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CONTROL THEORY

Asymptotics of the Solution of a Singular Optimal Distributed Control Problem with Essential Constraints in a Convex Domain

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Abstract—We consider an optimal distributed control problem in a convex planar domain with a quadratic performance functional and a small parameter multiplying the higher derivatives. Further, the characteristics of the limit equation of the problem are parallel to the y-axis. Using the method of matching asymptotic expansions in conjunction with the auxiliary parameter method, we derive a complete asymptotic expansion (up to any power of the small parameter) of the optimal state of the controlled system and the optimal control.

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1. STATEMENT OF THE PROBLEM AND AUXILIARY ASSERTIONS

This paper deals with studying the asymptotics of the solution of a bisingular [1] optimal distributed control [2] problem in a planar convex domain Ω with smooth boundary Γ and with a small parameter $\varepsilon > 0$ multiplying the higher derivatives in the elliptic operator $\mathcal{L}_\varepsilon$. The optimality conditions of the problem under study are stated in terms of a system of singularly perturbed equations depending on an additional parameter and an additional condition imposed on this parameter.

A feature of the differential operator $\mathcal{L}_\varepsilon$ in the problem is the fact that the degenerate operator $\mathcal{L}_0$ has characteristics tangent to the boundaries of the domain Ω. Even for boundary value problems with such an operator, the ordinary perturbation theory series has singularities in a neighborhood of the point of tangency, and one has to apply the method of matching asymptotic expansions [1] (or its analogs) (see, e.g., [3, 4]) for constructing the complete asymptotic expansion. It is even more so in distributed control problems (see, e.g., [5–8]). The asymptotics of the distributed control for an operator with a small parameter multiplying the highest derivative, though in an essentially different domain, was considered in [9]. A similar problem was considered in [10], where the authors studied the case where the control constraints degenerate. (The precise statement of this condition can be found in Sec. 2 of the present paper.)

In what follows, H^1 and H^2 are Sobolev spaces (see, e.g., [2, Ch. 1, Secs. 3.1–3.3]). Let us proceed to the rigorous statement of the problem under study. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma := \partial\Omega$ (Ω is a smooth manifold with boundary). Consider the following distributed control problem [2, Ch. 2, Sec. 2, relations (2.8), (2.9)]:

$$\mathcal{L}_\varepsilon z_\varepsilon := -\varepsilon^2\Delta z_\varepsilon + b(x)\frac{\partial z_\varepsilon}{\partial y} + a(x,y)z_\varepsilon = f(x,y) + u_\varepsilon(x,y), \quad (x,y)\in\Omega, \quad z_\varepsilon \in H_0^1(\Omega), \tag{1.1}$$

$$J(u) := \|z_\varepsilon - z_d\|^2 + \beta^{-1}\|u\|^2 \longrightarrow \inf, \quad u \in \mathcal{U}, \tag{1.2}$$

$$\mathcal{U} = \mathcal{U}(1), \quad \text{where} \quad \mathcal{U}(r) := \{u \in L_2(\Omega) : \|u\| \le r\}. \tag{1.3}$$

Here $\beta > 0$, $H_0^1(\Omega)$ is the Sobolev space of functions vanishing on the boundary Γ, $\|\cdot\|$ is the norm on the space $L_2(\Omega)$, and the given functions f, z_d, a, and b satisfy the conditions

$$\begin{gathered} f, z_d, a \in C^\infty(\overline{\Omega}), \quad a(x,y) \ge A > 0 \quad \text{for} \quad (x,y)\in\Omega, \\ b \in C^\infty(\overline{\Omega}), \quad b(x) \ge B > 0 \quad \text{for} \quad (x,y)\in\Omega. \end{gathered} \tag{1.4}$$

In this case, the existence of an optimal control $u_\varepsilon(\cdot)$ and the corresponding solution $z_\varepsilon(\cdot)$ is equivalent to the existence of a function $p_\varepsilon \in H_0^1(\Omega)$ such that (see [2, Sec. 2.2, relations (2.10); 10, relations (1.13), (1.14)])

$$\begin{gathered} \mathcal{L}_\varepsilon z_\varepsilon = f(x,y) + u_\varepsilon, \quad \mathcal{L}_\varepsilon^* p_\varepsilon - z_\varepsilon = -z_d(x,y), \quad (x,y)\in\Omega, \quad z_\varepsilon, p_\varepsilon \in H^1(\Omega), \\ z_\varepsilon = 0, \quad p_\varepsilon = 0, \quad (x,y)\in\Gamma, \\ (p + \beta^{-1} u_{\text{opt}}, (\widetilde{v} - u_{\text{opt}})) \ge 0 \quad \text{for all} \quad \widetilde{v} \in \mathcal{U}. \end{gathered} \tag{1.5}$$

Here

$$\mathcal{L}_\varepsilon^* p := -\varepsilon^2 \Delta p - b(x)\frac{\partial p}{\partial y} + a(x,y)p.$$

As was shown in [11, Lemma 1], in this case condition (1.5) amounts to the following system of conditions:

$$u_\varepsilon = -\lambda_\varepsilon p_\varepsilon, \quad \lambda_\varepsilon \in (0,\beta], \quad \lambda_\varepsilon \|p_\varepsilon\| \le 1 \quad \text{and} \quad (\beta - \lambda_\varepsilon)(1 - \lambda_\varepsilon\|p_\varepsilon\|) = 0. \tag{1.6}$$

The original problem has thus been reduced to the system of equations

$$\begin{gathered} \mathcal{L}_\varepsilon z_\varepsilon + \lambda_\varepsilon p_\varepsilon = f(x,y), \quad \mathcal{L}_\varepsilon^* p_\varepsilon - z_\varepsilon = -z_d(x,y), \quad (x,y)\in\Omega, \quad z_\varepsilon, p_\varepsilon \in H^1(\Omega), \\ z_\varepsilon = 0, \quad p_\varepsilon = 0, \quad (x,y)\in\Gamma, \end{gathered} \tag{1.7}$$

depending on a scalar parameter λ_ε with the additional condition (1.6).

The aim of the present paper is to study the behavior of z_ε, p_ε, and λ_ε as $\varepsilon \to 0$ and determine the complete asymptotic expansions of the indicated variables as $\varepsilon \to 0$.

In the sequel, we will often denote positive constants depending only on the domain Ω and the functions $b(x)$ and $a(x,y)$ by the same letter K (possibly, with indices).

Along with system (1.7), we will also consider a system of the more general form

$$\begin{gathered} \mathcal{L}_\varepsilon z + \lambda p = f_1(x,y), \quad \mathcal{L}_\varepsilon^* p - z = f_2(x,y), \quad (x,y)\in\Omega, \\ z = g_1, \quad p = g_2, \quad (x,y)\in\Gamma. \end{gathered} \tag{1.8}$$

Theorem 1. *Problem* (1.8) *is uniquely solvable for any* $f_i \in L_2(\Omega)$, $g_i \in H^{3/2}(\Gamma)$ $(i = 1,2)$, *and* $\varepsilon > 0$. *Its solution* (z,p) *belongs to the class* $H^2(\Omega)\times H^2(\Omega)$, *and if* $f_i \in C^\infty(\overline{\Omega})$ *and* $g_i \in C^\infty(\Gamma)$, *then the solution belongs to the class* $C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$.

Proof. By the trace theorems [12, Ch. 1, Theorem 8.3], the mapping $H^2(\Omega) \ni w \mapsto w|_\Gamma$ is a surjection. Therefore, there exist $\widetilde{g}_j \in H^2(\Omega)$ such that $\widetilde{g}_j|_\Gamma = g_j$. Passing to the new unknown functions $z - \widetilde{g}_1$ and $p - \widetilde{g}_2$, we arrive at a function with the zero boundary condition. After this, the proof of the above theorem almost verbatim reproduces that of Theorem 1 in [8]. The proof of the theorem is complete.

Note that if $g_1 = g_2 = 0$ and (z,p) is a solution of system (1.8), then for any $v, w \in H_0^1(\Omega)$ we have the relations

$$\begin{gathered} \varepsilon^2(\nabla z, \nabla v) + \left(b(x)\frac{\partial}{\partial y} z, v\right) + (a(x,y)z, v) + \lambda(p,v) = (f_1, v), \\ \varepsilon^2(\nabla p, \nabla w) - \left(b(x)\frac{\partial}{\partial y} p, w\right) + (a(x,y)p, w) - (z,w) = (f_2, w), \\ \left(b(x)\frac{\partial}{\partial y} z, z\right) = \left(b(x)\frac{\partial}{\partial y} p, p\right) = 0. \end{gathered} \tag{1.9}$$

Therefore, assuming that $v = p$ and $w = z$ in the first two relations in (1.9) and then subtracting the second from the first, we obtain

$$\|z\|^2 + \lambda\|p\|^2 = (f_1, p) - (f_2, z). \tag{1.10}$$

It was shown in [10] that if $(z_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ is a solution of system (1.7), (1.6), then the following properties hold:

$$\|z_\varepsilon\|_C = O(1), \quad \|p_\varepsilon\|_C = O(1) \quad \text{as} \quad \varepsilon \to 0, \quad \lambda_\varepsilon \geq \lambda_* > 0 \quad \text{for some} \quad \lambda_* > 0 \tag{1.11}$$

(Assertion 2), and the a priori estimates were derived for the solution to system (1.8) (Theorem 2): if $f_i \in C^\infty(\overline{\Omega})$, $g_i \in C^\infty(\Gamma)$, $i = 1, 2$, and $\lambda \in [\lambda_*, \lambda^*]$, $\lambda_* > 0$, then for the solutions (z, p) of problem (1.8) we have the following estimates uniform in λ:

$$\max\{\varepsilon^3\|z\|_C, \varepsilon^3\|p\|_C\} \leq K(\|f_1\|_C + \|f_2\|_C + \|g_1\|_C + \|g_2\|_C). \tag{1.12}$$

Here $\|\cdot\|_C$ is the norm on the space $C(\overline{\Omega})$.

2. APPROXIMATION THEOREMS

To justify the asymptotic expansions of solutions of problem (1.7), (1.6), we will need theorems on the estimate of the deviation of the exact solution $(z_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ to this problem from the solutions (Z_m, P_m, Λ_m) to the approximation problem

$$\begin{aligned} \mathcal{L}_\varepsilon Z_m + \Lambda_m P_m &= f(x) + f_{1,m}(x), \quad x \in \Omega, \\ \mathcal{L}_\varepsilon^* P_m - Z_m &= -z_d + f_{2,m}(x), \quad x \in \Omega, \\ Z_m = g_{1,m}(x), \quad P_m &= g_{2,m}(x), \quad x \in \Gamma, \end{aligned} \tag{2.1}$$

in the case where, as $\varepsilon \to 0$, the relations

$$f_{i,m} \in C^\infty(\overline{\Omega}), \quad g_{i,m} \in C^\infty(\Gamma), \quad \|f_{i,m}\|_C = O(\varepsilon^m), \quad \|g_{i,m}\|_C = O(\varepsilon^m), \quad i = 1, 2, \tag{2.2}$$

hold, and additionally, we will need an approximation to condition (1.6).

The paper [10] treats the case where the control constraints degenerate, i.e., $\lambda_\varepsilon\|p_\varepsilon\| < 1$ for all sufficiently small $\varepsilon > 0$. In this case, condition (1.6) transforms into the relation $\lambda_\varepsilon = \beta$ for all sufficiently small $\varepsilon > 0$.

In the present paper, we will assume that for all sufficiently small $\varepsilon > 0$ we have the inequality $0 < \lambda_\varepsilon < \beta$. In this case, condition (1.6) becomes the relation

$$\lambda_\varepsilon\|p_\varepsilon\| = 1. \tag{2.3}$$

Subject to condition (2.3), the approximation to condition (1.6) has the form

$$\Lambda_m\|P_m\| = 1 + O(\varepsilon^m), \tag{2.4}$$

and, to obtain an approximation theorem, we will need an auxiliary assertion on the dependence of the optimal solution $u_{\varepsilon,r}$ of problem (1.1)–(1.3) on r under the condition $\|u_{\varepsilon,r}\| = r$.

Assertion 1. *Let conditions* (1.4) *be satisfied, and let $u_{\varepsilon,r}$ be a solution of problem* (1.1), (1.2) *with $\mathcal{U} = \mathcal{U}(r)$ and $\|u_{\varepsilon,r}\| = r$ for all $r \in [r_*, r^*]$. Then, for some $K > 0$ the following estimate holds for all $r, r' \in [r_*, r^*]$:*

$$\|u_r - u_{r'}\| \leq K|r - r'|.$$

Proof. Let $z_{\varepsilon,0}$ be a solution of problem (1.1) with $u = 0$, and assume that the operator [4] $\mathcal{A} : L_2(\Omega) \to L_2(\Omega)$ takes the function u_ε to the solution of problem (1.1) with $f = 0$. Then $z_\varepsilon = z_{\varepsilon,0} + \mathcal{A}u_\varepsilon$ and the performance functional acquires the form $J(u_\varepsilon) = \|\mathcal{A}u_\varepsilon + v_0\|^2 + \beta^{-1}\|u_\varepsilon\|^2$, where $v_0 := z_{\varepsilon,0} - z_d$.

By Theorem 3 in [13] , we have the inequality $\|u_r - u_{r'}\| \leq K|r - r'|\,\|\mathcal{A}\|^2(\|\mathcal{A}\| + \|v_0\|)^4$. According to the definition of the norm $\|\mathcal{A}\|$, by virtue of (1.11), we obtain $\|\mathcal{A}\| \leq K_1$, but then also $\|v_0\| \leq \|z_{\varepsilon,0}\| + \|z_d\| \leq K_2$. The proof of the assertion is complete.

Theorem 2. *Let conditions* (1.4), (2.2), (2.3), *and* (2.4) *be satisfied. If $(z_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ is a solution of problem* (1.7), (1.6) *and (Z_m, P_m, Λ_m) is a solution of problem* (2.1), *then as $\varepsilon \to 0$ and for $m \geq 3$ one has the relation*

$$\max\{\|z_\varepsilon - Z_m\|, \|p_\varepsilon - P_m\|, |\lambda_\varepsilon - \Lambda_m|\} = O(\varepsilon^{m-3}).$$

Proof. The proof of this theorem is conducted by the scheme of proof of Theorem 4 in [13] with allowance for the estimate (1.12) and Assertion 1.

In what follows, we assume that the domain Ω is *strictly convex*.

Then there exist points $M_i = (x_i, y_i) \in \Gamma$, $i = 1, 2$, at which the equation of the tangent to Γ has the form $x = x_i$. The points M_i split the domain Γ into two parts Γ_j, the lower ($j = 1$) and the upper ($j = 2$) one, and the part Γ_j is the graph of the function $\varphi_j(x)$, $x \in [x_1, x_2]$, $j = 1, 2$. In this case,

$$\varphi_j(x) \in C([x_1, x_2]) \bigcap C^\infty((x_1, x_2)), \quad \varphi_j(x_i) = y_i, \quad \varphi_j'(x_i - (-1)^i 0) = \infty. \tag{2.5}$$

Moreover, in neighborhoods of the points M_i there exists one more parametrization of the boundary $\Gamma : x = \psi_i(y)$. Note that ψ_1 is a convex function ($\psi_1'' \geq 0$), ψ_2 is a concave function ($\psi_1'' \leq 0$), and $\psi_i'(y_i) = 0$.

To simplify the technicalities (condition (2.6) stated below affects only the form of the asymptotic expansions of the solution to the problem under consideration but not the method of producing these expansions), we will assume that

$$x_1 = y_1 = 0, \quad \psi_1''(y_1) > 0, \quad \psi_2''(y_2) < 0. \tag{2.6}$$

Note that the vertical lines $x = \text{const}$ are the characteristics of the operators $\mathcal{L}_0$ and $\mathcal{L}_0^*$ obtained, respectively, from $\mathcal{L}_\varepsilon$ and $\mathcal{L}_\varepsilon^*$ if we set $\varepsilon = 0$ in their definition.

Finding an asymptotic expansion of the solution to the boundary value problem with the operator $\mathcal{L}_\varepsilon$ in a domain with condition (2.6) is considered in detail in [1, Ch. IV, Sec. 3].

3. OUTER ASYMPTOTIC EXPANSION

By analogy with [9], the outer asymptotic expansion for the functions z_ε and p_ε has exponentially decaying boundary layers for each of them in neighborhoods of both curves Γ_1 and Γ_2.

We seek the outer expansion for z_ε and p_ε in the form

$$\begin{aligned} \overset{\text{out}}{z} &:= \sum_{k=0}^{\infty} \varepsilon^{2k} (z_{2k}(x, y) + \overset{1}{z}_{2k}(x, \eta_1) + \overset{2}{z}_{2k}(x, \eta_2)), \\ \overset{\text{out}}{p} &:= \sum_{k=0}^{\infty} \varepsilon^{2k} (p_{2k}(x, y) + \overset{1}{p}_{2k}(x, \eta_1) + \overset{2}{p}_{2k}(x, \eta_2)), \end{aligned} \tag{3.1}$$

where $\eta_j = (-1)^j (\varphi_j(x) - y)/\varepsilon^2$, and the expansion for λ_ε in the form

$$\lambda_\varepsilon := \sum_{k=0}^{\infty} \varepsilon^{2k} \lambda_{2k}. \tag{3.2}$$

Let us substitute the series (3.1) and (3.2) into system (1.7) and match the terms of the same form and of the same order of smallness. In this case, to find the coefficients of the functions $\overset{j}{z}_{2k}$ and $\overset{j}{p}_{2k}$ from the boundary layers, we should expand the function $a(x, y)$ into Taylor series with respect to the second variable in neighborhoods of the points $\varphi_j(x)$ and replace $(y - \varphi_j(x))$ with $(-1)^{j+1} \varepsilon^2 \eta_j$. As a result, for determining the functions z_{2k}, p_{2k}, $\overset{j}{z}_{2k}$, $\overset{j}{p}_{2k}$, and λ_{2k} we obtain the equations

$$\begin{aligned} &\mathcal{L}_0 z_0 + \lambda_0 p_0 = f(x, y), \quad \mathcal{L}_0^* p_0 - z_0 = -z_d(x, y), \\ &\mathcal{L}_0 z_{2k} + \lambda_0 p_{2k} = -\lambda_{2k} p_0 + \Delta z_{2k-2} - \sum_{s=1}^{k-1} \lambda_{2s} p_{2k-2s}, \quad k \geq 1, \\ &\mathcal{L}_0^* p_{2k} - z_{2k} = \Delta p_{2k-2}, \quad k \geq 1, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \overset{j}{\mathcal{M}}_1 \overset{j}{z}_0 = 0, \quad \overset{j}{\mathcal{M}}_2 \overset{j}{p}_0 = 0, \quad j = 1, 2, \\ \overset{j}{\mathcal{M}}_1 \overset{j}{z}_{2k} = \overset{j,1}{\mathcal{F}}_{2k}(x, \eta_j; \overset{j}{\mathbf{z}}_{2k-2}, \overset{j}{\mathbf{p}}\ j_{2k-2}), \quad \overset{j}{\mathcal{M}}_2 \overset{j}{p}\ j_{2k} = \overset{j,2}{\mathcal{F}}_{2k}(x, \eta_j; \overset{j}{\mathbf{z}}\ j_{2k-2}, \overset{j}{\mathbf{p}}\ j_{2k-2}), \quad k \geq 1, \end{aligned} \tag{3.4}$$

where

$$\overset{j}{\mathcal{M}}_m := -\gamma_j(x)\frac{\partial^2}{\partial\eta_j^2} + (-1)^{j+m}b(x)\frac{\partial}{\partial\eta_j}, \quad \gamma_j(x) := \varphi_j'(x)^2+1, \quad j,m=1,2, \tag{3.5}$$

$$\overset{j}{\mathbf{z}}_s := (\overset{j}{z}_0,\overset{j}{z}_1,\ldots,\overset{j}{z}_s), \quad \overset{j}{\mathbf{p}}_s := (\overset{j}{p}_0,\overset{j}{p}_1,\ldots,\overset{j}{p}_s),$$

$$\begin{aligned}\overset{j,1}{\mathcal{F}}_{2k}(x,\eta_j;\overset{j}{\mathbf{z}}_{2k-2},\overset{j}{\mathbf{p}}_{2k-2}) &:= (-1)^j\left(2\varphi_j'(x)\frac{\partial^2}{\partial\eta_j\partial x}\overset{j}{z}_{2k-2}+\varphi_j''(x)\frac{\partial}{\partial\eta_j}\overset{j}{z}_{2k-2}\right)\\ &\quad+\frac{\partial^2}{\partial x^2}\overset{j}{z}_{2k-4}-\sum_{s=0}\overset{j}{a}_s(x)\eta_j^s\overset{j}{z}_{2k-2-2s},\\ \overset{j,2}{\mathcal{F}}_{2k}(x,\eta_j;\overset{j}{\mathbf{z}}_{2k-2},\overset{j}{\mathbf{p}}_{2k-2}) &:= (-1)^j\left(2\varphi_j'(x)\frac{\partial^2}{\partial\eta_j\partial x}\overset{j}{p}_{2k-2}+\varphi_j''(x)\frac{\partial}{\partial\eta_j}\overset{j}{p}_{2k-2}\right)\\ &\quad+\frac{\partial^2}{\partial x^2}\overset{j}{p}_{2k-4}+\overset{j}{z}_{2k-2}-\sum_{s=0}\overset{j}{a}_s(x)\eta_j^s\overset{j}{p}_{2k-2-2s},\end{aligned} \tag{3.6}$$

while the $\overset{j}{a}_s(x)$ are known smooth functions that are the coefficients of the expansion of the function $a(x,y)$ in a neighborhood of the boundaries Γ_j,

$$a(x,\varphi_j(x)-(-1)^j\varepsilon^2\eta_j)=\sum_{s=0}^{\infty}\overset{j}{a}_s(x)\varepsilon^{2s}\eta_j^s.$$

In this case, it is assumed that if one of the indices of a function is negative, then the function is identically zero.

Performing a similar procedure with the boundary conditions in (1.7), we obtain the relations

$$z_{2k}(x,\varphi_j(x))+\overset{j}{z}_{2k}(x,0)=0, \quad p_{2k}(x,\varphi_j(x))+\overset{j}{p}_{2k}(x,0)=0, \quad j=1,2. \tag{3.7}$$

Since both characteristic numbers of the operator $\overset{1}{\mathcal{M}}_1$ are nonnegative (see (3.5)), the equation $\overset{1}{\mathcal{M}}_1\overset{1}{z}=e^{-\eta_2 b(x)/\gamma_1(x)}R_s(\eta_1;x)$, where $R_s(\eta_1;x)$ is a polynomial in η_1 of degree s with coefficients smoothly depending on x, has a unique solution of the similar form $\overset{1}{z}=e^{-\eta_2 b(x)/\gamma_1(x)}\widetilde{R}_s(\eta_1;x)$ with the polynomial $\widetilde{R}_s(\eta_1;x)$ in η_1 of the same degree s.

At the same time, the equation $\overset{1}{\mathcal{M}}_2\overset{1}{p}=e^{-\eta_1 b(x)/\gamma_1(x)}R_s(\eta_1;x)$ has a general solution of the form

$$\overset{1}{p}=e^{-\eta_1 b(x)/\gamma_1(x)}(C(x)+\eta_1\widetilde{R}_s(\eta_1;x)),$$

where $\widetilde{R}_s(\eta_1;x)$ is a known similar polynomial in η_1 of degree s and the function $C(x)$ (a polynomial of the zero degree) is to be determined. A similar situation also takes place on Γ_2 (with the replacement of $\overset{2}{z}$ with $\overset{2}{p}$ and vice versa). Allowing for the form (3.6) of the functions $\overset{j,m}{\mathcal{F}}_{2k}$ $(j,m=1,2)$, we find that the functions $\overset{j}{z}_{2k}$ and $\overset{j}{p}_{2k}$ have the following structure:

$$\begin{aligned}\overset{1}{z}_{2k}&=e^{-\eta_1 b(x)/\gamma_1(x)}\overset{1}{P}_{2k-2}(\eta_1;x), \quad \overset{1}{p}_{2k}=e^{-\eta_1 b(x)/\gamma_1(x)}\overset{1}{Q}_{2k}(\eta_1;x),\\ \overset{2}{z}_{2k}&=e^{-\eta_2 b(x)/\gamma_2(x)}\overset{2}{P}_{2k}(\eta_2;x), \quad \overset{2}{p}_{2k}=e^{-\eta_2 b(x)/\gamma_2(x)}\overset{2}{Q}_{2k-2}(\eta_2;x).\end{aligned} \tag{3.8}$$

Here, by analogy with the preceding, $\overset{1}{Q}_{2k}(\eta_1;x)$ and $\overset{2}{P}_{2k}(\eta_2;x)$ ($\overset{1}{P}_{2k-2}(\eta_1;x)$ and $\overset{2}{Q}_{2k-2}(\eta_2;x)$) are polynomials in η_j of degree $2k$ (degree $2k-2$) with coefficients that smoothly depend on x.

Note that $\overset{1}{z}_{2k}$ and $\overset{2}{p}_{2k}$ are uniquely determined by the previous members of the series in (3.1), while $\overset{2}{z}_{2k}$ and $\overset{1}{p}_{2k}$ have the form

$$\begin{aligned}\overset{2}{z}_{2k}&=e^{-\eta_2 b(x)/\gamma_2(x)}D_{2k}(x)+e^{-\eta_2 b(x)/\gamma_2(x)}\eta_2\widetilde{P}_{2k-1}(\eta_2;x),\\ \overset{1}{p}_{2k}&=e^{-\eta_1 b(x)/\gamma_1(x)}C_{2k}(x)+e^{-\eta_1 b(x)/\gamma_1(x)}\eta_1\widetilde{Q}_{2k-1}(\eta_1;x),\end{aligned} \tag{3.9}$$

where $\widetilde{P}_{2k-1}(\eta_2;x)$ and $\widetilde{Q}_{2k-1}(\eta_1;x)$ are uniquely determined by the previous members of the series in (3.1).

Thus, for system (3.3), (3.4), (3.7) to be solvable, it is necessary to consider a system of the form

$$\begin{gathered}\mathcal{L}_0 z+\lambda_0 p=f_1(x,y),\quad \mathcal{L}_0^* p-z=f_2(x,y),\\ z(x,\varphi_1(x))=g_1(x),\quad p(x,\varphi_2(x))=g_2(x).\end{gathered} \tag{3.10}$$

The paper [10, Lemma 2] proved that if $f_i(x,y)\in C(\overline{\Omega}\setminus\{M_1,M_2\})$ and $g_i(x)\in C((x_1,x_2))$, then system (3.10) is uniquely solvable for each $\lambda_0>0$. Further, if [4] $f_i(x,y)\in C^\infty(\overline{\Omega}\setminus\{M_1,M_2\})$ and $g_i(x)\in C^\infty((x_1,x_2))$, then also $z(x,y),p(x,y)\in C^\infty(\overline{\Omega}\setminus\{M_1,M_2\})$.

This, together with (3.7)–(3.9), implies the following assertion.

Theorem 3. *Let conditions* (1.4) *and* (2.5) *be satisfied. Then problem* (3.3), (3.4), (3.7) *is uniquely solvable for whichever collection* $\{\lambda_{2k}\}$ $(\lambda_0>0)$, *and all of its solutions are infinitely differentiable in* $\overline{\Omega}\setminus\{M_1,M_2\}$.

Note that the algorithm for constructing solutions of the above-indicated systems is as follows:

(1) Find $\overset{1}{z}_{2k}$ and $\overset{2}{p}_{2k}$.
(2) Set

$$z_{2k}(x,\varphi_1(x))=-\overset{1}{z}_{2k}(x,0),\quad p_{2k}(x,\varphi_2(x))=-\overset{2}{p}_{2k}(x,0). \tag{3.11}$$

(3) Solve problem (3.3) with conditions (3.11).
(4) Find $\overset{2}{z}_{2k}$ and $\overset{1}{p}_{2k}$ from the conditions

$$\overset{2}{z}_{2k}(x,0)=-z_{2k}(x,\varphi_2(x)),\quad \overset{1}{p}_{2k}(x,0)=-p_{2k}(x,\varphi_1(x)).$$

The outer expansion for a given collection $\{\lambda_{2k}\}$ has thus been constructed. By construction, it is a formal asymptotic solution of problem (1.7) in those subdomains of the domain Ω where the series (3.1) do not lose their asymptotic property. Note that this expansion also fails to approximate relation (2.3).

It turns out that these series lose their asymptotic nature in some small neighborhoods of the points M_1 and M_2. By virtue of complete similarity in considering the neighborhoods of these points, we consider in detail only the neighborhood of point $M_1=(0,0)$. Denote $c:=\sqrt{2\psi_1''(0)}$; then the functions φ_j, by virtue of (2.6), have, as $x\to+0$, the asymptotic expansions

$$\varphi_j(x)=(-1)^j c x^{1/2}+\sum_{s=2}^{\infty}c_s x^{s/2},\quad x\to+0. \tag{3.12}$$

By $\sigma(x)$ (possibly, with indices) we will denote functions that are smooth in a neighborhood of the point $x=+0$ and have an asymptotic expansion as $x\to+0$ of the form $\sum_{s=0}^{\infty}q_s x^{s/2}$, which can be differentiated term-by-term infinitely many times.

By $\sigma(x,y)$ (possibly, with indices) we will denote functions that are smooth in a neighborhood of the point $(+0,0)$ and have an asymptotic expansion uniform in y as $x\to+0$ of the form $\sum_{s=0}^{\infty}x^{s/2}q_s(y/\sqrt{x})$, where $q_s(\theta)\in C^\infty((-1-\gamma_2,1+\gamma_2))$ and γ_2 is a sufficiently small positive constant that can be differentiated term-by term infinitely many times.

Assertion 2. *For each collection* $\{\lambda_{2k}\}$ $(\lambda_0>0)$, *the coefficients of the outer expansion in* (3.1) *have the following asymptotic expansions as* $x\to x_i-(-1)^i 0$:

$$z_{2k}(x,y)=|x-x_i|^{(1-3k)/2}\sigma(|x-x_i|,y),\quad p_{2k}(x,y)=|x-x_i|^{(1-3k)/2}\sigma(|x-x_i|,y),$$

$$\overset{1}{z}_{2k}(x,y)=|x-x_i|^{(2-3k)/2}e^{-\eta_1|x-x_i|\sigma(|x-x_i|)}\sum_{s=0}^{2k-2}(|x-x_i|\eta_1)^s\sigma_s(|x-x_i|),\quad k\ge 1,$$

$$\overset{2}{z}_{2k}(x,y)=|x-x_i|^{(1-3k)/2}e^{-\eta_2|x-x_i|\sigma(|x-x_i|)}\sum_{s=0}^{2k}(|x-x_i|\eta_2)^s\sigma_s(|x-x_i|),$$

$$\begin{aligned} \overset{1}{p}_{2k}(x,y) &= |x-x_i|^{(1-3k)/2} e^{-\eta_1|x-x_i|\sigma(|x-x_i|)} \sum_{s=0}^{2k} (|x-x_i|\eta_1)^s \sigma_s(|x-x_i|), \\ \overset{2}{p}_{2k}(x,y) &= |x-x_i|^{(2-3k)/2} e^{-\eta_2|x-x_i|\sigma(|x-x_i|)} \sum_{s=0}^{2k-2} (|x-x_i|\eta_2)^s \sigma_s(|x-x_i|), \quad k \ge 1. \end{aligned} \tag{3.13}$$

Proof. By virtue of (3.5) and (3.12), we have

$$\begin{gathered} \varphi_j(x) = (-1)^j c x^{1/2} + x\sigma(x), \quad \gamma_j(x) = \frac{c^2}{4x} + x^{-1/2}\sigma(x) = x^{-1}\sigma(x), \\ \frac{b(x)}{\gamma_j(x)} = \frac{4b(0)x}{c^2} + x^{3/2}\sigma(x) = x\sigma(x). \end{gathered} \tag{3.14}$$

Since we can term-by-term differentiate and integrate the series in the definition of the function $\sigma(x,y)$, and also allowing for the relation $x^{-3/2}y = x^{-1}(y/\sqrt{x})$ and relations (3.14), we obtain

$$\begin{gathered} \frac{\partial}{\partial x}\sigma(x,y) = x^{-1}\sigma(x,y), \quad \frac{\partial}{\partial y}\sigma(x,y) = x^{-1/2}\sigma(x,y), \\ \int\limits_{\varphi_1(x)}^{y} \sigma(x,\eta)d\eta = x^{1/2}\sigma(x,y) + x^{1/2}\sigma(x). \end{gathered} \tag{3.15}$$

Note that it follows from these formulas and formulas (3.13) and (3.14) in [10] that for the $\overline{z}_\lambda$ and $\overline{p}_\lambda$, which are solutions of the problem

$$\mathcal{L}_0\overline{z} + \lambda\overline{p} = f_1(x,y), \quad \mathcal{L}_0^*\overline{p} - \overline{z} = f_1(x,y), \quad \overline{z}(x,\varphi_1(x)) = g_1(x), \quad \overline{p}_\lambda(x,\varphi_2(x)) = g_2(x), \tag{3.16}$$

where $\lambda > 0$, the following property holds:

$$\text{if } f_i = x^\alpha\sigma(x,y), \ g_i = x^\alpha\sigma(x), \text{ then } \overline{z}_\lambda, \overline{p}_\lambda = x^{\alpha+1/2}\sigma(x,y). \tag{3.17}$$

The proof is conducted further by induction with respect to k using formulas (3.3), (3.6), (3.15), and (3.17), as well as formulas (3.13) and (3.14) in [10]. The proof of the assertion is complete.

Note that, as follows from the expansions (3.13), for $k > 0$ the coefficients of the outer expansion do not belong to the space $L_2(\Omega)$, while the series in (3.1) stop being asymptotic for $|x - x_i| \ll \varepsilon$.

It can readily be verified by a straightforward computation that the coefficients of the outer expansion in (3.1) belong to $L_2(\Omega)$ at $k = 0$.

4. INNER ASYMPTOTIC EXPANSION

In the previous section, for a given collection $\{\lambda_{2k}\}$ we constructed a formal asymptotic solution (FAS) of problem (1.7) in those subdomains of the domain Ω where the series in (3.1) do not lose their asymptotic property.

Since the outer expansion is unsuitable in a small neighborhood of the points M_i, we must consider a new, "inner," expansion in terms of stretched variables in neighborhoods of these points.

To avoid writing fractional powers of ε, in this part of the paper we introduce the new small parameter $\mu := \varepsilon^{1/3}$. Also, we consider in detail only a neighborhood of the point $M_1 = (0,0)$, because the inner expansion of the problem under consideration in a neighborhood of the point M_2 is similar.

In a neighborhood of the point M_1, we introduce new stretched variables, similar to how it was done in [1, Ch. IV, Sec. 3, (3.13)]: $x = \mu^4\xi$, $y = \mu^2\tau$.

In terms of these variables, the functions $V_\varepsilon(\xi,\tau) := z_\varepsilon(\mu^4\xi,\mu^2\tau)$ and $W_\varepsilon(\xi,\tau) := p_\varepsilon(\mu^4\xi,\mu^2\tau)$ will satisfy the system

$$\begin{aligned} -\frac{\partial^2}{\partial\xi^2}V_\varepsilon + b(\mu^4\xi)\frac{\partial}{\partial\tau}V_\varepsilon + \mu^2 a(\mu^4\xi,\mu^2\tau)V_\varepsilon + \mu^2\lambda_\varepsilon W_\varepsilon + \mu^4\frac{\partial^2}{\partial\tau^2}V_\varepsilon &= \mu^2 f(\mu^4\xi,\mu^2\tau), \\ -\frac{\partial^2}{\partial\xi^2}W_\varepsilon - b(\mu^4\xi)\frac{\partial}{\partial\tau}W_\varepsilon + \mu^2 a(\mu^4\xi,\mu^2\tau)W_\varepsilon - \mu^2 V_\varepsilon + \mu^4\frac{\partial^2}{\partial\tau^2}W_\varepsilon &= -\mu^2 z_d(\mu^4\xi,\mu^2\tau) \end{aligned} \tag{4.1}$$

in the domain $\mu^4\xi \geq \psi_1(\mu^2\tau)$, $\xi < \mu^{\widetilde{\alpha}}$ for some $\widetilde{\alpha} > 0$ and the boundary conditions

$$V_\varepsilon(\psi_1(\mu^2\tau), \mu^2\tau) = 0 = W_\varepsilon(\psi_1(\mu^2\tau), \mu^2\tau). \tag{4.2}$$

We seek the inner expansion for z_ε and p_ε in the form

$$\overset{\text{in},1}{z} := \sum_{l=1}^{\infty} \mu^{2l} \overset{1}{v}_{2l}(\xi, \tau), \quad \overset{\text{in},1}{p} := \sum_{l=1}^{\infty} \mu^{2l} \overset{1}{w}_{2l}(\xi, \tau). \tag{4.3}$$

In a standard manner, we obtain the following system for the functions $\overset{1}{v}_{2l}$ and $\overset{1}{w}_{2l}$:

$$\begin{gathered} \frac{\partial^2}{\partial \xi^2} \overset{1}{v}_{2l} - b(0) \frac{\partial}{\partial \tau} \overset{1}{v}_{2l} = \overset{1}{\mathcal{G}}_{2k}(\xi, \tau; \overset{1}{\mathbf{v}}_{2k-2}, \overset{1}{\mathbf{w}}_{2l-2}), \\ \frac{\partial^2}{\partial \xi^2} \overset{1}{w}_{2l} + b(0) \frac{\partial}{\partial \tau} \overset{1}{w}_{2l} = \overset{2}{\mathcal{G}}_{2l}(\xi, \tau; \overset{1}{\mathbf{v}}_{2k-2}, \overset{1}{\mathbf{w}}_{2k-2}), \end{gathered} \tag{4.4}$$

where $\overset{1}{\mathbf{v}}_s := (\overset{1}{v}_2, \overset{1}{v}_4, \ldots, \overset{1}{v}_s)$, $\overset{1}{\mathbf{w}}_s := (\overset{1}{w}_2, \overset{1}{w}_4, \ldots, \overset{1}{w}_s)$,

$$\overset{1}{\mathcal{G}}_2(\xi, \tau) = -f(0,0), \quad \overset{2}{\mathcal{G}}_2(\xi, \tau) = z_d(0,0),$$

$$\overset{1}{\mathcal{G}}_4(\xi, \tau) = -f_2(\xi, \tau) + a(0,0) \overset{1}{v}_2 + \lambda_0 \overset{1}{w}_2, \quad \overset{2}{\mathcal{G}}_4(\xi, \tau) = z_{d,2}(\xi, \tau) + a(0,0) \overset{1}{w}_2 - \overset{1}{v}_2,$$

$$\begin{aligned} \overset{1}{\mathcal{G}}_{2k}(\xi, \tau; \overset{1}{\mathbf{v}}_{2l-2}, \overset{1}{\mathbf{w}}_{2k-2}) &= -f_{2l-2}(\xi, \tau) - \frac{\partial^2}{\partial \tau^2} \overset{1}{v}_{2l-4} \\ &\quad + \sum_{s=0} \lambda_{2s} \overset{1}{w}_{2l-4} + \sum_{s=1} b_s \xi^s \frac{\partial}{\partial \tau} \overset{1}{v}_{2k-4s} + \sum_{s=0}^{l-1} d_{2l,2s}(\xi, \tau) \overset{1}{v}_{2k-2-2s}, \\ \overset{2}{\mathcal{G}}_{2k}(\xi, \tau; \overset{1}{\mathbf{v}}_{2l-2}, \overset{1}{\mathbf{w}}_{2k-2}) &= z_{d,2l-2}(\xi, \tau) - \frac{\partial^2}{\partial \tau^2} \overset{1}{w}_{2l-4} \\ &\quad - \sum_{s=1} b_s \xi^s \frac{\partial}{\partial \tau} \overset{1}{w}_{2k-4s} + \sum_{s=0}^{l-1} \widetilde{d}_{2l,2s}(\xi, \tau) \overset{1}{w}_{2k-2-2s}, \end{aligned} \tag{4.5}$$

and $f_{2s}(\xi, \tau)$, $z_{d,2s}(\xi, \tau)$, $d_{2l,2s}(\xi, \tau)$, and $\widetilde{d}_{2l,2s}(\xi, \tau)$ are known (homogeneous of $2s$-*parabolic degree*, i.e., when the degree of the monomial $\xi^n \tau^m$ is taken to be equal to $2n + m$) polynomials obtained from the expansions of the functions f, z_d, a, and b in a neighborhood of the point $M_1 = (0,0)$.

In this case, each of systems (4.4), by virtue of (3.12), is considered in the unbounded domain $D = \{(\xi, t) : \xi \geq \tau^2, \tau \in \mathbb{R}\}$ with the boundary conditions

$$\overset{1}{v}_2(\tau^2, \tau) = 0 = \overset{1}{w}_2(\tau^2, \tau), \quad \overset{1}{v}_{2l}(\tau^2, \tau) = g_{v,2l}(\tau), \quad \overset{1}{w}_{2l}(\tau^2, \tau) = g_{w,2l}(\tau) \tag{4.6}$$

defined by the preceding $\overset{1}{v}_{2s}$ and $\overset{1}{w}_{2s}$ by virtue of (4.2).

The solutions of systems (4.4), (4.5) are unbounded in the domain concerned and thereby nonunique. However, we are only interested in solutions that are consistent with the outer expansion.

As was shown in [10], for $\xi \in (\varepsilon^{\overline{\alpha}}; \varepsilon^{\widetilde{\alpha}})$, $4/3 > \overline{\alpha} > \widetilde{\alpha} > 1$, the series in (3.1) can be re-expanded in ξ and τ. Having performed such a procedure, we will find that

$$\begin{aligned} \sum_{k=0}^{\infty} \varepsilon^{2k} (z_{2k}(x, y) + \overset{1}{z}_{2k}(x, \eta_1) + \overset{2}{z}_{2k}(x, \eta_2)) &= \sum_{l=1}^{\infty} \mu^{2l} (\overset{z}{H}_{0,2l}(\xi, \tau) + \overset{z}{H}_{1,2l}(\xi, \tau) + \overset{z}{H}_{2,2l}(\xi, \tau)), \\ \sum_{k=0}^{\infty} \varepsilon^{2k} (p_{2k}(x, y) + \overset{1}{p}_{2k}(x, \eta_1) + \overset{2}{p}_{2k}(x, \eta_2)) &= \sum_{l=1}^{\infty} \mu^{2l} (\overset{p}{H}_{0,2l}(\xi, \tau) + \overset{p}{H}_{1,2l}(\xi, \tau) + \overset{p}{H}_{2,2l}(\xi, \tau)), \end{aligned}$$

where

$$\overset{z}{H}_{0,2l} = \xi^{l/2}\sum_{s=0}^{\infty}\xi^{-3s/2}\widetilde{q}_{l,s}(\tau/\sqrt{\xi}),$$
$$\overset{z}{H}_{1,2} = 0, \quad \overset{z}{H}_{1,2l} = e^{-(4b(0)\xi(c\sqrt{\xi}+\tau)/c^2)}\xi^{2l-9/2}\sum_{s=0}^{\infty}\xi^{-3s/2}\sum_{s_1=0}^{2s-2}F_1(\xi,\tau)^{s_1}\widetilde{\sigma}_{l,1,s_1}(\xi), \tag{4.7}$$
$$\overset{z}{H}_{2,2l} = e^{-(4b(0)\xi(c\sqrt{\xi}-\tau)/c^2)}\xi^{2l-3/2}\sum_{s=0}^{\infty}\xi^{-3s/2}\sum_{s_1=0}^{2s}F_2(\xi,\tau)^{s_1}\widetilde{\sigma}_{l,2,s_1}(\xi),$$

$$\overset{p}{H}_{0,2l} = \xi^{l/2}\sum_{s=0}^{\infty}\xi^{-3s/2}\widetilde{q}_{l,s}(\tau/\sqrt{\xi}),$$
$$\overset{p}{H}_{1,2l} = e^{-(4b(0)\xi(c\sqrt{\xi}+\tau)/c^2)}\xi^{2l-3/2}\sum_{s=0}^{\infty}\xi^{-3s/2}\sum_{s_1=0}^{2s}F_1(\xi,\tau)^{s_1}\widetilde{\sigma}_{l,1,s_1}(\xi), \tag{4.8}$$
$$\overset{p}{H}_{2,2} = 0, \quad \overset{p}{H}_{2,2l} = e^{-(4b(0)\xi(c\sqrt{\xi}-\tau)/c^2)}\xi^{2l-9/2}\sum_{s=0}^{\infty}\xi^{-3s/2}\sum_{s_1=0}^{2s-2}F_2(\xi,\tau)^{s_1}\widetilde{\sigma}_{l,1,s_1}(\xi),$$

the $F_j(\xi,\tau) = \xi(c\sqrt{\xi}-(-1)^j\tau)$, $\widetilde{q}_{l,s}$ are similar to the functions q_s, while $\widetilde{\sigma}(\xi)$ are linear combinations of degrees $\xi^{-\widetilde{s}/2}$, $\widetilde{s} = 0, 1, \ldots$ Here the resulting series are a formal asymptotic solution of system (4.1) as $\xi \to +\infty$.

Theorem 4. *There exist functions* $\overset{1}{v}_{2l}\ (\xi,\tau)$ *and* $\overset{1}{w}_{2l}\ (\xi,\tau)$ *such that they are solutions of system* (4.4), (4.6) *and have the asymptotic expansions* $\overset{z}{H}_{0,2l}(\xi,\tau) + \overset{z}{H}_{1,2l}(\xi,\tau) + \overset{z}{H}_{2,2l}(\xi,\tau)$ *and* $\overset{p}{H}_{0,2l}(\xi,\tau) + \overset{p}{H}_{1,2l}(\xi,\tau) + \overset{p}{H}_{2,2l}(\xi,\tau)$, *respectively, as* $\xi \to +\infty$.

Proof. Since for each l the system in question splits into two independent equations, and, in this case, because of the form of the domain, the second equation is transformed by the change $\tau_1 := -\tau$ into an equation of the first form in the same domain, by following the proof of Theorem 3.1 in [1, Ch. IV, Sec. 3], we arrive at the existence of the desired solution. The proof of the theorem is complete.

By construction, the outer expansion in (3.1) is consistent in a neighborhood of the point $M_1 = (0,0)$ with the inner expansion in (4.3) (see [1, formula (0.9)]); i.e., for $N_1 \geq 1$ and $N_2 \geq 1$ we have

$$\begin{aligned}\mathcal{A}_{N_2,\xi,\tau}(\mathcal{A}_{N_1,x,y,\eta_1,\eta_2}\overset{\text{out}}{z}) &= \mathcal{A}_{N_1,x,y,\eta_1,\eta_2}(\mathcal{A}_{N_2,\xi,\tau}\overset{\text{in},1}{z}),\\ \mathcal{A}_{N_2,\xi,\tau}(\mathcal{A}_{N_1,x,y,\eta_1,\eta_2}\overset{\text{out}}{p}) &= \mathcal{A}_{N_1,x,y,\eta_1,\eta_2}(\mathcal{A}_{N_2,\xi,\tau}\overset{\text{in},1}{p}),\end{aligned} \tag{4.9}$$

where $\mathcal{A}_{N,(\cdot)}$ is the operator of taking the Nth partial sum of the respective series. (Here both parts of relations (4.9) must be reduced to the same variables.)

In a neighborhood of the point M_2, in a similar way, we construct the second inner expansion

$$\overset{\text{in},2}{z} := \sum_{l=1}^{\infty}\mu^{2l}\overset{2}{v}_{2l}\ (\xi_2,\tau_2), \quad \overset{\text{in},2}{p} := \sum_{l=1}^{\infty}\mu^{2l}\overset{2}{w}_{2l}\ (\xi_2,\tau_2), \quad x_2 - x := \mu^4\xi_2, \quad y_2 - y := \mu^2\tau_2,$$

consistent with the expansion (3.1) in a neighborhood of the point M_2. In view of the consistency of the series under consideration, in a standard manner (see, e.g., the proof of Theorem 1.4 in [1, Ch. IV, Sec. 1]), it can be shown that, in the domain Ω, we have the estimates

$$|\mathcal{L}_\varepsilon Z_{2N} + \Lambda_N P_{2N} - f(x,y)| < K_{2N}\varepsilon^{N_1}, \quad |\mathcal{L}_\varepsilon^* P_{2N} - Z_{2N} + z_d(x,y)| < K_{2N}\varepsilon^{N_1},$$

and, on the boundary Γ, the estimates

$$|Z_{2N}| < K\varepsilon^{N_1}, \quad |P_{2N}| < K\varepsilon^{N_1},$$

where $N_1 \to +\infty$ as $N \to +\infty$. Here $\Lambda_{2N} := \sum_{l=0}^{N} \lambda_{2l}$, while

$$\begin{aligned} Z_{2N} &:= \mathcal{A}_{2N,x,y,\eta_1,\eta_2} \overset{\text{out}}{z} + \mathcal{A}_{2N,\xi,\tau} \overset{\text{in},1}{z} + \mathcal{A}_{2N,\xi_2,\tau_2} \overset{\text{in},2}{z} \\ &\quad - \mathcal{A}_{2N,\xi,\tau}(\mathcal{A}_{2N,x,y,\eta_1,\eta_2} \overset{\text{out}}{z}) - \mathcal{A}_{2N,\xi_2,\tau_2}(\mathcal{A}_{2N,x,y,\eta_1,\eta_2} \overset{\text{out}}{z}), \\ P_{2N} &:= \mathcal{A}_{2N,x,y,\eta_1,\eta_2} \overset{\text{out}}{p} + \mathcal{A}_{2N,\xi,\tau} \overset{\text{in},1}{p} + \mathcal{A}_{2N,\xi_2,\tau_2} \overset{\text{in},2}{p} \\ &\quad - \mathcal{A}_{2N,\xi,\tau}(\mathcal{A}_{2N,x,y,\eta_1,\eta_2} \overset{\text{out}}{p}) - \mathcal{A}_{2N,\xi_2,\tau_2}(\mathcal{A}_{2N,x,y,\eta_1,\eta_2} \overset{\text{out}}{p}). \end{aligned} \tag{4.10}$$

5. COMPLETE ASYMPTOTICS OF THE SOLUTION OF THE PROBLEM

Thus, for a fixed collection $\{\lambda_n\}$ we have constructed consistent outer and inner formal asymptotic solutions of system (1.7). The compound asymptotic expansions Z_N and P_N produced from these series approximate system (1.7) uniformly in the domain Ω. However, they do not satisfy the approximation condition.

First, we find the zeroth approximation to the original problem. Consider a system of the form (3.16)

$$\begin{gathered} \mathcal{L}_0 \overline{z}_\lambda + \lambda \overline{p}_\lambda = f_1(x,y) := f(x,y), \quad \mathcal{L}_0^* \overline{p}_\lambda - \overline{z}_\lambda = f_1(x,y) := -z_d(x,y), \\ \overline{z}_\lambda(x,\varphi_1(x)) = 0, \quad \overline{p}_\lambda(x,\varphi_2(x)) = 0, \end{gathered} \tag{5.1}$$

depending on a parameter $\lambda \in (0,\beta]$.

Since $f, z_d = \sigma(x,y)$, it follows from property (3.17) that $\overline{z}_\lambda, \overline{p}_\lambda = x^{1/2}\sigma(x,y)$ and $\overline{z}_\lambda, \overline{p}_\lambda \in L_2(\Omega)$.

Lemma 1. *Let conditions* (1.4) *and* (2.5), *as well as the conditions*

$$z_d \text{ is not a solution to the problem } \mathcal{L}_0 z = f(x,y),\ z(x,\varphi_1(x)) = 0, \tag{5.2}$$

$$\beta \|p_\beta\| > 1 \tag{5.3}$$

be satisfied. Then there exists a unique λ_0 *such that the relation* $\lambda_0 \|\overline{p}_{\lambda_0}\| = 1$ *holds for the solution of problem* (5.1) *with* $\lambda = \lambda_0$.

Proof. Let us introduce the notation $\widetilde{p}_\lambda := \lambda \overline{p}_\lambda$. Then $(\overline{z}_\lambda, \widetilde{p}_\lambda)$ is a solution of the system

$$\begin{gathered} \mathcal{L}_0 \overline{z}_\lambda + \widetilde{p}_\lambda = f(x,y), \quad \mathcal{L}_0^* \widetilde{p}_\lambda - \lambda \overline{z}_\lambda = -\lambda z_d(x,y), \\ \overline{z}_\lambda(x,\varphi_1(x)) = 0, \quad \widetilde{p}_\lambda(x,\varphi_2(x)) = 0. \end{gathered} \tag{5.4}$$

According to the theorem on the differentiability of solutions of ordinary differential equations with respect to a parameter, the function $\|\widetilde{p}_\lambda\|^2$ is differentiable with respect to λ, with $d\|\widetilde{p}_\lambda\|^2/d\lambda = 2\|\widetilde{p}_\lambda\|(\widetilde{p}_\lambda, \widetilde{P}_\lambda)$, where $\widetilde{P}_\lambda(x,y) := \partial \widetilde{p}_\lambda(x,y)/\partial\lambda$. Let $\overline{Z}_\lambda(x,y) := \partial \overline{z}_\lambda(x,y)/\partial\lambda$. Then $(\overline{Z}_\lambda, \widetilde{P}_\lambda)$ is a solution of the system

$$\mathcal{L}_0 \overline{Z}_\lambda + \widetilde{P}_\lambda = 0, \quad \mathcal{L}_0^* \widetilde{P}_\lambda - \lambda \overline{Z}_\lambda = \overline{z}_\lambda - \lambda z_d, \quad \overline{Z}_\lambda(x,\varphi_1(x)) = 0, \quad \widetilde{P}_\lambda(x,\varphi_2(x)) = 0. \tag{5.5}$$

Here, since $\overline{z}_\lambda - \lambda z_d = \sigma(x,y)$, the relations $\overline{Z}_\lambda, \widetilde{P}_\lambda = x^{1/2}\sigma(x,y)$ hold in view of property (3.17). Thereby $\overline{Z}_\lambda, \widetilde{P}_\lambda \in L_2(\Omega)$.

Therefore, by virtue of systems (5.4) and (5.5), we have

$$(\widetilde{p}_\lambda, \widetilde{P}_\lambda) = -(\widetilde{p}_\lambda, \mathcal{L}_0 \overline{Z}_\lambda) = -(\mathcal{L}_0^* \widetilde{p}_\lambda, \overline{Z}_\lambda) = -\lambda(\overline{Z}_\lambda, \overline{z}_\lambda - z_d)$$

and

$$\|\widetilde{P}_\lambda\|^2 = -(\mathcal{L}_0 \overline{Z}_\lambda, \widetilde{P}_\lambda) = -\lambda\|\overline{Z}_\lambda\| - (\overline{Z}_\lambda, \overline{z}_\lambda - z_d),$$

and hence $d\|\widetilde{p}_\lambda\|^2/d\lambda \geq 0$.

If $(\widetilde{p}_\lambda, \widetilde{P}_\lambda) = 0$ for some $\lambda > 0$, then $\widetilde{P}_\lambda = 0$ and $\overline{Z}_\lambda = 0$, and therefore, $\overline{z}_\lambda - z_d = 0$. Then $\widetilde{p}_\lambda = 0$ and $\mathcal{L}_0 z_d = f(x,y)$. Consequently,

$$\frac{d}{d\lambda}\|\widetilde{p}_\lambda\|^2 > 0; \tag{5.6}$$

therefore, the function $\|\widetilde{p}_\lambda\|^2$ strictly increases and, in particular, is bounded. In this case, $\|\widetilde{p}_\beta\|^2 > 1$.

Since $\overline{z}_\lambda$ is a solution of the problem $\mathcal{L}_0\overline{z}_\lambda = f(x,y) - \widetilde{p}_\lambda$, we have, according to the well-known a priori estimates (see, e.g., [14, Ch. 3, formula (1.5)]), $\|\overline{z}_\lambda\| \le K\|f(x,y) - \widetilde{p}_\lambda\| \le K_1$. Consequently, $\lambda\|\overline{z}_\lambda\| \to 0$ as $\lambda \to 0$.

Since $\mathcal{L}_0^*\widetilde{p}_\lambda = \lambda\overline{z}_\lambda - \lambda z_d(x,y)$, we have $\|\widetilde{p}_\lambda\| \le K_3\lambda\|\overline{z}_\lambda - z_d\| \to 0$ as $\lambda \to 0$. The proof of the lemma is complete.

Lemma 2. *Assume that conditions* (1.4) *and* (2.5), *as well as the condition* $\lambda_{\varepsilon_n} \to \widetilde{\lambda}$ *for some* $\varepsilon_n \to 0$, *are satisfied. Then* $\|z_{\varepsilon_n} - \widetilde{z}_n\| \to 0$ *and* $\|p_{\varepsilon_n} - \widetilde{p}_n\| \to 0$, *where* $(\widetilde{z}_n, \widetilde{p}_n)$ *is the solution of the system*

$$\mathcal{L}_{\varepsilon_n}\widetilde{z}_n + \widetilde{\lambda}\widetilde{p}_n = f(x,y), \quad \mathcal{L}_{\varepsilon_n}^*\widetilde{p}_n - \widetilde{z}_n = -z_d(x,y), \quad (x,y) \in \Omega,$$
$$\widetilde{z}_n = 0, \quad \widetilde{p}_n = 0, \quad (x,y) \in \Gamma.$$

Proof. Denoting $\widehat{z}_n := z_{\varepsilon_n} - \widetilde{z}_n$, $\widehat{p}_n := p_{\varepsilon_n} - \widetilde{p}_n$, and $\widehat{\lambda}_n := \widetilde{\lambda} - \lambda_{\varepsilon_n}$, we obtain $\mathcal{L}_{\varepsilon_n}\widehat{z}_n + \widetilde{\lambda}\widehat{p}_n = \widehat{\lambda}_n p_{\varepsilon_n}$ and $\mathcal{L}_{\varepsilon_n}^*\widehat{p}_n - \widehat{z}_n = 0$. By virtue of (1.10), the relation $\|\widehat{z}_n\|^2 + \widetilde{\lambda}\|\widehat{p}_n\|^2 = \widehat{\lambda}_n(p_{\varepsilon_n}, \widehat{p}_n)$ holds. It follows from this relation that $\widetilde{\lambda}\|\widehat{p}_n\|^2 \le \widehat{\lambda}_n\|p_{\varepsilon_n}\|\|\widehat{p}_n\|$. However, according to (1.11), the sequence $\{\|p_{\varepsilon_n}\|\}$ is bounded and $\widetilde{\lambda} \ge \lambda_* > 0$, and therefore, $\|\widehat{p}_n\| \to 0$ and hence $\|\widehat{z}_n\| \to 0$. The proof of the lemma is complete.

Theorem 5. *Let conditions* (1.4), (2.5), *and* (5.3) *be satisfied. Then* $\lambda_\varepsilon \to \lambda_0$, $\|z_\varepsilon - z_0\| \to 0$, *and* $\|p_\varepsilon - p_0\| \to 0$ *as* $\varepsilon \to +0$, *where the number* λ_0 *has been defined in Lemma* 1, $z_0 := \overline{z}_{\lambda_0}$, $p_0 := \overline{p}_{\lambda_0}$, *and for all sufficiently small* $\varepsilon > 0$ *relation* (2.3) *holds.*

Proof. Let us show that λ_0 is the only limit point of the set $\{\lambda_\varepsilon\}$.

Let $\lambda_{\varepsilon_n} \to \widetilde{\lambda}$ for some $\varepsilon_n \to 0$. Then by Lemma 2 we have $\|z_{\varepsilon_n} - \widetilde{z}_n\| \to 0$ and $\|p_{\varepsilon_n} - \widetilde{p}_n\| \to 0$. However, according to Theorem 5 in [10], we have $\|\widetilde{z}_n - \widehat{z}_{\widetilde{\lambda}}\| \to 0$ and $\|\widetilde{p}_n - \widehat{p}_{\widetilde{\lambda}}\| \to 0$. Therefore, $\lambda_{\varepsilon_n}\|p_{\varepsilon_n}\| \to \widetilde{\lambda}\|\widehat{p}_{\widetilde{\lambda}}\|$. By virtue of inequality (5.3) from Lemma 1 we find that $\widetilde{\lambda} < \beta$. Consequently, we also have $\lambda_{\varepsilon_n} < \beta$ for all sufficiently large n. Then, by virtue of (1.6), $1 = \lambda_{\varepsilon_n}\|p_{\varepsilon_n}\| \to \widetilde{\lambda}\|\widehat{p}_{\widetilde{\lambda}}\|$, i.e., $\widetilde{\lambda} = \lambda_0$. The remaining assertions in the theorem follow from Lemma 2 and [10, Theorem 5]. The proof of the theorem is complete.

Let us proceed to constructing a complete asymptotic expansion of the solution of the problem in (1.7), (2.3). By Theorem 2, we must find a sequence $\{\lambda_{2k}\}$ such that the asymptotic expansions P_{2N} constructed on its basis satisfy the approximation condition (2.4).

To determine $\{\lambda_{2k}\}$ from this condition, by analogy with [7, Sec. 4] and [9], we use the method of auxiliary parameter (see [15, Sec. 30; 16, Lemma 2.1]).

Let us introduce a positive parameter δ and split the domain Ω into the following three domains: $\Omega_{0,\delta} := \{(x,y) \in \Omega : \delta < x < x_2 - \delta\}$, $\Omega_{1,\delta} := \{(x,y) \in \Omega : 0 < x < \delta\}$, and [4] $\Omega_{2,\delta} := \{(x,y) \in \Omega : x_2 - \delta < x < x_2\}$. Then, in $\Omega_{0,\delta}$, the asymptotic expansion P_{2N} will coincide with the $2N$th partial sum of the outer asymptotic expansion $\overset{\text{out}}{p}$, while in the domains $\Omega_{j,\delta}$, with $\overset{\text{in},j}{p}$. Using the asymptotics of the coefficients of these expansions (see (3.13), (4.7), and (4.8)), we derive asymptotic (in terms of even powers of ε) representations for the variable $\|P_{2N}\|^2$.

If all $\{\lambda_{2k}\}$ for $k < N$ have already been constructed, then the relation that equates to zero at ε^{2N} the expression $\lambda_{2N}^2\|P_{2N}\|^2$ in the asymptotic representation contains, of the so far unknown variables, λ_{2N} and the terms generated by the function p_{2N}. Since by (3.3) the functions z_{2N}, p_{2N} can be expanded into the sum $z_{2N} = z_{1,2N} + \lambda_{2N}\widehat{z}$, $p_{2N} = p_{1,2N} + \lambda_{2N}\widehat{p}$, where

$$\mathcal{L}_0 z_{1,2N} + \lambda_0 p_{1,2N} = \Delta z_{2N-2} - \sum_{s=1}^{N-1}\lambda_{2s}p_{2N-2s}, \quad \mathcal{L}_0^* p_{1,2N} - z_{1,2N} = \Delta p_{2N-2}$$

with the boundary conditions defined according to (3.11), and

$$\mathcal{L}_0\widehat{z}+\lambda_0\widehat{p}=-p_0,\quad \mathcal{L}_0^*\widehat{p}-\widehat{z}=0,\quad \overset{1}{w}\,\widehat{z}(x,\varphi_1(x))=0,\quad \widehat{p}(x,\varphi_2(x))=0, \tag{5.7}$$

the p_{2N} is completely determined by the variable λ_{2N} and by the functions that have been unambiguously determined by the current moment. Since, as follows from the expansions (3.13), the asymptotics as $x\to x_i-(-1)^i0$ has the form $p_0=|x-x_i|^{1/2}\sigma(|x-x_i|,y)$, in view of property (3.17), for the solution $(\widehat{z},\widehat{p})$ of problem (5.7) we have the representation

$$\widehat{z}=|x-x_i|\sigma(|x-x_i|,y),\quad \widehat{p}=|x-x_i|\sigma(|x-x_i|,y); \tag{5.8}$$

in particular, $\widehat{p}\in L_2(\Omega)$. Consequently, the equation for finding λ_{2N} acquires the form

$$\lambda_{2N}(\|p_0\|^2+\lambda_0(p_0,\widehat{p}))=c_{2N}, \tag{5.9}$$

in which c_{2N} is known and defined by previously determined members of outer and inner expansions.

Lemma 3. *Under the conditions in Lemma* 1, *the inequality* $\|p_0\|^2+\lambda_0(p_0,\widehat{p})\neq 0$ *holds, and Eq.* (5.9) *is thereby uniquely solvable.*

Proof. Since $\widehat{p}=\partial\overline{p}_\lambda/\partial\lambda|_{\lambda=\lambda_0}$, taking inequality (5.6) into account, we have

$$0<\frac{\partial}{\partial\lambda}(\lambda^2\|\overline{p}_\lambda\|^2)_{\lambda=\lambda_0}=2\lambda_0(\|p_0\|^2+\lambda_0(p_0,\widehat{p})).$$

The proof of the lemma is complete.

Finally, note that replacing the old value of λ_{2N} with a new one, by virtue of (5.8), will not alter the first $1+3k$ terms in the asymptotics of the functions z_{2k} and p_{2k} for $k\geq N$. Thus, for $l<N$ the functions $\overset{j}{v}_{2l}$ and $\overset{j}{w}_{2l}$ participating in the definition of λ_{2N} will not change either.

Acting this way, we will construct all λ_{2k} and the corresponding compound asymptotic expansions, which will be, according to Theorem 2, asymptotic expansions of the functions z_ε and p_ε uniform in the domain Ω. We have thus proved the following assertion, central to the present paper.

Theorem 6. *Let conditions* (1.4), (2.5), (2.6), (5.2), *and* (5.3) *be satisfied. Then there exists a sequence* $\{\lambda_{2k}\}$ *and the corresponding solutions of problems* (3.3)–(3.6), (4.2)–(4.4) *such that* $\sum_{k=0}^{\infty}\varepsilon^{2k}\lambda_{2k}$ *is an asymptotic expansion of* λ_ε, *while the compound asymptotic expansions constructed from the outer and inner expansions by formulas* (4.10) *are asymptotic expansions of the functions* z_ε *and* z_ε *uniform in* Ω.

In this case, the series $\overset{\text{out}}{z}$ *and* $\overset{\text{out}}{p}$ *are uniform asymptotic expansions of the functions* z_ε *and* p_ε *in the domain* $\varepsilon^{\widetilde{\alpha}}<x<x_2-\varepsilon^{\widetilde{\alpha}}$, $\varphi_1(x)<y<\varphi_2(x)$, $0<\widetilde{\alpha}<4/3$, *respectively, while the series* $\overset{\text{in},1}{z}$, $\overset{\text{in},1}{p}$ ($\overset{\text{in},2}{z}$, $\overset{\text{in},2}{p}$) *are uniform asymptotic expansions of the functions* z_ε *and* p_ε *in the domain* $0<x<\varepsilon^{\widetilde{\alpha}}$, $(x_2-\varepsilon^{\widetilde{\alpha}}<x<x_2)$, $\widetilde{\alpha}>1$, $\varphi_1(x)<y<\varphi_2(x)$, *respectively.*

REFERENCES

1. Il'in, A.M., *Soglasovanie asimptoticheskikh razlozhenii reshenii kraevykh zadach* (Matching of Asymptotic Expansions of Solutions to Boundary Value Problems), Moscow: Nauka, 1989.
2. Lions, J.L., *Optimal Control of Systems Governed by Partial Differential Equations*, New York-Heidelberg: Spinger, 1971. Translated under the title: *Optimal'noe upravlenie sistemami, opisyvaemymi uravneniyami s chastnymi proizvodnymi*, Moscow: Mir, 1972.
3. Il'in, A.M. and Lelikova, E.F., A method of joining asymptotic expansions for the equation $\varepsilon\Delta u - a(x,y)u_y=f(x,y)$ in a rectangle, *Math. USSR-Sb.*, 1975, vol. 96, no. 4, pp. 533–548.
4. Lelikova, E.F., On the asymptotic behavior of a solution to an equation with a small parameter in a neighborhood of a boundary inflection point, *Dokl. Math.*, 2012, vol. 86, pp. 756–759.
5. Kapustyan, V.E., Asymptotics of bounded controls in optimal elliptic problems, *Dokl. Akad. Nauk Ukr. Ser. Mat. Estestvozn. Tekh. Nauki*, 1992, no. 2, pp. 70–74.

6. Kapustyan, V.E., Optimal bisingular elliptic problems with bounded control, *Dokl. Akad. Nauk Ukr.*, 1993, no. 6, pp. 81–85.
7. Danilin, A.R., Asymptotic behaviour of bounded controls for a singular elliptic problem in a domain with a small cavity, *Sb. Math.*, 1998, vol. 189, no. 11, pp. 1611–1642.
8. Danilin, A.R., Optimal boundary control in domain with a small cavity, *Ufim. Mat. Zh.*, 2012, vol. 4, no. 2, pp. 87–100.
9. Danilin, A.R., Asymptotic behaviour of solutions of a singular elliptic system in a rectangle, *Sb. Math.*, 2003, vol. 194, no. 1, pp. 31–61.
10. Danilin, A.R., Asymptotics of the solution of a singular optimal distributed control problem in a convex domain, *Proc. Steklov Inst. Math. (Suppl.)*, 2018, vol. 300, pp. 72–87.
11. Danilin, A.R. and Zorin, A.R., Asymptotics of a solution to an optimal boundary control problem, *Proc. Steklov Inst. Math. (Suppl.)*, 2010, vol. 269, pp. S81–S94.
12. Lions, J.L. and Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications*, New York–Heidelberg: Spinger, 1971. Translated under the title: *Neodnorodnye granichnye zadachi i ikh prilozheniya*, Moscow: Mir, 1971.
13. Danilin, A.R., Asymptotics of the solution in a problem of optimal boundary control of a flow through a part of the boundary, *Proc. Steklov Inst. Math. (Suppl.)*, 2016, vol. 292, pp. 55–66.
14. Ladyzhenskaya, O.A. and Ural'tseva, N.N., *Lineinye i kvazilineinye uravneniya ellipticheskogo tipa* (Linear and Quasilinear Equations of Elliptic Type), Moscow: Nauka, 1964.
15. Il'in, A.M. and Danilin, A.R., *Asimptoticheskie metody v analize* (Asymptotic Methods in Analysis), Moscow: Fizmatlit, 2009.
16. Danilin, A.R., Asymptotic behavior of the optimal cost functional for a rapidly stabilizing indirect control in the singular case, *Comput. Math. Math. Phys.*, 2006, vol. 46, no. 12, pp. 2068–2079.