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PARTIAL DIFFERENTIAL EQUATIONS

Existence and Asymptotic Stability of a Stationary Boundary-Layer Solution of the Two-Dimensional Reaction–Diffusion–Advection Problem

N. T. Levashova[1*], N. N. Nefedov[1], and O. A. Nikolaeva[1***]**

[1] *Faculty of Physcis, Lomonosov Moscow State University, Moscow, 119991 Russia*
*e-mail: *natasha@npanalytica.ru, **nefedov@phys.msu.ru, ***o.a.nikolaewa@gmail.com*

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Abstract— We prove the existence and the Lyapunov asymptotic stability of a stationary boundary layer solution of the initial–boundary value problem for a two-dimensional singularly perturbed reaction—diffusion—advection equation. We construct an asymptotic approximation to this solution using the boundary function method. The proofs are based on the applicability of the asymptotic method of differential inequalities.

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INTRODUCTION

We study issues related to the existence and asymptotic Lyapunov stability of a stationary solution with a large gradient in a neighborhood of the boundary (solution with a boundary layer) of the two-dimensional initial–boundary value problem for a singularly perturbed reaction—diffusion—advection equation. An analysis of boundary layer type solutions is an important part of research into reaction—diffusion—advection problems with solutions in the form of front (internal transition layer), playing an important role in mathematical physics when modeling transfer or combustion processes, as well as nonlinear waves [1, 2]. In many such processes important for applications, the higher derivatives in a natural manner contain a small parameter multiplying them, whose presence enables the usage of asymptotic research methods.

In the present paper, an asymptotic approximation to the solution of the problem is constructed using Vasil'eva's boundary function method (see, e.g., [3, pp. 36–43; 4]). The existence of a solution, as well as its local existence and asymptotic stability as a stationary solution of the corresponding initial–boundary value problem, is proved using a modification of the asymptotic method of differential inequalities for problems with boundary and internal transition layers [5–7]. (Note that for problems incorporating an advection term one uses theorems from the method of upper and lower solutions, which can be found in, e.g., [8–11].)

Asymptotic methods and the method of differential inequalities have been used previously in [12–14] for justifying the existence of solutions with internal transition or boundary layers in multidimensional problems of the reaction—diffusion—advection type. These papers considered the case where the advection term is small compared with the reaction term. What distinguishes this paper is that the advection and reaction terms are comparable ("large" advection). In the papers [15–18], we have considered solutions in the form of a front for one-dimensional problems in the case of "large" advection, while in the paper [19], an asymptotic approximation was constructed for a solution in the form of a front of the two-dimensional reaction—diffusion—advection problem with a linear "large" advection term. The periodic spatially one-dimensional problem in the case of "large" advection was considered in [20].

1. STATEMENT OF THE PROBLEM

Consider the following initial–boundary value problem in the domain $(x, y, t) \in \mathbb{R} \times [0, a] \times \mathbb{R}^+$ with periodic conditions with respect to the variable x:

$$\begin{gathered}\varepsilon\Delta v-\frac{\partial v}{\partial t}=(\mathbf{A}(v,x,y),\nabla)v+B(v,x,y),\quad x\in\mathbb{R},\quad y\in(0,a),\quad t>0,\\ v(x,0,t,\varepsilon)=u^0(x),\quad v(x,a,t,\varepsilon)=u^a(x),\quad x\in\mathbb{R},\quad t>0,\\ v(x,y,t,\varepsilon)=v(x+L,y,t,\varepsilon),\quad x\in\mathbb{R},\quad y\in[0,a],\quad t>0,\\ v(x,y,0,\varepsilon)=v_{\text{init}}(x,y,\varepsilon),\quad x\in\mathbb{R},\quad y\in[0,a].\end{gathered}\tag{1}$$

Here $\varepsilon\in(0,\varepsilon_0]$ is a small parameter, L is a positive number, $\mathbf{A}(v,x,y)=\{A_1(v,x,y),A_2(v,x,y)\}$; the functions $A_i(v,x,y)$, $i=1,2$, and $B(v,x,y)$ are L-periodic in variable x and sufficiently smooth in the domain $(v,x,y,t)=I\times\overline{D}\times\mathbb{R}^+$, where I is the possible interval of variation of the variable v, $\overline{D}=\{(x,y):\mathbb{R}\times[0,a]\}$, and the functions $u^0(x)$, $u^a(x)$ and $v_{\text{init}}(x,y,\varepsilon)$ are continuous and L-periodic in variable x. The initial and boundary conditions are assumed to be continuity consistent.

The aim of the present paper is to study the existence and asymptotic stability of a stationary solution of problem (1), i.e., a solution $u_\varepsilon(x,y)$ of the boundary value problem

$$\begin{gathered}\varepsilon\Delta u=(\mathbf{A}(u,x,y),\nabla)u+B(u,x,y),\quad x\in\mathbb{R},\quad y\in(0,a),\\ u(x,0,\varepsilon)=u^0(x),\quad u(x,a,\varepsilon)=u^a(x),\quad x\in\mathbb{R},\\ u(x,y,\varepsilon)=u(x+L,y,\varepsilon),\quad x\in\mathbb{R},\quad y\in[0,a].\end{gathered}\tag{2}$$

Assume that the following conditions are satisfied.

Condition A1. The partial differential equation

$$(\mathbf{A}(u,x,y),\nabla)u+B(u,x,y)=0$$

with the additional condition $u(x,0)=u^0(x)$ has a solution $u=\varphi(x,y)$, where the function $\varphi(x,y)$ is sufficiently smooth in $\overline{D}$ and L-periodic in variable x.

Condition A2. The inequality $A_2(\varphi(x,y),x,y)>0$ holds everywhere in $\overline{D}$.

Remark. It is obvious that if the functions $A_i(v,x,y)$, $i=1,2$, belong to the space $C^1(\overline{D})$, then the function $F(x,y):=A_1(\varphi(x,y),x,y)/A_2(\varphi(x,y),x,y)$ satisfies the Lipschitz condition in the variable x in the domain $\overline{D}$.

1.1. Associated Equation

The function $\varphi(x,y)$ satisfies the boundary condition for $y=0$. To describe the solution behavior in a neighborhood of the boundary $y=a$, let us introduce the stretched variable

$$\xi=\frac{y-a}{\varepsilon}.\tag{3}$$

Consider the so-called associated equation for problem (2),

$$\frac{\partial^2\tilde{u}}{\partial\xi^2}=A_2(\tilde{u},x,a)\frac{\partial\tilde{u}}{\partial\xi},\quad \xi\le 0,\tag{4}$$

considering the variable x as a parameter. This equation is equivalent to the system of two equations of the first order (associated system)

$$\frac{\partial\tilde{u}}{\partial\xi}=\Phi,\quad \frac{\partial\Phi}{\partial\xi}=A_2(\tilde{u},x,a)\Phi.\tag{5}$$

Let us switch from the associated system to an equation for the function $\Phi(\tilde{u},x)$,

$$\frac{\partial\Phi}{\partial\tilde{u}}=A_2(\tilde{u},x,a),$$

which determines the trajectories on the phase plane $(\tilde{u},\Phi)$.

The point $(\varphi(x,a),0)$ on the phase plane is an equilibrium point of system (5). Since the function $A_2(\tilde{u},x,a)$ is continuous, there exists a phase trajectory entering this equilibrium point as $\xi\to-\infty$. This phase trajectory is determined by the expression

$$\Phi(\tilde{u},x)=\int\limits_{\varphi(x,a)}^{\tilde{u}} A_2(u,x,a)\,du.$$

Condition A3. For all $x\in\mathbb{R}$, one of the following inequalities is satisfied:

$$\int\limits_{\varphi(x,a)}^{\tilde{u}} A_2(u,x,a)\,du>0 \quad \text{if} \quad \varphi(x,a)<\tilde{u}\le u^a$$

or

$$\int\limits_{\varphi(x,a)}^{\tilde{u}} A_2(u,x,a)\,du<0 \quad \text{if} \quad u^a\le\tilde{u}<\varphi(x,a).$$

It follows from Condition A3 that, for each $x\in\mathbb{R}$, Eq. (4) with the additional conditions $\tilde{u}(x,a)=u^a(x)$ and $\tilde{u}(x,\infty)=\varphi(x,a)$ has a solution. (In this case, one says that the boundary value $u^a(x)$ belongs to the influence domain of the solution $\varphi(x,y)$ of the reduced equation mentioned in Condition A1.)

2. ASYMPTOTIC APPROXIMATION TO THE SOLUTION

Let us construct an asymptotic approximation $U(x,y,\varepsilon)$ to the solution of problem (2) in the form of a sum of two terms

$$U(x,y,\varepsilon)=\bar{u}(x,y,\varepsilon)+\Pi(\xi,x,\varepsilon). \tag{6}$$

Here $\bar{u}(x,y,\varepsilon)$ is the regular part of the asymptotic representation, $\Pi(\xi,x,\varepsilon)$ is the boundary layer function describing the solution in a neighborhood of the line $y=a$, and the variable ξ is defined by the expression (3). Each term in the sum (6) can be represented as an expansion in powers of the small parameter ε,

$$\bar{u}(x,y,\varepsilon)=\bar{u}_0(x,y)+\varepsilon\bar{u}_1(x,y)+\ldots,\qquad \Pi(\xi,x,\varepsilon)=\Pi_0(\xi,x)+\varepsilon\Pi_1(\xi,x)+\ldots \tag{7}$$

2.1. Regular Part

Substituting the sum (7) for $\bar{u}$ into the equality

$$\varepsilon\left(\frac{\partial^2\bar{u}}{\partial x^2}+\frac{\partial^2\bar{u}}{\partial y^2}\right)=A_1(\bar{u},x,y)\frac{\partial\bar{u}}{\partial x}+A_2(\bar{u},x,y)\frac{\partial\bar{u}}{\partial y}+B(\bar{u},x,y),$$

expanding the functions on the right-hand side in the resulting relation by the Taylor formula in powers of the small parameter, and matching the coefficients of like powers of ε, we arrive at first-order partial differential equations for the functions $\bar{u}_i(x,y)$, $i=0,1,\ldots$. We will seek solutions L-periodic in x of these equations in the domain D. The additional conditions for $\bar{u}_i(x,y)$ at $y=0$ will be determined from the respective boundary condition of problem (2).

For the functions in the regular part of the zero order, we obtain the problem

$$A_1(\bar{u}_0,x,y)\frac{\partial\bar{u}_0}{\partial x}+A_2(\bar{u}_0,x,y)\frac{\partial\bar{u}_0}{\partial y}+B(\bar{u}_0,x,y)=0,\quad (x,y)\in D,$$
$$\bar{u}_0(x,y)=\bar{u}_0(x+L,y),\quad (x,y)\in\overline{D},\quad \bar{u}_0(x,0)=u^0(x),\quad x\in\mathbb{R}.$$

According to Condition A1, the function $\varphi(x,y)$ is a solution L-periodic in x of this problem. Thus, $\bar{u}_0=\varphi(x,y)$.

Here and throughout the following, for brevity we use the notation

$$\bar{A}_i(x,y):=A_i(\varphi(x,y),x,y),\quad i=1,2,\quad \bar{B}(x,y):=B(\varphi(x,y),x,y) \tag{8}$$

and similar notation for the derivatives of the functions $A_i(\varphi(x,y),x,y)$ and $B(\varphi(x,y),x,y)$.

The functions $\bar{u}_i(x,y)$, $i=1,2,\ldots$, will be defined as the solutions of the problems

$$\begin{gathered}\bar{A}_1(x,y)\frac{\partial \bar{u}_i}{\partial x}+\bar{A}_2(x,y)\frac{\partial \bar{u}_i}{\partial y}+W\bar{u}_i=\bar{f}_i(x,y),\quad (x,y)\in D,\\ \bar{u}_i(x,y)=\bar{u}_i(x+L,y),\quad (x,y)\in\overline{D},\quad \bar{u}_i(x,0)=0,\quad x\in\mathbb{R},\end{gathered}\tag{9}$$

where

$$W(x,y)=\frac{\partial \bar{A}_1}{\partial u}(x,y)\frac{\partial\varphi}{\partial x}(x,y)+\frac{\partial \bar{A}_2}{\partial u}(x,y)\frac{\partial\varphi}{\partial y}(x,y)+\frac{\partial \bar{B}}{\partial u}(x,y)\tag{10}$$

and the $\bar{f}_i(x,y)$ are known functions, in particular, $\bar{f}_1(x,y)=\dfrac{\partial^2\varphi}{\partial x^2}+\dfrac{\partial^2\varphi}{\partial y^2}$.

Equations (9) are first-order partial differential equations. Let us write the equations of characteristics for these equations,

$$\bar{A}_2(x,y)\,dx=\bar{A}_1(x,y)\,dy,\quad (\bar{f}_i(x,y)-W(x,y)\bar{u}_i)dy=\bar{A}_2(x,y)\,d\bar{u}_i.\tag{11}$$

By virtue of the continuity of functions $\bar{A}_i(x,y)$ (see the Remark), there exists a first integral

$$\Psi(x,y)=C_1\tag{12}$$

of the first equation in (11), and the function $x=X(y,C_1)$ can be found on the interval $y\in[0,a]$.

Solving the equations

$$\frac{d\bar{u}_i}{dy}=\frac{\bar{f}_i(X(y,C_1),y)-W(X(y,C_1),y)\bar{u}_i}{\bar{A}_2(X(y,C_1),y)}$$

with the additional conditions $\bar{u}_i(x,0)=0$, we derive equations for $\bar{u}_i$

$$\bar{u}_i(X(y,C_1),y)=\int\limits_0^y\exp\biggl(-\int\limits_{y_1}^y\frac{W(X(y_2,C_1),y_2)}{\bar{A}_2(X(y_2,C_1),y_2)}\,dy_2\biggr)\frac{\bar{f}_i(X(y_1,C_1),y_1)}{\bar{A}_2(X(y_1,C_1),y_1)}\,dy_1.$$

Having replaced C_1 with $\Psi(x,y)$ in these equations, we arrive at the solutions $\bar{u}_i(x,y)$ of problems (9).

2.2. Boundary Layer Functions

Equations for the boundary layer functions $\Pi_i(\xi,x)$, $i=0,1,\ldots$, $\xi\le 0$, $x\in\mathbb{R}$, can be derived from the equality

$$\begin{gathered}\varepsilon\frac{\partial^2\Pi}{\partial x^2}+\frac{1}{\varepsilon}\frac{\partial^2\Pi}{\partial\xi^2}=A_1(\xi,x,\varepsilon)\frac{\partial\Pi}{\partial x}+\frac{1}{\varepsilon}A_2(\xi,x,\varepsilon)\frac{\partial\Pi}{\partial\xi}+\Pi A_1(\xi,x,\varepsilon)\frac{\partial\bar{u}}{\partial x}(x,a+\varepsilon\xi)\\ +\Pi A_2(\xi,x,\varepsilon)\frac{\partial\bar{u}}{\partial y}(x,a+\varepsilon\xi)+\Pi B(\xi,x,\varepsilon),\end{gathered}\tag{13}$$

where

$$\begin{aligned}A_i(\xi,x,\varepsilon)&=A_i(\bar{u}(x,a+\varepsilon\xi)+\Pi(\xi,x,\varepsilon),x,a+\varepsilon\xi),\quad i=1,2,\\ \Pi A_i(\xi,x,\varepsilon)&=A_i(\xi,x,\varepsilon)-A_i(\bar{u}(x,a+\varepsilon\xi),x,a+\varepsilon\xi),\\ \Pi B(\xi,x,\varepsilon)&=B(\bar{u}(x,a+\varepsilon\xi)+\Pi(\xi,x,\varepsilon),x,a+\varepsilon\xi)-B(\bar{u}(x,a+\varepsilon\xi),x,a+\varepsilon\xi).\end{aligned}\tag{14}$$

By substituting the sums (7) into Eq. (13), expanding the functions on the right-hand side in the resulting relation in Taylor series in powers of the small parameter, and matching the coefficients of like powers of ε, we arrive at equations for the functions $\Pi_i(\xi,x)$, $i=0,1,\ldots$, $\xi\le 0$, $x\in\mathbb{R}$.

As an additional condition, we will require that the boundary layer functions decay at infinity,

$$\Pi_i(-\infty, x) = 0. \tag{15}$$

The condition at $\xi = 0$ for the functions $\Pi_i(\xi, x)$ follows from the boundary condition at $y = a$ for problem (2),

$$\bar{u}_0(x,a) + \varepsilon\bar{u}_1(x,a) + \ldots + \Pi_0(0,x) + \varepsilon\Pi_1(0,x) + \ldots = u^a(x). \tag{16}$$

2.2.1. Boundary layer function of the zero order. By matching the coefficients of ε^{-1} in relation (13) and of ε^0 in relation (16), with allowance for condition (15), we obtain the following problem for the function $\Pi_0(\xi, x)$:

$$\frac{\partial^2\Pi_0}{\partial\xi^2} = A_2(\varphi(x,a) + \Pi_0(\xi,x), x, a)\frac{\partial\Pi_0}{\partial\xi}, \quad \xi < 0,$$
$$\varphi(x,a) + \Pi_0(0,x) = u^a(x), \quad \Pi_0(-\infty, x) = 0,$$

which is solvable by Condition A3. Moreover, we have the following standard estimate for the function $\Pi_0(\xi, x)$:

$$|\Pi_0(\xi,x)| < Ce^{-\kappa|\xi|}, \tag{17}$$

where C and κ are some positive constants.

2.2.2. Boundary layer function of the first order. By matching the coefficients of ε^0 in Eq. (13) and by using the boundary conditions (16) and the condition of decay at infinity, we arrive at the following problem for the function $\Pi_1(\xi, x)$:

$$\begin{gathered}\frac{\partial^2\Pi_1}{\partial\xi^2} - \tilde{A}_2(\xi,x)\frac{\partial\Pi_1}{\partial\xi} - \frac{\partial\tilde{A}_2}{\partial u}(\xi,x)\Phi(\xi,x)\Pi_1 = f_1(\xi,x), \quad \xi < 0,\\ \Pi_1(0,x) + \bar{u}_1(x,a) = 0, \quad \Pi_1(-\infty, x) = 0,\end{gathered} \tag{18}$$

where $\tilde{u}(\xi,x) = \varphi(x,a) + \Pi_0(\xi,x)$, $\tilde{A}_i(\xi,x) := A_i(\tilde{u}(\xi,x),x,a)$, $i = 1,2$, $\tilde{B}(\xi,x) := B(\tilde{u}(\xi,x),x,a)$, $\Phi(\xi,x) = \partial\tilde{u}/\partial\xi$ (similar notation will be used for the derivatives of the functions $A_i(\tilde{u}(\xi,x),x,a)$ and $B(\tilde{u}(\xi,x),x,a)$), and

$$\begin{aligned}f_1(\xi,x) = {} & \left(\frac{\partial\tilde{A}_2}{\partial u}(\xi,x)\left(\bar{u}_1(x,a) + \frac{\partial\varphi}{\partial y}(x,a)\xi\right) + \frac{\partial\tilde{A}_2}{\partial y}(\xi,x)\xi\right)\Phi(\xi,x)\\ & + \tilde{A}_1(\xi,x)\frac{\partial\Pi_0}{\partial x} + \Pi_0A_1(\xi,x,0)\frac{\partial\varphi}{\partial x}(x,a) + \Pi_0A_2(\xi,x,0)\frac{\partial\varphi}{\partial y}(x,a) + \Pi_0B(\xi,x,0).\end{aligned}$$

Here the functions Π_0A_i, $i = 1,2$, and Π_0B are the zero approximations to the Taylor series expansions in powers of ε for the functions ΠA_i and ΠB defined by the expressions (14).

The solution of problem (18) has the form

$$\Pi_1(\xi,x) = -\bar{u}_1(x,a)\frac{\Phi(\xi,x)}{\Phi(0,x)} - \Phi(\xi,0)\int\limits_0^{\xi}\frac{ds}{\Phi(s,x)}\int\limits_{-\infty}^{s} f_1(\eta,x)\,d\eta.$$

An exponential estimate similar to the estimate (17) holds for the function $\Pi_1(\xi, x)$.

2.2.3. Boundary layer functions of higher orders. Boundary layer functions of order $k = 2,3,\ldots$ are defined as the solutions of the equations

$$\begin{gathered}\frac{\partial^2\Pi_k}{\partial\xi^2} - \tilde{A}_2(\xi,x)\frac{\partial\Pi_k}{\partial\xi} - \frac{\partial\tilde{A}_2}{\partial u}(\xi,x)\Phi(\xi,x)\Pi_k = f_k(\xi,x), \quad \xi < 0,\\ \Pi_k(0,x) + \bar{u}_k(x,a) = 0, \quad \Pi_k(-\infty, x) = 0,\end{gathered} \tag{19}$$

where the functions $f_k(\xi, x)$ are known (determined from the representation (13)). Exponential estimates similar to those in (17) hold for the functions $\Pi_k(\xi, x)$.

2.2.4. Asymptotic approximation to the solution. Having determined all terms up to order k inclusive in (7), we compose the sum

$$U_k(x,y,\varepsilon)=\sum_{i=0}^{k}\varepsilon^i(\bar{u}_i(x,y)+\Pi_i(\xi,x)),\quad (x,y)\in\overline{D},\quad \xi\le 0. \tag{20}$$

The function $U_k(x,y,\varepsilon)$ thus constructed satisfies Eq. (2) up to $O(\varepsilon^k)$ and exactly satisfies the boundary condition at $y=a$ of problem (2). Moreover, this function satisfies the boundary condition at $y=0$ up to exponentially small terms. The standard procedure of multiplying the boundary layer terms by a cutoff function makes it possible to satisfy the boundary condition at $y=0$ exactly.

3. EXISTENCE OF THE STATIONARY SOLUTION

To prove the existence of a boundary layer solution of problem (2) and estimate the error of its asymptotic approximation, we use the method of upper and lower solutions [8, pp. 67–87; 9; 10].

Definition 1. Two functions $\beta(x,y,\varepsilon),\alpha(x,y,\varepsilon)\in C(\overline{D})\cap C^2(D)$ L-periodic in x are called the *upper and lower solutions*, respectively, *of problem* (2) if for sufficiently small ε they satisfy the following conditions:

1. The order of the upper and lower solutions,

$$\alpha(x,y,\varepsilon)\le\beta(x,y,\varepsilon),\quad (x,y)\in\overline{D}.$$

2. The differential inequalities,

$$L_\varepsilon[\beta]:=\varepsilon\Delta\beta-(\mathbf{A}(\beta,x,y),\nabla)\beta-B(\beta,x,y)\le 0\le L_\varepsilon[\alpha],\quad (x,y)\in D.$$

3. The inequalities at the boundary,

$$\alpha(x,0,\varepsilon)\le u^0(x)\le\beta(x,0,\varepsilon),\quad \alpha(x,a,\varepsilon)\le u^a(x)\le\beta(x,a,\varepsilon),\quad x\in\mathbb{R}.$$

It is well known (see [8, pp. 67–87; 9; 10]) that if there exist upper and lower solutions of problem (2), then this problem has a solution $u_\varepsilon(x,y)$ confined between the upper and lower solutions,

$$\alpha(x,y,\varepsilon)\le u_\varepsilon(x,y)\le\beta(x,y,\varepsilon),\quad (x,y)\in\overline{D}.$$

3.1. Constructing Upper and Lower Solutions

We will construct upper and lower solutions of problem (2) according to the asymptotic method of differential inequalities [5–7] as a modification of the asymptotic approximation (20) for $k=n+1$,

$$\begin{gathered}\beta_{n+1}(x,y,\varepsilon)=U_{n+1}(x,y,\varepsilon)+\varepsilon^{n+1}(\mu(x,y)+\pi_0(\xi,x)+\varepsilon\pi_1(\xi,x)),\\ \alpha_{n+1}(x,y,\varepsilon)=U_{n+1}(x,y,\varepsilon)-\varepsilon^{n+1}(\mu(x,y)+\pi_0(\xi,x)+\varepsilon\pi_1(\xi,x)),\quad (x,y)\in\overline{D},\quad \xi\le 0.\end{gathered} \tag{21}$$

The functions $\mu(x,y)$ and $\pi_i(\xi,x)$, $i=0,1$, are defined so that inequalities 1–3 in Definition 1 hold.

The function $\mu(x,y)$ is defined as the solution of the problem

$$\begin{gathered}\bar{A}_1(x,y)\frac{\partial\mu}{\partial x}+\bar{A}_2(x,y)\frac{\partial\mu}{\partial y}+W\mu=R,\quad (x,y)\in D,\\ \mu(x,y)=\mu(x+L,y),\quad (x,y)\in\overline{D},\quad \mu(x,0)=R^0,\quad x\in\mathbb{R}.\end{gathered} \tag{22}$$

Here R and R^0 are sufficiently large positive constants; the functions $\bar{A}_i(x,y)$, $i=1,2$, and $W(x,y)$ are defined by the expressions (8) and (10), respectively.

Earlier, in Sec. 2.1, we have considered a similar problem for the functions $\bar{u}_i(x,y)$. By reproducing the argument therein, we write the solution of problem (22) in closed form as

$$\mu(x,y) = R^0 \exp\Bigg(-\int\limits_0^y \frac{W(X(y_1,C_1),y_1)}{\bar{A}_2(X(y_1,C_1),y_1)}\,dy_1\Bigg) + \int\limits_0^y \exp\Bigg(-\int\limits_{y_1}^y \frac{W(X(y_2,C_1),y_2)}{\bar{A}_2(X(y_2,C_1),y_2)}\,dy_2\Bigg)\frac{R}{\bar{A}_2(X(y_1,C_1),y_1)}\,dy_1,$$

where C_1 is the left-hand side of the first integral (12).

Since R and R_0 are positive constants, and, by Condition A2, the inequality $\bar{A}_2(x,y) > 0$ holds for all $(x,y) \in \overline{D}$, it follows that the function $\mu(x,y)$ assumes positive values in $\overline{D}$.

Let us define the function $\pi_0(\xi,x)$ as the solution of the problem

$$\begin{gathered}\frac{\partial^2\pi_0}{\partial\xi^2} - \tilde{A}_2(\xi,x)\frac{\partial\pi_0}{\partial\xi} - \frac{\partial\tilde{A}_2}{\partial u}(\xi,x)\Phi(\xi,x)\pi_0 = \frac{\partial\tilde{A}_2}{\partial u}(\xi,x)\Phi(\xi,x)\mu(x,a) - \nu e^{\gamma\xi}, \quad \xi < 0, \\ \pi_0(0,x) = 0, \quad \pi_0(-\infty,x) = 0,\end{gathered} \tag{23}$$

where ν and γ are positive constants chosen so that the function $\pi_0(\xi,x)$ takes positive values for all $\xi < 0$ and $x \in \mathbb{R}$. The solution of problem (23) can be written in closed form as

$$\pi_0(\xi,x) = -\Phi(\xi,x)\int\limits_0^\xi \frac{ds}{\Phi(s,x)}\int\limits_{-\infty}^s \Bigg(\frac{\partial\tilde{A}_2}{\partial u}(\eta,x)\Phi(\eta,x)\mu(x,a) - \nu e^{\gamma\eta}\Bigg)\,d\eta.$$

Choose the constants ν and γ such that the inequality

$$\frac{\partial\tilde{A}_2}{\partial u}(\xi,x)\Phi(\xi,x)\mu(x,a) - \nu e^{\gamma\xi} < 0$$

is satisfied for $\xi \le 0$ and $x \in \mathbb{R}$. (Since the function $\Phi(\xi,x)$ decays exponentially, it follows that the last inequality is satisfied provided that ν is sufficiently large and γ is sufficiently small.) We thereby find that the function $\pi_0(\xi,x)$ is nonnegative.

Let us define the function $\pi_1(\xi,x)$ as the solution of the problem

$$\begin{gathered}\frac{\partial^2\pi_1}{\partial\xi^2} - \tilde{A}_2(\xi,x)\frac{\partial\pi_1}{\partial\xi} - \frac{\partial\tilde{A}_2}{\partial u}(\xi,x)\Phi(\xi,x)\pi_1 = \pi_1 f(\xi,x), \quad \xi < 0, \\ \pi_1(0,x) = 0, \quad \pi_1(-\infty,x) = 0,\end{gathered}$$

where the expression $\pi_1 f(\xi,x)$ comprises the terms of the order of ε^{n+1} that emerge due to adding the functions $\mu(x,y)$ and $\pi_0(\xi,x)$ in the expression $L_\varepsilon[\beta]$ as well as the terms Π_{n+1} and $\bar{u}_{n+1}$ occurring in the asymptotic approximation U_{n+1}. It can readily be noted that the same terms (but with the opposite sign) occur in the expression $L_\varepsilon[\alpha]$.

Just as all boundary layer functions, $\pi_0(\xi,x)$ and $\pi_1(\xi,x)$ exponentially decay as $\xi \to -\infty$.

Lemma 1. *The functions $\beta_{n+1}(x,y,\varepsilon)$, $\alpha_{n+1}(x,y,\varepsilon)$ satisfy inequalities* 1–3 *in Definition* 1.

Proof. 1. To establish the order of the functions $\beta_{n+1}(x,y,\varepsilon)$ and $\alpha_{n+1}(x,y,\varepsilon)$, consider their difference

$$\beta_{n+1}(x,y,\varepsilon) - \alpha_{n+1}(x,y,\varepsilon) = 2\varepsilon^{n+1}(\mu(x,y) + \pi_0(\xi,x)) + O(\varepsilon^{n+2}). \tag{24}$$

The function $\mu(x,y)$ assumes positive values for all $(x,y) \in \overline{D}$, and $\pi_0(\xi,x) \ge 0$ for all $\xi \le 0$ and $x \in \mathbb{R}$; therefore, the right-hand side of the last expression is positive.

2. It follows from the definition of the functions $\mu(x,y)$, $\pi_0(\xi,x)$, and $\pi_1(\xi,x)$ that

$$L_\varepsilon[\beta_{n+1}] = -\varepsilon^n d e^{\gamma\xi} - \varepsilon^{n+1}R + O(\varepsilon^{n+2}), \quad L_\varepsilon[\alpha_{n+1}] = \varepsilon^n d e^{\gamma\xi} + \varepsilon^{n+1}R + O(\varepsilon^{n+2}), \tag{25}$$

where $R > 0$ is the constant on the right-hand side in Eq. (22). Consequently, the differential inequalities 2 in Definition 1 are satisfied for sufficiently small ε.

Inequality 3 in Definition 1 holds by virtue of the boundary conditions of problem (22) and the positivity of the function $\mu(x,y)$. The proof of the lemma is complete.

The result of this subsection can be stated as the following theorem.

Theorem 1. *Assume that conditions* A1–A3 *are satisfied. Then for sufficiently small* ε *there exists a solution* $u_\varepsilon(x,y)$ *of problem* (2) *for which the function* $U_n(x,y,\varepsilon)$ *is a uniform asymptotic approximation in* $\overline{D}$ *up to* $O(\varepsilon^{n+1})$; *i.e., the inequality*

$$|u_\varepsilon(x,y) - U_n(x,y,\varepsilon)| < C\varepsilon^{n+1} \tag{26}$$

holds in $\overline{D}$, *where* C *is a positive constant.*

According to [8, pp. 67–87; 9; 10], the existence of upper and lower solutions implies the existence of a solution $u_\varepsilon(x,y)$ of problem (2) for which the inequalities

$$\alpha_{n+1}(x,y,\varepsilon) \le u_\varepsilon(x,y) \le \beta_{n+1}(x,y,\varepsilon) \tag{27}$$

hold. It follows from this expression, as well as the fact that $\beta_{n+1}(x,y,\varepsilon) - \alpha_{n+1}(x,y,\varepsilon) = O(\varepsilon^{n+1})$ (see (24)) that the estimate (26) holds.

4. LOCAL UNIQUENESS AND ASYMPTOTIC STABILITY OF THE STATIONARY SOLUTION

The proof of the local uniqueness and asymptotic stability of the stationary solution of problem (1), whose existence was proved in the preceding section, is based on the method of upper and lower solutions. Let us recall their definition.

Definition 2. Two functions $\hat\alpha(x,y,t,\varepsilon), \hat\beta(x,y,t,\varepsilon) \in C(\overline{D}\times\mathbb{R}^+) \cap C^{2,1}(D\times\mathbb{R}^+)$ that are L-periodic in the variable x are called an *upper* and a *lower solution of problem* (1), respectively, if the following conditions are satisfied:

1°. $\hat\alpha(x,y,t,\varepsilon) \le \hat\beta(x,t,y,\varepsilon)$, $(x,y)\in\overline{D}$, $t>0$.

2°. $L_t[\hat\beta] := \varepsilon\Delta\hat\beta - \dfrac{\partial\hat\beta}{\partial t} - (\mathbf{A}(\hat\beta,x,y),\nabla)\hat\beta - B(\hat\beta,x,y) \le 0 \le L_t[\hat\alpha]$, $(x,y)\in D$, $t>0$.

3°. $\hat\alpha(x,0,t,\varepsilon) \le u^0 \le \hat\beta(x,0,t,\varepsilon)$ and $\hat\alpha(x,a,t,\varepsilon) \le u^a \le \hat\beta(x,a,t,\varepsilon)$, $x\in\mathbb{R}$, $t>0$.

4°. $\hat\alpha(x,y,0,\varepsilon) \le v_{\text{init}}(x,y,\varepsilon) \le \hat\beta(x,y,0,\varepsilon)$, $(x,y)\in\overline{D}$.

It is known (see, e.g., [11]) that if there exists an upper and a lower solution of problem (1), then this problem has a unique solution $v_\varepsilon(x,y,t)$ confined between the upper and lower solutions,

$$\hat\alpha(x,y,t,\varepsilon) \le v_\varepsilon(x,y,t) \le \hat\beta(x,y,t,\varepsilon). \tag{28}$$

The upper and lower solutions of problem (1) will be constructed by analogy with [6, 7, pp. 12–14] in the following form:

$$\begin{aligned}\hat\beta(x,y,t,\varepsilon) &= u_\varepsilon(x,y) + (\beta_{n+1}(x,y,\varepsilon) - u_\varepsilon(x,y))e^{-\lambda t},\\ \hat\alpha(x,y,t,\varepsilon) &= u_\varepsilon(x,y) + (\alpha_{n+1}(x,y,\varepsilon) - u_\varepsilon(x,y))e^{-\lambda t},\end{aligned} \tag{29}$$

where $u_\varepsilon(x,y)$ is a solution of problem (2), which exists according to Theorem 1, the functions $\beta_{n+1}(x,y,\varepsilon)$ and $\alpha_{n+1}(x,y,\varepsilon)$ are defined by the expressions (21), and λ is a positive constant.

Note that these functions satisfy conditions 1° and 3° in Definition 2, because conditions 1 and 3 in Definition 1 are satisfied for the functions α_{n+1} and β_{n+1}. Condition 4° is satisfied provided that

$$\hat\alpha(x,y,0,\varepsilon) = \alpha_{n+1}(x,y,\varepsilon) \le v_{\text{init}}(x,y,\varepsilon) \le \hat\beta(x,y,0,\varepsilon) = \beta_{n+1}(x,y,\varepsilon).$$

We will need the following assertion to prove that inequality 2° in Definition 2 holds.

Lemma 2. *If $\alpha_{n+1}(x,y,\varepsilon)$ and $\beta_{n+1}(x,y,\varepsilon)$ are the lower and upper solutions of problem* (2) *defined by the expressions* (21) *and $u_\varepsilon(x,y)$ is a solution of problem* (2), *which exists according to Theorem* 1, *then the following estimates hold everywhere in the domain $\overline{D}$*:

$$\left|\frac{\partial(\beta_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial x}\right| = O(\varepsilon^n), \quad \left|\frac{\partial(\beta_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial y}\right| = O(\varepsilon^n),$$
$$\left|\frac{\partial(\alpha_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial x}\right| = O(\varepsilon^n), \quad \left|\frac{\partial(\alpha_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial y}\right| = O(\varepsilon^n). \tag{30}$$

Proof. The proof of the lemma will be conducted using the relations

$$\frac{\partial(\beta_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial x} = \frac{\partial(U_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial x} + O(\varepsilon^{n+1}),$$
$$\frac{\partial(\beta_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial y} = \frac{\partial(U_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial y} + O(\varepsilon^{n+1}), \tag{31}$$

which can derived by analogy with how it was done in [21], and the corresponding relations for the lower solution.

Let us derive the estimates

$$\left|\frac{\partial(U_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial x}\right| \le C\varepsilon^n, \quad \left|\frac{\partial(U_{n+1}(x,y,\varepsilon)-u_\varepsilon(x,y))}{\partial y}\right| \le C\varepsilon^n. \tag{32}$$

We introduce the notation $z_{n+1}(x,y,\varepsilon) = U_{n+1}(x,y,\varepsilon) - u_\varepsilon(x,y)$. By construction, the function $U_{n+1}(x,y,\varepsilon)$ satisfies the equation in problem (2) up to $O(\varepsilon^{n+1})$ and exactly satisfies the boundary conditions at $y = 0$ and $y = a$. Therefore, the function z_{n+1} can be represented as a solution of the problem

$$\begin{aligned}\varepsilon\Delta z_{n+1} - \left(A_1(U_{n+1},x,y)\frac{\partial U_{n+1}}{\partial x} + A_2(U_{n+1},x,y)\frac{\partial U_{n+1}}{\partial y} - A_1(u,x,y)\frac{\partial u}{\partial x} - A_2(u,x,y)\frac{\partial u}{\partial y}\right)&\\ -(B(U_{n+1},x,y) - B(u,x,y)) = \varepsilon^{n+1}\psi(x,y), \quad (x,y)\in D,&\\ z_{n+1}(x,0,\varepsilon) = z_{n+1}(x,a,\varepsilon) = 0, \quad x\in\mathbb{R},&\\ z_{n+1}(x,y,\varepsilon) = z_{n+1}(x+L,y,\varepsilon), \quad (x,y)\in\overline{D},&\end{aligned} \tag{33}$$

where $|\psi(x,y)| < c$ and c is a positive constant.

Using the relations

$$A_1(U_{n+1},x,y)\frac{\partial U_{n+1}}{\partial x} - A_1(u,x,y)\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\int\limits_u^{U_{n+1}} A_1(s,x,y)\,ds - \int\limits_u^{U_{n+1}} \frac{\partial}{\partial x}A_1(s,x,y)\,ds,$$
$$A_2(U_{n+1},x,y)\frac{\partial U_{n+1}}{\partial y} - A_2(u,x,y)\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int\limits_u^{U_{n+1}} A_2(s,x,y)\,ds - \int\limits_u^{U_{n+1}} \frac{\partial}{\partial y}A_2(s,x,y)\,ds,$$

we bring Eq. (33) to the form

$$\Delta z_{n+1} = \frac{1}{\varepsilon}\frac{\partial}{\partial x}\int\limits_u^{U_{n+1}} A_1(s,x,y)\,ds + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\int\limits_u^{U_{n+1}} A_2(s,x,y)\,ds + p(x,y,\varepsilon),$$

where

$$\begin{aligned}p(x,y,\varepsilon) = -\frac{1}{\varepsilon}\int\limits_u^{U_{n+1}} \frac{\partial}{\partial x}A_1(s,x,y)\,ds - \frac{1}{\varepsilon}\int\limits_u^{U_{n+1}} \frac{\partial}{\partial y}A_2(s,x,y)\,ds&\\ + \frac{1}{\varepsilon}(B(U_{n+1},x,y) - B(u,x,y)) + \varepsilon^n\psi(x,y).&\end{aligned}$$

Inequality (26) implies the estimate

$$|p(x, y, \varepsilon)| \leq C\varepsilon^n, \quad C > 0. \tag{34}$$

Using the representation of the solution of problem (33) via Green's function (see, e.g., [22, p. 26 of the Russian translation]), we derive the relation

$$z_{n+1}(x, y) = \int_0^L \int_0^a G(x, y; \xi, \eta) p(\xi, \eta, \varepsilon)\, d\xi\, d\eta + \frac{1}{\varepsilon} \int_0^L \int_0^a G(x, y; \xi, \eta) \left(\frac{\partial}{\partial \xi} \int_{u(\xi,\eta,\varepsilon)}^{U_{n+1}(\xi,\eta,\varepsilon)} A_1(s, \xi, \eta)\, ds \right) d\xi\, d\eta$$
$$+ \frac{1}{\varepsilon} \int_0^L \int_0^a G(x, y; \xi, \eta) \left(\frac{\partial}{\partial \eta} \int_{u(\xi,\eta,\varepsilon)}^{U_{n+1}(\xi,\eta,\varepsilon)} A_2(s, \xi, \eta)\, ds \right) d\xi\, d\eta.$$

Integrating by parts and taking into account the boundary conditions for Green's function of problem (33), we make the transformation

$$\int_0^L \int_0^a G(x, y, \xi, \eta) \left(\frac{\partial}{\partial \xi} \int_{u(\xi,\eta,\varepsilon)}^{U_{n+1}(\xi,\eta,\varepsilon)} A_1(s, \xi, \eta)\, ds + \frac{\partial}{\partial \eta} \int_{u(\xi,\eta,\varepsilon)}^{U_{n+1}(\xi,\eta,\varepsilon)} A_2(s, \xi, \eta)\, ds \right) d\xi\, d\eta$$
$$= -\int_0^L \int_0^a \left[G_\xi(x, y; \xi, \eta) \int_{u(\xi,\eta,\varepsilon)}^{U_{n+1}(\xi,\eta,\varepsilon)} A_1(s, \xi, \eta)\, ds \right] d\xi\, d\eta \tag{35}$$
$$- \int_0^L \int_0^a \left[G_\eta(x, y; \xi, \eta) \int_{u(\xi,\eta,\varepsilon)}^{U_{n+1}(\xi,\eta,\varepsilon)} A_2(s, \xi, \eta)\, ds \right] d\xi\, d\eta.$$

In view of the estimate (26) and the smoothness of the functions $A_i(u, x, y)$, $i = 1, 2$, the following relations hold:

$$\left| \int_{u(\xi,\eta,\varepsilon)}^{U_{n+1}(\xi,\eta,\varepsilon)} A_i(s, \xi, \eta)\, ds \right| = c_i(\xi, \eta, \varepsilon), \quad i = 1, 2, \tag{36}$$

where the $c_i(\xi, \eta, \varepsilon) \leq c\varepsilon^{n+2}$, $i = 1, 2$, are smooth functions.

Considering relations (36) and transformations (35) as well as the estimate (34), we derive the following estimates for the derivatives $\partial z_{n+1}/\partial x$ and $\partial z_{n+1}/\partial y$:

$$\left| \frac{\partial z_{n+1}}{\partial x}(x, y) \right| \leq C\varepsilon^n \left| \int_0^L \int_0^a G_x(x, y; \xi, \eta)\, d\xi\, d\eta \right| + \left(\left| \int_0^L \int_0^a G_{x\xi}(x, y; \xi, \eta) c_1(\xi, \eta, \varepsilon)\, d\xi\, d\eta \right| \right.$$
$$\left. + \left| \int_0^L \int_0^a G_{x\eta}(x, y; \xi, \eta) c_2(\xi, \eta, \varepsilon)\, d\xi\, d\eta \right| \right),$$
$$\left| \frac{\partial z_{n+1}}{\partial y}(x, y) \right| \leq C\varepsilon^n \left| \int_0^L \int_0^a G_y(x, y; \xi, \eta)\, d\xi\, d\eta \right| + \left(\left| \int_0^L \int_0^a G_{y\xi}(x, y; \xi, \eta) c_1(\xi, \eta, \varepsilon)\, d\xi\, d\eta \right| \right. \tag{37}$$
$$\left. + \left| \int_0^L \int_0^a G_{y\eta}(x, y; \xi, \eta, \varepsilon) c_2(\xi, \eta, \varepsilon)\, d\xi\, d\eta \right| \right).$$

Using the estimates for the derivatives of Green's function (see [23]), we arrive at the following estimates of the integrals:

$$\left|\int_0^L\int_0^a G_x(x,y;\xi,\eta)\,d\xi\,d\eta\right| < c, \qquad \left|\int_0^L\int_0^a G_y(x,y;\xi,\eta)\,d\xi\,d\eta\right| < c,$$

where c is a constant.

It is well known that the Green's function of the Laplace operator can be represented as the sum of two terms [24, pp. 423–426], $G(x,y;\xi,\eta) = \mathcal{E}(r) + g(x,y;\xi,\eta)$, where $\mathcal{E}(r)$ is the fundamental solution of the Laplace equation, $r = ((x-\xi)^2+(y-\eta)^2)^{1/2}$, and $g(x,y;\xi,\eta)$ is a harmonic function of the variables (x,y) and (ξ,η). Using this representation, we make the transformation

$$\int_0^L\int_0^a G_{x\xi}(x,y;\xi,\eta)c_1(\xi,\eta,\varepsilon)\,d\xi\,d\eta = \int_0^L\int_0^a \mathcal{E}_{x\xi}(r)c_1(\xi,\eta,\varepsilon)\,d\xi\,d\eta + \int_0^L\int_0^a g_{x\xi}(x,y;\xi,\eta)c_1(\xi,\eta,\varepsilon)\,d\xi\,d\eta.$$

The second term on the right-hand side in the last expression is a double integral of a smooth function bounded in absolute value. The first term is transformed as follows:

$$\int_0^L\int_0^a \mathcal{E}_{x\xi}(r)c_1(\xi,\eta,\varepsilon)\,d\xi\,d\eta = -\frac{\partial^2}{\partial x^2}\int_0^L\int_0^a \mathcal{E}(r)c_1(\xi,\eta,\varepsilon)\,d\xi\,d\eta.$$

The right-hand side of the last relation is the second derivative of Newton's potential and is continuous in the rectangle $\{(x,y) : 0 \le x \le L,\ 0 \le y \le a\}$ [25, p. 171]. The following estimate thus holds:

$$\left|\int_0^L\int_0^a G_{x\xi}(x,y;\xi,\eta)c_1(\xi,\eta,\varepsilon)\,d\xi\,d\eta\right| < c\varepsilon^{n+2}.$$

In a similar manner, we prove the estimate of all the other double integrals on the right-hand sides in inequalities (37) containing the second derivatives of the Green's function.

The estimates for the derivatives $\partial z_{n+1}/\partial x$ and $\partial z_{n+1}/\partial y$ imply that inequalities (32) hold, and hence, in view of relations (31), the assertion of the lemma is true.

Let us proceed to proving the main result of this section of the present paper. Acting by the operator L_t on the function $\hat\beta$, we arrive at the relation

$$L_t[\hat\beta] = \varepsilon\Delta\hat\beta - (\mathbf{A}(\hat\beta,x,y),\nabla)\hat\beta - B(\hat\beta,x,y) + \lambda(\beta_{n+1} - u_\varepsilon)e^{-\lambda t}.$$

(We omit the arguments of the functions $\hat\beta$ and u_ε for brevity.)

We add the following terms to the right-hand side of the last relation:

$$(\mathbf{A}(u_\varepsilon,x,y),\nabla)u_\varepsilon, \quad e^{-\lambda t}(\mathbf{A}(\beta_{n+1},x,y),\nabla)\beta_{n+1}, \quad e^{-\lambda t}(\mathbf{A}(u_\varepsilon,x,y),\nabla)u_\varepsilon, \quad e^{-\lambda t}(\mathbf{A}(u_\varepsilon,x,y),\nabla)\beta_{n+1},$$
$$(\mathbf{A}(u_\varepsilon,x,y),\nabla)\beta_{n+1}, \quad (\mathbf{A}(\hat\beta,x,y),\nabla)\beta_{n+1}, \quad B(u_\varepsilon,x,y), \quad B(u_\varepsilon,x,y)e^{-\lambda t}, \quad B(\beta_{n+1},x,y)e^{-\lambda t},$$

and then we subtract them so that the relation still holds. After some transformations, we obtain the relation

$$\begin{aligned} L_t[\hat\beta] &= L_\varepsilon[u_\varepsilon] + e^{-\lambda t}(L_\varepsilon[\beta_{n+1}] - L_\varepsilon[u_\varepsilon]) + \lambda e^{-\lambda t}(\beta_{n+1} - u_\varepsilon) \\ &+ (e^{-\lambda t} - 1)(\mathbf{A}(u_\varepsilon,x,y) - \mathbf{A}(\hat\beta,x,y),\nabla)(\beta_{n+1} - u_\varepsilon) \\ &+ e^{-\lambda t}(\mathbf{A}(\beta_{n+1},x,y) - \mathbf{A}(u_\varepsilon,x,y),\nabla)\beta_{n+1} + (\mathbf{A}(u_\varepsilon,x,y) - \mathbf{A}(\hat\beta,x,y),\nabla)\beta_{n+1} \\ &+ e^{-\lambda t}(B(\beta_{n+1},x,y) - B(u_\varepsilon,x,y)) + (B(u_\varepsilon,x,y) - B(\hat\beta,x,y)), \end{aligned}$$

where the operator L_ε has been defined in condition 2 of Definition 1.

Let us take into account the relation $L_\varepsilon[u_\varepsilon] = 0$, which holds by virtue of Eq. (2), the first relation in (25), and some corollaries of the Lagrange formula. Then the expression for $L_t[\hat{\beta}]$ acquires the form

$$\begin{aligned} L_t[\hat{\beta}] &= e^{-\lambda t}(-\varepsilon^n d e^{\gamma\xi} - \varepsilon^{n+1} R + \lambda(\beta_{n+1} - u_\varepsilon) \\ &\quad - (e^{-\lambda t} - 1)(\beta_{n+1} - u_\varepsilon)(\mathbf{A}_u^*, \nabla)(\beta_{n+1} - u_\varepsilon) + (\beta_{n+1} - u_\varepsilon)^2(\theta_1 - \theta_2 e^{-\lambda t})(\mathbf{A}_{uu}^*, \nabla)\beta_{n+1}) \\ &\quad + e^{-\lambda t}((\beta_{n+1} - u_\varepsilon)^2(\theta_4 - \theta_5 e^{-\lambda t}) B_{uu}^* + O(\varepsilon^{n+2})), \end{aligned} \tag{38}$$

where

$$\begin{aligned} \mathbf{A}_u^* &= \mathbf{A}_u(u_\varepsilon + \theta_0(\beta_{n+1} - u_\varepsilon)e^{-\lambda t}), \\ \mathbf{A}_{uu}^* &= \mathbf{A}_{uu}(u_\varepsilon + \theta_1(\beta_{n+1} - u_\varepsilon) + \theta_3(\beta_{n+1} - u_\varepsilon)(\theta_1 - \theta_2 e^{-\lambda t})), \\ B_{uu}^* &= B_{uu}(u_\varepsilon + \theta_4(\beta_{n+1} - u_\varepsilon) + \theta_6(\beta_{n+1} - u_\varepsilon)(\theta_4 - \theta_5 e^{-\lambda t})), \quad 0 < \theta_i < 1, \quad i = 0, \ldots, 6. \end{aligned}$$

By inequalities (27) and (26), for all $(x, y) \in \overline{D}$ we have the estimate

$$\beta_{n+1}(x, y, \varepsilon) - u_\varepsilon(x, y) = O(\varepsilon^{n+1}).$$

Moreover, it follows from the estimates (30) that $(\mathbf{A}_u^*, \nabla)(\beta_{n+1}(x, y, \varepsilon) - u_\varepsilon(x, y)) = O(\varepsilon^n)$, $(x, y) \in D$, and therefore, for $n \geq 1$ and sufficiently small ε the expression on the right-hand side in relation (38) assumes negative values given a sufficiently large positive R, and hence condition 2° is satisfied for the function $\hat{\beta}$. Thus, the function $\hat{\beta}(x, y, t, \varepsilon)$ is an upper solution of problem (1). In a similar way, we can prove that the function $\hat{\alpha}(x, y, t, \varepsilon)$ is a lower solution of problem (1).

The form (29) for the upper and lower solutions of problem (1) for $n = 1$, as well as inequalities (28), implies that the following limit relation holds for any initial function $v_{\text{init}}(x, y)$ of problem (1) for which we have the inequalities $\alpha_2(x, y, \varepsilon) \leq v_{\text{init}}(x, y) \leq \beta_2(x, y, \varepsilon)$ for all $(x, y) \in \overline{D}$:

$$\lim_{t \to +\infty} |v_\varepsilon(x, y, t) - u_\varepsilon(x, y)| = 0, \tag{39}$$

where $v_\varepsilon(x, y, t)$ is a solution of problem (1). The fact that this limit relation holds implies the (Lyapunov) asymptotic stability of the solution $u_\varepsilon(x, y)$ of problem (2) as a time-invariant solution of problem (1). Moreover, the uniqueness of the solution $v_\varepsilon(x, y, t)$ of problem (1) and relation (39) imply the uniqueness of the stationary solution $u_\varepsilon(x, y)$ on the interval $[\alpha_2(x, y, \varepsilon), \beta_2(x, y, \varepsilon)]$. Thus, the following assertion holds.

Theorem 2. *Let conditions* A1–A3 *be satisfied. Then, given sufficiently small* ε, *the solution* $u_\varepsilon(x, y)$ *of problem* (2) *for which the function* $U_n(x, y, \varepsilon)$ *is an asymptotic approximation is locally unique and asymptotically Lyapunov stable as the stationary solution of problem* (1), *with the stability domain being at least* $[\alpha_2, \beta_2]$.

CONCLUSIONS

In the present paper, we have studied the boundary layer solution of a two-dimensional boundary value problem of the reaction—diffusion—advection type in the case where the advection term is comparable in the order of magnitude with the reaction one. The existence and stability of the solution have been proved, and its asymptotic approximation has been constructed. The results obtained can be used for further research into the solutions of the advancing front type in problems arising in various applications, for example, when modeling combustion or nonlinear acoustic processes.

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