j+1/2}_{i_1-1,i_2}-C_{1(i_1,i_2)}y^{j+1/2}_{i_1,i_2}+B_{1(i_1,i_2)}y^{j+1/2}_{i_1+1,i_2}=-F^{j+1/2}_{1(i_1,i_2)},\quad 0<i_1<N_1, \tag{6.4}$$

$$y^{j+1/2}_{0,i_2}=\varkappa_{11}(i_2h_2,t_{j+1/2})y^{j+1/2}_{1,i_2}+\widetilde{\varkappa}_{11}(i_2h_2,t_{j+1/2})y^{j+1/2}_{N_1,i_2}+\mu_{11}(i_2h_2,t_{j+1/2}),$$

$$y^{j+1/2}_{N_1,i_2}=\varkappa_{12}(i_2h_2,t_{j+1/2})y^{j+1/2}_{N_1-1,i_2}+\mu_{12}(i_2h_2,t_{j+1/2}),$$

where

$$A_{1(i_1,i_2)}=\frac{(a_1)_{i_1,i_2}}{h_1^2},\quad B_{1(i_1,i_2)}=\frac{(a_1)_{i_1+1,i_2}}{h_1^2},$$

$$C_{1(i_1,i_2)}=A_{1(i_1,i_2)}+B_{1(i_1,i_2)}+\frac{1}{\tau}+\frac{1}{p}(d_1)_{i_1,i_2},\quad F^{j+1/2}_{1(i_1,i_2)}=\frac{1}{\tau}y^j_{i_1,i_2}+(\varphi_1)_{i_1,i_2},$$

$$\varkappa_{11}(i_2h_2,t_{j+1/2})=\frac{(a_1)_{1,i_2}}{h_1}\left(\frac{(a_1)_{1,i_2}}{h_1}+0.5hd^{j+1/2}_{1,i_2}+\frac{0.5h_1}{\tau}\right)^{-1},$$

$$\widetilde{\varkappa}_{11}(i_2h_2,t_{j+1/2})=\beta_{-1,i_2}\left(\frac{(a_1)_{1,i_2}}{h_1}+0.5hd^{j+1/2}_{1,i_2}+\frac{0.5h_1}{\tau}\right)^{-1},$$

$$\varkappa_{12}(i_2h_2,t_{j+1/2})=\frac{(a_1)_{N_1,i_2}}{h_1}\left(\frac{(a_1)_{N_1,i_2}}{h_1}+\overline{\beta}^{j+1/2}_{+1,i_2}+\frac{0.5h_1}{\tau}\right)^{-1},$$

$$\mu_{11}(i_2h_2,t_{j+1/2})=\left(\bar{\mu}_{-1}(i_2h_2,t_{j+1/2})+\frac{0.5h_1}{\tau}y^j_0+\sum_{s=0}^{j}\rho_{s,j}y^s_{N_1,i_2}\tau\right)\left(\frac{(a_1)_{1,i_2}}{h_1}+0.5hd^{j+1/2}_{1,i_2}+\frac{0.5h_1}{\tau}\right)^{-1},$$

$$\mu_{12}(i_2h_2,t_{j+1/2})=\left(\bar{\mu}_{+1}(i_2h_2,t_{j+1/2})+\frac{0.5h_1}{\tau}y^j_{N_1}\right)\left(\frac{(a_1)_{N_1,i_2}}{h_1}+\overline{\beta}^{j+1/2}_{+1,i_2}+\frac{0.5h_1}{\tau}\right)^{-1}.$$

At the second stage, we find the solution $y^{j+1}_{i_1,i_2}$. To this end, by analogy with the first stage, for various $i_1=1,\dots,N_1-1$ we solve the problem

$$A_{2(i_1,i_2)}y^{j+1}_{i_1,i_2-1}-C_{2(i_1,i_2)}y^{j+1}_{i_1,i_2}+B_{2(i_1,i_2)}y^{j+1}_{i_1,i_2+1}=-F^{j+1}_{2(i_1,i_2)},\quad 0<i_2<N_2, \tag{6.5}$$

$$y^{j+1/2}_{i_1,0}=\varkappa_{21}(i_1h_1,t_{j+1})y^{j+1}_{i_1,1}+\widetilde{\varkappa}_{21}(i_1h_1,t_{j+1})y^{j+1}_{i_1,N_2}+\mu_{21}(i_1h_1,t_{j+1}),$$

$$y^{j+1}_{i_1,N_2} = \varkappa_{22}(i_1h_1, t_{j+1})y^{j+1}_{i_1,N_2-1} + \mu_{22}(i_1h_1, t_{j+1}),$$

$$A_{2(i_1,i_2)} = \frac{(a_2)_{i_1,i_2}}{h_2^2}, \quad B_{2(i_1,i_2)} = \frac{(a_2)_{i_1,i_2+1}}{h_2^2},$$

$$C_{2(i_1,i_2)} = A_{2(i_1,i_2)} + B_{2(i_1,i_2)} + \frac{1}{\tau} + \frac{1}{p}(d_2)_{i_1,i_2}, \quad F^{j+1/2}_{2(i_1,i_2)} = \frac{1}{\tau}y^j_{i_1,i_2} + (\varphi_1)_{i_1,i_2},$$

$$\varkappa_{21}(i_1h_1, t_{j+1}) = \frac{(a_2)_{i_1,1}}{h_2}\left(\frac{(a_2)_{i_1,1}}{h_1} + 0.5hd^{j+1}_{i_1,2} + \frac{0.5h_2}{\tau}\right)^{-1},$$

$$\widetilde{\varkappa}_{21}(i_1h_1, t_{j+1}) = \beta_{i_1,-2}\left(\frac{(a_2)_{i_1,1}}{h_1} + 0.5hd^{j+1}_{i_1,2} + \frac{0.5h_2}{\tau}\right)^{-1},$$

$$\varkappa_{22}(i_1h_1, t_{j+1}) = \frac{(a_2)_{i_1,N_2}}{h_2}\left(\frac{(a_2)_{i_1,N_2}}{h_2} + \overline{\beta}^{j+1}_{i_1,+2} + \frac{0.5h_2}{\tau}\right)^{-1},$$

$$\mu_{21}(i_1h_1, t_{j+1}) = \left(\bar{\mu}_{-2}(i_1h_1, t_{j+1}) + \frac{0.5h_2}{\tau}y^j_0 + \sum_{s=0}^{j}\rho_{s,j}y^s_{i_1,N_2}\tau\right)\left(\frac{(a_2)_{i_1,1}}{h_2} + 0.5hd^{j+1}_{i_1,2} + \frac{0.5h_2}{\tau}\right)^{-1},$$

$$\mu_{22}(i_1h_1, t_{j+1}) = \left(\bar{\mu}_{+2}(i_1h_1, t_{j+1}) + \frac{0.5h_2}{\tau}y^j_{N_2}\right)\left(\frac{(a_2)_{i_1,N_2}}{h_2} + \overline{\beta}^{j+1}_{i_1,+2} + \frac{0.5h_2}{\tau}\right)^{-1}.$$

The bordering method [16, p. 187] is used to solve problems (6.4) and (6.5). With this method, solving each problem reduces to solving two systems of linear algebraic equations with a tridiagonal coefficient matrix. This is easy to do by the Thomas method.

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