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ORDINARY DIFFERENTIAL EQUATIONS

Study of a Nonlinear Eigenvalue Problem by the Integral Characteristic Equation Method

D. V. Valovik
Penza State University, Penza, 440026 Russia
e-mail: dvalovik@mail.ru

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Abstract—We consider an eigenvalue problem for a quasilinear nonautonomous second-order differential equation with a cubic nonlinearity. The problem is posed on an interval with boundary conditions of the first kind and with an auxiliary (local) condition at one of the endpoints of the interval. We prove that the problem in question has infinitely many negative and infinitely many positive eigenvalues. The corresponding linear problem has infinitely many negative and finitely many (or none) positive eigenvalues. Moreover, the first terms of the asymptotics of the negative eigenvalues of the nonlinear and linear problems coincide, while the asymptotics of the positive eigenvalues of the nonlinear problem is expressed in terms of a transcendental function of the eigenvalue number. The results are derived with the use of a nonclassical approach.

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1. STATEMENT OF THE PROBLEM AND INTRODUCTORY REMARKS

Let $\mathrm{I} = (0, h)$ and $\bar{\mathrm{I}} = [0, h]$, where $h > 0$, let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = (0, +\infty)$, and let $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$ be parameters. In addition, let $a(x) \in C^1(\bar{\mathrm{I}})$ be a given nonnegative function that is not identically zero and satisfies $a'(x) \geq 0$; we set $a_- = \min\{a(x) : x \in \bar{\mathrm{I}}\}$ and $a_+ = \max\{a(x) : x \in \bar{\mathrm{I}}\}$.

Problem $\mathcal{P}$ consists in finding the parameter values $\lambda = \widehat{\lambda}$ such that there exist solutions $u \equiv u(x; \widehat{\lambda}, \alpha) \in C^2(\bar{\mathrm{I}})$ of the equation

$$u'' = -(a(x) - \lambda)u - \alpha u^3 \tag{1}$$

with the boundary conditions

$$u|_{x=0} = 0, \quad u'|_{x=0} = A \neq 0, \tag{2}$$

$$u|_{x=h} = 0, \tag{3}$$

where $(x, \lambda, \alpha) \in \bar{\mathrm{I}} \times \mathbb{R} \times \mathbb{R}_+$ and $A \neq 0$ is a real constant. It is obvious that, without loss of generality, we can assume that $A > 0$.

Definition 1. A number $\lambda = \widehat{\lambda}$ such that there exists a function $u \equiv u(x; \widehat{\lambda}, \alpha) \in C^2(\bar{\mathrm{I}})$ satisfying Eq. (1) and the boundary conditions (2) and (3) is called an *eigenvalue* of problem $\mathcal{P}$, and the corresponding function u is called an *eigenfunction* of problem $\mathcal{P}$.

Concerning this definition, note that Definition 1 is a nonclassical analog of the well-known definition of a characteristic number of a linear operator function nonlinearly depending on the spectral parameter [1, p. 324]. We point out that throughout the following, when speaking of eigenvalues, we mean eigenvalues in the sense of Definition 1. In other words, the assertions and theorems given below do not pertain to the notion of eigenvalue in the conventional sense.

The cubic nonlinearity occurring in Eq. (1) is rather simple; this permits one, without obscuring the exposition with technical details, to isolate some essential points arising in eigenvalue problems of the type considered in the present paper (see, e.g., [2, 3]). Moreover, last but not least, cubic nonlinearities arise in problems of nonlinear mathematical physics, and hence equations with such nonlinearities provide mathematical models of physical processes and are important to study from the viewpoint of applications (see, e.g., [4, 5]).

For $\alpha = 0$, we obtain the (linear) problem $\mathcal{P}^0$ of finding the parameter values $\lambda = \widetilde{\lambda}$ such that there exist nontrivial solutions $v \equiv v(x; \widetilde{\lambda}) \in C^2(\bar{\mathrm{I}})$ of the equation

$$v'' = -(a(x) - \lambda)v \tag{4}$$

with the boundary conditions

$$v|_{x=0} = 0, \quad v|_{x=h} = 0, \tag{5}$$

where $(x, \lambda) \in \bar{\mathrm{I}} \times \mathbb{R}$.

The main method for studying problem $\mathcal{P}$ is the integral characteristic equation method developed in this paper. Note that widely known approaches used in nonlinear analysis like the variational method [6–8] or methods of solution branching theory [9, 10] do not apply to problem $\mathcal{P}$ (see also the remarks in [2]).

For $\lambda \in \mathbb{R}_+$, problem $\mathcal{P}$ describes the propagation of a monochromatic electromagnetic TE-wave in a planar shielded dielectric waveguide with permittivity $\varepsilon = \varepsilon_y + \beta|\mathbf{E}|^2$. Here $\varepsilon_y \equiv \varepsilon_y(x)$ is a real continuous function that characterizes the linear component of the permittivity ε, $\beta > 0$ is a real constant, and $\mathbf{E}$ is the electric field vector of the TE-wave (see [2, 3]). In the above notation, $\lambda = \omega^2\mu_0\varepsilon_0\gamma^2$, where $\gamma \in \mathbb{R}$ is the TE-wave propagation constant; $a = \omega^2\mu_0\varepsilon_0\varepsilon_y$; $\alpha = \omega^2\mu_0\varepsilon_0\beta$, where ω is the frequency of the TE-wave; finally, μ_0 is the permeability and ε_0 the permittivity of vacuum.

The present paper is organized as follows. Section 2 states the results and provides additional remarks and discussion of these results (Sec. 2.1 briefly presents known results on the eigenvalues of problem $\mathcal{P}^0$, and Sec. 2.2 studies problem $\mathcal{P}$); the proofs are presented in Sec. 3. Section 2.2 also contains auxiliary constructions as well as a discussion of the results.

2. MAIN RESULTS

The positive and negative eigenvalues $\widehat{\lambda}$ of problem $\mathcal{P}$ will be denoted by $\widehat{\lambda}_k^-$ and $\widehat{\lambda}_{k'}^+$, respectively, and the eigenvalues $\widetilde{\lambda}$ of problem $\mathcal{P}^0$ will be denoted by $\widetilde{\lambda}_k$, where k and k' are nonnegative integer indices. Here we assume that $\widehat{\lambda}_k^-$ and $\widetilde{\lambda}_k$ are arranged in descending order and the $\widehat{\lambda}_{k'}^+$ are arranged in ascending order.

2.1. Problem $\mathcal{P}^0$

Problem $\mathcal{P}^0$ has been well studied, with the following assertion holding true [11, p. 277].

Theorem 1. *Problem $\mathcal{P}^0$ has infinitely many eigenvalues $\widetilde{\lambda}_k$; moreover, there exist only finitely many (or none at all) positive and infinitely many negative eigenvalues, and $\widetilde{\lambda}_k \to -\infty$ as $k \to +\infty$. All eigenvalues are real and simple (i.e., of multiplicity one), and the asymptotics $\widetilde{\lambda}_k = O^*(k^2)$ holds as $k \to +\infty$.*

One possible approach to studying problem $\mathcal{P}$ is to use problem $\mathcal{P}^0$ as an "unperturbed" one. Using the approach based on perturbation theory, for example, by inverting the linear part of the differential operator defined by problem $\mathcal{P}$ with the use of Green's function, one can prove that, for sufficiently small α, some neighborhood of each eigenvalue $\widetilde{\lambda}_k$ of problem $\mathcal{P}^0$ contains an eigenvalue $\widehat{\lambda}_{k'}$ of problem $\mathcal{P}$ (see, e.g., [12] for $\widehat{\lambda}_{k'}^+$), with the size of this neighborhood tending to zero as $\alpha \to 0$.

This approach has a natural and substantial limitation; namely, it allows finding only those solutions of the nonlinear problem which are close to solutions of the "unperturbed" linear problem. As will be shown in the sequel, problem $\mathcal{P}$ has infinitely many positive eigenvalues with an accumulation point at infinity (even for $a \equiv 0$), while problem $\mathcal{P}^0$, in the general case, according to Theorem 1, has only finitely many positive eigenvalues (and does not have them at all if $a \equiv 0$). However, it is the positive eigenvalues of problem $\mathcal{P}$ that are important in some applications (see Sec. 1). In other words, the perturbation method essentially fails to provide a possibility for studying problems of type $\mathcal{P}$ for all $\lambda \in \mathbb{R}_+$. The above-described situation necessitates developing other techniques for solving such problems.

Despite the above-indicated limitation, the approach based on perturbation theory is realized in the present paper as a simple consequence of the method developed here (see Theorem 4 below).

2.2. Problem $\mathcal{P}$

Let $u \equiv u(x;\lambda,\alpha)$ be the solution of the Cauchy problem (1), (2). To simplfy the notation, we will also write $u(x)$ instead of $u(x;\lambda,\alpha)$. It is obvious that this solution is defined locally (in a neighborhood of the point $x=0$). Assume that it is defined globally, i.e., for all $x \in \bar{\mathrm{I}}$. This assumption, which will be proved below, will enable us to carry out constructions forming an important part of the approach proposed in the present paper.

It can readily be seen that Eq. (1), with allowance for conditions (2), implies the relations

$$u'^2(x) = A^2 - \frac{\alpha}{2}u^4(x) + \lambda u^2(x) - a(x)u^2(x) + \int\limits_0^x a'(s)u^2(s)\,ds. \tag{6}$$

Since the left-hand side of identity (6) is nonnegative, it follows that so is its right-hand side. This implies the boundedness of both the function u (for a given λ) and the function u'.

Consider the function

$$\theta(x) = u^2(x), \quad \mu(x) = \frac{u'(x)}{u(x)}.$$

The function $\theta(x)$ is defined and continuous for $x \in \bar{\mathrm{I}}$, and the function $\mu(x)$ is defined and continuous for those $x \in \bar{\mathrm{I}}$ at which the function $u(x)$ is nonzero. It follows from the uniqueness theorem for the solution of the Cauchy problem that the function $u \not\equiv 0$ cannot be zero together with its derivative at any point.

By (1) and (2), the functions $\theta(x)$ and $\mu(x)$ satisfy the system of equations

$$\theta' = 2\theta\mu, \quad \mu' = -(\mu^2 + a(x) - \lambda + \alpha\theta). \tag{7}$$

Moreover, in view of (6), the functions θ and μ are related by the formula

$$\frac{1}{2}\alpha\theta^2 + (\mu^2 - \lambda + a(x))\theta - \left(A^2 + \int\limits_0^x a'(s)\theta(s)\,ds\right) = 0. \tag{8}$$

It can readily be verified that $\mu' < 0$ for all $\lambda \in \mathbb{R}$. Indeed, in view of Eq. (1) and identity (6), we obtain

$$\mu' = \left(\frac{u'}{u}\right)' = \frac{u''u - (u')^2}{u^2} = -\frac{A^2}{u^2} - \frac{\alpha}{2}u^2 - \frac{1}{u^2}\int\limits_0^x a'(s)u^2(s)\,ds < 0$$

for $u(x) \neq 0$.

Since $\mu' < 0$, we make the following conclusion, which is important in the sequel: for each $\lambda \in \mathbb{R}$, the function $\mu \equiv \mu(x;\lambda)$ is monotone in x on each interval $\mathrm{I}' \subset \mathrm{I}$ that does not contain points where $u(x)$ is zero.

Let the solution u have n' zeros $x_i' \in \mathrm{I}$, where $1 \le i \le n'$. If $n' = 0$, then u does not vanish for $x \in \mathrm{I}$. As was noted above, if $u \not\equiv 0$, then $u'(x_k') \neq 0$. We assume that $x_k' < x_{k+1}'$, $1 \le k \le n'-1$.

Define the intervals $\mathrm{I}_{i+1}' = (x_i', x_{i+1}')$, $i = 0,\dots,n'$, where $x_0' = 0$ and $x_{n'+1}' = h$; if $n' = 0$, then $\mathrm{I}_1' = \mathrm{I}$.

Relation (8) has been derived from identity (6), which holds for the functions u and u' on the entire domain where they are defined regardless of whether u vanishes or not. However, then relation (8) also holds for all $x \in \mathrm{I}_{i+1}'$ for each $i = 0,\dots,n'$.

The fact that $\mu' < 0$, together with conditions (2), implies the relations

$$\lim_{x\to+0}\theta(x) = 0, \quad \lim_{x\to+0}\mu(x) = +\infty, \tag{9}$$

$$\lim_{x\to x_i'}\theta(x)=0,\qquad \lim_{x\to x_i'\pm 0}\mu(x)=\pm\infty,\qquad i=1,\ldots,n'. \tag{10}$$

Thus, the functions $\theta(x)$ and $\mu(x)$ can be defined on each of the intervals I_i' as the solutions of system (7) with conditions appropriately selected from (9) and (10).

In addition, taking into account condition (3), we obtain

$$\lim_{x\to h-0}\theta(x)=0,\qquad \lim_{x\to h-0}\mu(x)=-\infty. \tag{11}$$

It follows from the above reasoning that $\mu'<0$ for $(x,\lambda,\alpha)\in \mathrm{I}_i'\times\mathbb{R}\times\mathbb{R}_+$, where $i=1,\ldots,n'+1$. Consequently, the function μ is monotone decreasing from $+\infty$ to $-\infty$ as x runs through the interval I_i', where $i=1,\ldots,n'+1$. In other words, for each $i=1,\ldots,n'+1$ the restriction $\mu|_{\mathrm{I}_i'}$ of the function μ to the interval I_i' has an inverse function $g_i:\mathbb{R}\to\mathrm{I}_i'$; i.e., $g_i=(\mu|_{\mathrm{I}_i'})^{-1}$, which is a continuous bijection of the line $\mathbb{R}$ onto the interval I_i'. On each of the intervals I_i', this bijection allows one to consider the (continuous) function $\theta(x)$ as a function of the variable μ by setting $x\equiv g_i(\mu)$ (see Fig. 1).

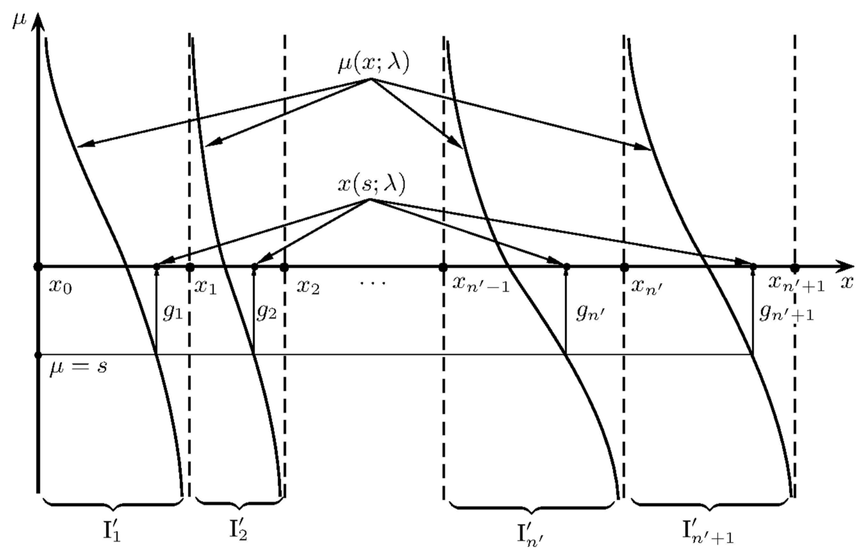

Fig. 1. The function $\mu(x;\lambda)$ and the mappings g_i.

Now, on each of the intervals I_i', consider the equation

$$\mu'=-w_i(\mu;\lambda), \tag{12}$$

where

$$w_i(\mu;\lambda)\equiv\mu^2+a(x)-\lambda+\alpha\theta(x)\quad\text{and}\quad x\equiv g_i(\mu). \tag{13}$$

Using the mappings g_i permits one to study Eq. (12) as autonomous on the interval I_i'.

Thus, for each point $s\in\mathbb{R}$ and each number $i=1,\ldots,n'+1$ there exists a unique point $x\in\mathrm{I}_i'$ determined by the mapping g_i (see Fig. 1). This allows one to give a meaning to the relations

$$T(\lambda;\mathrm{I}_i)=\int\limits_{-\infty}^{+\infty}\frac{ds}{w_i(s;\lambda)}, \tag{14}$$

where w_i is defined on each interval I_i', $i=1,\ldots,n'+1$ (see formula (13)).

Integrating Eq. (12) on each of the intervals I_i' and using conditions (9) and (10) and formula (14), we arrive at the following assertion.

Assertion 1. *The solution* $u \equiv u(x;\lambda,\alpha)$ *of the Cauchy problem* (1), (2) *is uniquely determined for all* $x \in \bar{\mathrm{I}}$ *and continuously depends on the point* $(x,\lambda,\alpha) \in \bar{\mathrm{I}} \times \mathbb{R} \times \mathbb{R}_+$. *Moreover, the following formula holds*:

$$\sum_{i=1}^{n'} T(\lambda;\mathrm{I}_i) + \int\limits_{\mu(h)}^{+\infty} \frac{ds}{w_{n'+1}(s;\lambda)} = h. \tag{15}$$

If the upper limit in the sum in (15) is less than the lower one, then the sum is zero.

Under condition (11), formula (15) gives an equation for which we have the following assertion.

Theorem 2 (on equivalence). *A number* $\widehat{\lambda}$ *is a solution of problem* $\mathcal{P}$ *if and only if there exists an integer* $n' = \widehat{n} \geq 0$ *such that, for* $n' = \widehat{n}$, $\lambda = \widehat{\lambda}$ *is a solution of the equation*

$$\Phi(\lambda; n'+1) \equiv \sum_{i=1}^{n'+1} T(\lambda;\mathrm{I}_i) = h; \tag{16}$$

further, the eigenfunction $u \equiv u(x;\widehat{\lambda},\alpha)$ *has* $\widehat{n}$ (*simple*) *zeros* $x'_i \in \mathrm{I}$, *where* $x'_i = \sum_{k=1}^{i} T(\lambda;\mathrm{I}_k)$, $i = 1,\ldots,\widehat{n}$.

The notion of characteristic equation [13] is widely used in the (linear) theory of Sturm–Liouville problems. For example, in the case of problem $\mathcal{P}^0$, the characteristic equation occurs when substituting the solution $v \equiv v(x;\lambda)$ of the Cauchy problem for Eq. (4) with the initial data $v|_{x=0} = 0$ and $v'|_{x=0} = A$, where $A \neq 0$ is a constant, into the second condition in (5).

In a similar way, we can also introduce the characteristic equation for a nonlinear eigenvalue problem. For linear problems, the use of such an equation is completely justified owing to a vast variety of methods for studying linear differential equations, but with nonlinear equations, even the question about the global unique solvability of the corresponding Cauchy problem has no definite answer in the general case. In relation to what has been said above and in view of Theorem 2 on equivalence, Eq. (16) can be called the *integral characteristic equation* of problem $\mathcal{P}$ in the sense that it determines the eigenvalues; moreover, Eq. (16) explicitly contains a parameter that is responsible for the number of zeros of the eigenfunction. Then it is natural to refer to the function $\Phi(\lambda; n'+1) - h$ as the *integral characteristic function* of problem $\mathcal{P}$.

Theorem 2 is the key result of the present paper. This theorem introduces a new object, the integral characteristic equation of problem $\mathcal{P}$, and establishes the (spectral) equivalence between this equation and the problem under study. Although Eq. (16) actually contains the (unknown!) solution of the auxiliary Cauchy problem, the solvability of Eq. (16) can still be studied under certain conditions. One tool enabling such an investigation is the asymptotic analysis of the behavior of the function $\Phi(\lambda, n')$ as $\lambda \to \pm\infty$. We provide such an analysis in what follows.

It is also important that we have managed to obtain Eq. (16) for the nonautonomous equation (1). In the case of eigenvalue problems for some autonomous nonlinear equations, the integral characteristic equations are rather simple to obtain [2, 3, 14], but it is absolutely unclear whether these results can be extended to the case of nonautonomous equations. One important fact is that Eq. (16) has been derived without any restrictions on α; it is only the positivity of this coefficient that matters.

In view of the preceding, it is natural to introduce the following definition.

Definition 2. We say that an eigenvalue $\widehat{\lambda}$ of problem $\mathcal{P}$ has *multiplicty* l if $\lambda = \widehat{\lambda}$ is a root of multiplicity l of Eq. (16) for $n' = \widehat{n}$.

It is the properties of the function $T(\lambda;\mathrm{I}_i)$ of the variable λ that are the key to studying the solvability of problem $\mathcal{P}$. It is thus convenient to state the following separate assertion.

Assertion 2. *The function* $T(\lambda;\mathrm{I}_i)$ *is defined, continuous, and positive for any* $i = 1,\ldots,n'+1$ *and* $\lambda \in \mathbb{R}$. *Moreover, if* $\lambda < -\delta < 0$, *then*

$$\pi(\delta_1 + |\lambda|)^{-1/2} < T(\lambda;\mathrm{I}_i) < \pi|\lambda|^{-1/2}, \tag{17}$$

where $\delta, \delta_1 > 0$ *are fixed positive constants independent of* λ, *and if* $\lambda > 0$ *is sufficiently large*, *then*

$$\lambda^{-1/2}\ln\lambda + O(\lambda^{-1/2}) < T(\lambda; \mathrm{I}_i) < 2\sqrt{2}\lambda^{-1/2}\ln\lambda + O(\lambda^{-1/2}). \tag{18}$$

Remark 1. The proof of Assertion 2 contains a stronger result; namely, the inequalities

$$T(\lambda; \mathrm{I}_i) < \pi|\lambda|^{-1/2} \quad \text{and} \quad T(\lambda; \mathrm{I}_i) < 2\sqrt{2}\lambda^{-1/2}\ln\lambda$$

hold for all $\lambda < 0$ and all $\lambda > 0$, respectively.

Taking into account formula (16), we derive the following assertion from Assertion 2.

Corollary 1. *The asymptotic formulas*

$$\Phi(\lambda; n') = \begin{cases} \pi n'|\lambda|^{-1/2} + O(\lambda^{-1}) & \text{as} \quad \lambda \to -\infty, \\ \beta n'\lambda^{-1/2}\ln\lambda + O(\lambda^{-1/2}) & \text{as} \quad \lambda \to +\infty \end{cases} \tag{19}$$

hold, *where* $1 < \beta < 2\sqrt{2}$ *is some constant.*

Based on properties (17)–(19), we can establish the solvability of Eq. (16) and hence of problem $\mathcal{P}$. Namely, the following assertion holds.

Theorem 3. *Problem* $\mathcal{P}$ *has infinitely many negative eigenvalues* $\widehat{\lambda}_k^-$ *and infinitely many positive eigenvalues* $\widehat{\lambda}_{k'}^+$, $k, k' = 0, 1, \ldots$ *Further*, *for sufficiently large indices* k *and any* $\Delta > 0$ *one has the relations*

$$\widehat{\lambda}_{k-1}^- = \frac{\pi^2}{h^2}k^2 + O(1) \quad \text{and} \quad (1-\Delta)g\left(\frac{h}{\beta k}\right) < \widehat{\lambda}_{k-1}^+ < (1+\Delta)g\left(\frac{h}{\beta k}\right),$$

where $g(t)$ *is the inverse function of the function* $\lambda^{-1/2}\ln\lambda$; *moreover*, *for all sufficiently large* $\widehat{\lambda}_k^\pm$ *one has the formulas*

$$\max_{x\in(0,h)} |u(x; \widehat{\lambda}_k^-)| = O(|\widehat{\lambda}_k^-|^{-1/2}) \quad \text{and} \quad \max_{x\in(0,h)} |u(x; \widehat{\lambda}_k^+)| = O(|\widehat{\lambda}_k^+|^{1/2}).$$

Taking into account Corollary 1 and Theorem 3, we arrive at the following statement.

Corollary 2. *The sequences* $\{\widehat{\lambda}_k^-\}$ *and* $\{\widehat{\lambda}_{k'}^+\}$ *contain infinite subsequences* $\{\widehat{\lambda}_{k_i}^-\}$ *and* $\{\widehat{\lambda}_{k'_i}^+\}$, *respectively*, *such that any element* $\widehat{\lambda}$ *of either of these subsequences has a neighborhood* $U_{\widehat{\lambda},\delta} = (\widehat{\lambda} - \delta, \widehat{\lambda} + \delta)$ *such that*

$$(\Phi(\widehat{\lambda} - \delta; \widehat{n} + 1) - h)(\Phi(\widehat{\lambda} + \delta; \widehat{n} + 1) - h) < 0; \tag{20}$$

further, *the neighborhood* $U_{\widehat{\lambda},\delta}$ *can be selected to contain no other elements of the sequences* $\{\widehat{\lambda}_k^-\}$ *and* $\{\widehat{\lambda}_{k'}^+\}$.

The number $\widehat{n}$ is the value of n' for which $\lambda = \widehat{\lambda}$ is a solution of Eq. (16).
One also has the following assertion.

Theorem 4. *Let* $\widetilde{\lambda}_k$ *be the eigenvalues of problem* $\mathcal{P}^0$, *and let*

$$\ldots < \widetilde{\lambda}_{p+p'} < 0 \le \widetilde{\lambda}_{p+p'-1} < \ldots < \widetilde{\lambda}_p \le a_- < \widetilde{\lambda}_{p-1} < \ldots < \widetilde{\lambda}_0,$$

where $p, p' \ge 0$ *are some integers. There exists a constant* $\alpha'' > 0$ *such that for each positive parameter value* $\alpha = \alpha' < \alpha''$ *and each* $k \ge 0$ *there exists a number* j *such that the following relations hold*:

$$\lim_{\alpha'\to+0} \widehat{\lambda}_j^- = \widetilde{\lambda}_{k+p+p'} \quad \text{and} \quad \lim_{\alpha'\to+0} \widehat{\lambda}_{j'}^+ = \widetilde{\lambda}_{p+p'-1-j'} \quad \text{for} \quad j' = 0, \ldots, p+p'-1.$$

Remark 2. If $p = p' = 0$, then problem $\mathcal{P}^0$ does not have positive eigenvalues. If $p = 0$ and $p' > 0$, then problem $\mathcal{P}^0$ does not have (positive) eigenvalues $\widetilde{\lambda} > a_-$. If $p > 0$ and $p' = 0$, then problem $\mathcal{P}^0$ does not have (nonnegative) eigenvalues $\widetilde{\lambda} \in [0, a_-]$.

Let us give some remarks concerning the results stated in Theorems 2–4 and Corollary 2.

An equation similar to (16) can also be derived for the classical (linear) Sturm–Liouville problem (for the nonautonomous equation). Such an equation permits one to prove the solvability of this problem, find the eigenvalue asymptotics, and establish comparison theorems for two equations with different coefficients. Although the Sturm–Liouville problem has been thoroughly studied and these results are long known, the integral characteristic equation method is likely to be a tool that can also deliver new results in the linear theory.

Theorem 3 shows the fundamental distinction in the behavior of both eigenvalues and eigenfunctions for large $|\widehat{\lambda}|$. There arise infinitely many positive eigenvalues $\widehat{\lambda}_k^+$ for any positive value of the coefficient α. In other words, a regular perturbation of the linear differential operator defined by problem $\mathcal{P}^0$ by terms αv^3 with arbitrarily small fixed $\alpha > 0$ leads to an irregular change in the set of eigenvalues of the problem. Moreover, in the case where the function a is a constant, it can be proved that all positive eigenvalues $\widehat{\lambda}_k^+$ larger than a certain fixed number possess the following property [15]:

$$\lim_{\alpha \to +0} \widehat{\lambda}_k^+ = +\infty.$$

Corollary 2 has been stated to make explicit the presence of infinitely many eigenvalues $\widehat{\lambda}_k^-$ and $\widehat{\lambda}_k^+$ for which property (20) is satisfied. Property (20) plays a key role when attempting to construct a "nonlinear" perturbation method for some multiparameter nonlinear eigenvalue problems arising, in particular, in electrodynamics [16]. Such a method is based on using problems like $\mathcal{P}$ as "unperturbed" problems. Some features of the implementation of this concept follow from the argument presented in the proof of Theorem 4 but with substantial additional constructions. Applying the described approach to multiparameter problems of the indicated type allows one to prove the existence of those (vector) eigenvalues of multiparameter problems which are not related to solutions of the corresponding linear problems of type $\mathcal{P}^0$.

Theorem 4 is of interest because its proof is based on the straightforward use of the integral characteristic equation (16) without resorting to the classical methods of the theory of boundary value problems.

3. PROOFS

Proof of Assertion 1. Let

$$\mu_i^- = \mu(x_i' - 0) = -\infty, \quad i = 1, \ldots, n', \quad \text{and} \quad \mu_i^+ = \mu(x_i' + 0) = +\infty, \quad i = 0, \ldots, n'.$$

Since the function $w_i(\mu; \lambda)$ in (12) does not vanish, we can integrate (12). Thus, by integrating Eq. (12) on each interval I_i', we obtain

$$\begin{aligned} \int\limits_{\mu(x)}^{\mu_1^-} \frac{ds}{w_1(s;\lambda)} &= x + c_1', \quad x \in \mathrm{I}_1'; \\ -\int\limits_{\mu_{i-1}^+}^{\mu(x)} \frac{ds}{w_i(s;\lambda)} &= x + c_i', \quad x \in \mathrm{I}_i', \quad i = 2, \ldots, n'; \\ -\int\limits_{\mu_{n'}^+}^{\mu(x)} \frac{ds}{w_{n'+1}(s;\lambda)} &= x + c_{n'+1}', \quad x \in \mathrm{I}_{n'+1}'. \end{aligned} \tag{21}$$

Substituting $x = x'_0 + 0$ $(= 0 + 0)$, $x = x'_i - 0$, and $x = x'_{n'+1} - 0$ $(= h - 0)$ into the first, second, and third rows, respectively, of relations (21), we obtain

$$c'_1 = \int\limits_{\mu_0^+}^{\mu_1^-} \frac{ds}{w_1(s;\lambda)};$$

$$c'_i = -\int\limits_{\mu_{i-1}^+}^{\mu_i^-} \frac{ds}{w_i(s;\lambda)} - x'_i, \quad i = 2, \dots, n';$$

$$c'_{n'+1} = -\int\limits_{\mu_{n'}^+}^{\mu(h)} \frac{ds}{w_{n'+1}(s;\lambda)} - h.$$

Using the resulting values of c'_i, we write (21) in the form

$$\int\limits_{\mu(x)}^{\mu_1^-} \frac{ds}{w_1(s;\lambda)} = x + \int\limits_{\mu_0^+}^{\mu_1^-} \frac{ds}{w_1(s;\lambda)}, \quad x \in \mathrm{I}'_1;$$

$$-\int\limits_{\mu_{i-1}^+}^{\mu(x)} \frac{ds}{w_i(s;\lambda)} = x - \int\limits_{\mu_{i-1}^+}^{\mu_i^-} \frac{ds}{w_i(s;\lambda)} - x'_i, \quad x \in \mathrm{I}'_i, \quad i = 2, \dots, n';$$

$$-\int\limits_{\mu_{n'}^+}^{\mu(x)} \frac{ds}{w_{n'+1}(s;\lambda)} = x - \int\limits_{\mu_{n'}^+}^{\mu(h)} \frac{ds}{w_{n'+1}(s;\lambda)} - h, \quad x \in \mathrm{I}'_{n'+1}.$$

Substituting $x = x'_1 - 0$, $x = x'_{i-1} + 0$, and $x = x'_{n'} + 0$ into the first, second, and third rows, respectively, of the preceding system, we obtain

$$0 = x'_1 + \int\limits_{\mu_0^+}^{\mu_1^-} \frac{ds}{w_1(s;\lambda)};$$

$$0 = x'_{i-1} - \int\limits_{\mu_{i-1}^+}^{\mu_i^-} \frac{ds}{w_i(s;\lambda)} - x'_i, \quad i = 2, \dots, n';$$

$$0 = x'_{n'} - \int\limits_{\mu_{n'}^+}^{\mu(h)} \frac{ds}{w_{n'+1}(s;\lambda)} - h.$$

Replacing $\mu_i^\pm$ with $\pm\infty$ in the preceding formulas, we arrive at the relations

$$0 < x'_1 = \int\limits_{-\infty}^{+\infty} \frac{ds}{w_1(s;\lambda)},$$

$$0 < x'_i - x'_{i-1} = \int\limits_{-\infty}^{+\infty} \frac{ds}{w_i(s;\lambda)}, \quad i = 2, \dots, n',$$

$$0 < h - x'_{n'} = \int\limits_{\mu(h)}^{+\infty} \frac{ds}{w_{n'+1}(s;\lambda)}. \tag{22}$$

Formulas (22) give the distances between neighboring zeros of the function u. Since the left-hand sides in (22) are finite, so are the right-hand sides. This implies the convergence of all improper integrals on the right-hand sides in formulas (22). The convergence of the indicated improper integrals implies the existence of the function $\mu \equiv \mu(x)$ on each interval I'. In view of the above and the fact that the functions $u(x)$ and $u'(x)$ are bounded (see identity (6)), we conclude that the function $u(x)$, as well as $u'(x)$, is defined for all $x \in \bar{\mathrm{I}}$. In other words, we have proved that the solution $u \equiv u(x;\lambda,\alpha)$ of the Cauchy problem (1), (2) exists and is defined for all $x \in \bar{\mathrm{I}}$. The uniqueness of this solution and its continuity (and differentiability) with respect to $x \in \bar{\mathrm{I}}$, $\lambda \in \mathbb{R}$, and $\alpha \in \mathbb{R}_+$ follows from the smoothness of the right-hand side of Eq. (1) with respect to u, λ, and α [17, 18].

Summing all terms in (22), we obtain

$$x'_1 + x'_2 - x'_1 + x'_3 - x'_2 + \ldots + x'_{n'} - x'_{n'-1} + h - x'_{n'} = \sum_{i=1}^{n'} \int\limits_{-\infty}^{+\infty} \frac{ds}{w_i(s;\lambda)} + \int\limits_{\mu(h)}^{+\infty} \frac{ds}{w_{n'+1}(s;\lambda)}$$

and arrive at formula (15). The proof of Assertion 1 is complete.

Proof of Theorem 2. According to condition (3), we have $u(h) = 0$, while an analysis of the second equation in system (7), together with relation (8), implies that the function μ is decreasing. Then, as can readily be seen, $\lim\limits_{x\to h-0} \mu(x) = -\infty$. Now Eq. (16) can be derived from formula (15) under condition (11).

The fact that every solution (eigenvalue) of problem $\mathcal{P}$ satisfies Eq. (16) with some $n' = \widehat{n}$ follows from the computations in the proof of Assertion 1.

Let us prove that each solution of Eq. (16) is an eigenvalue. Let $\lambda = \widehat{\lambda}$ be a solution of Eq. (16) for $n' = \widehat{n}$. Consider the Cauchy problem (1), (2) for $\lambda = \widehat{\lambda}$. Taking into account Assertion 1, we conclude that there exists a unique solution $u \equiv u(x;\widehat{\lambda},\alpha)$ of this Cauchy problem, which is defined for all $x \in \bar{\mathrm{I}}$.

Let us use this solution to construct the functions $\theta = u^2$ and $\mu = u'/u$. It is clear that $\theta(0) = 0$ and $\lim\limits_{x\to 0+0} \mu(x) = +\infty$. At this point, we do not claim that the condition $\lim\limits_{x\to h-0} \mu(x;\widehat{\lambda}) = -\infty$ is satisfied. To be definite, set

$$\mu(h) = u'(h)/u(h) = a > -\infty.$$

Using the resulting functions θ and μ, we construct an expression similar to (16) and obtain

$$\sum_{i=1}^{\widehat{n}} T(\lambda;\mathrm{I}_i) + \int\limits_{a}^{+\infty} \frac{ds}{w_{\widehat{n}+1}(s;\widehat{\lambda})} = h. \tag{23}$$

By virtue of the uniqueness of the solution of the Cauchy problem (1), (2), the integrand in (23) coincides with the similar expression in (16). At the same time, $\lambda = \widehat{\lambda}$ satisfies Eq. (16) for $n' = \widehat{n}$. Subtracting (16) from (23), we obtain

$$\int\limits_{a}^{+\infty} \frac{ds}{w_{\widehat{n}+1}(s;\widehat{\lambda})} - T(\widehat{\lambda};\mathrm{I}_{\widehat{n}+1}) = 0. \tag{24}$$

In view of the obvious estimate

$$T(\widehat{\lambda};\mathrm{I}_{\widehat{n}+1}) > \int\limits_{a}^{+\infty} \frac{ds}{w_{\widehat{n}+1}(s;\widehat{\lambda})} > 0,$$

we conclude that relation (24) holds only if $a = -\infty$, but then $\widehat{\lambda}$ is an eigenvalue.

The formula for the zeros x_i' of the eigenfunction u follows from the calculations in the proof of Assertion 2 (see formula (22)). The proof of Theorem 2 is complete.

Proof of Assertion 2. The existence, continuity, and positivity of the functions $T(\lambda; \mathrm{I}_i)$ follow from the existence of a solution $u \equiv u(x; \lambda, \alpha)$ of the Cauchy problem (1), (2), which is continuous in the arguments $(x, \lambda, \alpha) \in \bar{\mathrm{I}} \times \mathbb{R} \times \mathbb{R}_+$, and from the positivity of the functions $w_i(s; \lambda)$ (see formulas (7), (8) and (12)–(14)).

Consider relation (6). It has been said above that (6) implies the boundedness of the functions u and u' (for a fixed λ); i.e., $\max\limits_{x\in\bar{\mathrm{I}}} u^2 = c$ and $\max\limits_{x\in\bar{\mathrm{I}}} u'^2 = c'$, where the positive constants c and c' depend on λ. In addition, relation (6) implies the existence of a constant $\delta > 0$ such that the quantity $\max\limits_{x\in\bar{\mathrm{I}}} u^2(x)$ is bounded by one and the same constant c_- for all $\lambda < -\delta$, where c_- is independent of λ. At the same time, it can readily be seen from (6) that the maximum $\max\limits_{x\in\bar{\mathrm{I}}} u^2$ increases with $\lambda > 0$.

We can extract more precise results on the behavior of the function u from identity (6). Indeed, replacing u^2 with $\lambda\bar{u}^2$ for $\lambda \in \mathbb{R}$, we obtain

$$\bar{u}'^2(x) = \lambda\left(\frac{A^2}{\lambda^2} - \frac{\alpha}{2}\bar{u}^4(x) + \bar{u}^2(x) - \frac{a(x)}{\lambda}\bar{u}^2(x) + \int\limits_0^x \frac{a'(s)}{\lambda}\bar{u}^2(s)\,ds\right).$$

The bracketed expression on the right-hand side in the resulting identity should be nonnegative. Then for sufficiently large λ we obtain

$$\bar{u}^4(x) - \frac{2}{\alpha}\bar{u}^2(x) + O(\lambda^{-1}) \le 0. \tag{25}$$

By replacing u^2 with $|\lambda|^{-1}\bar{u}^2$ for $\lambda < 0$, we conclude from (6) that

$$\frac{\bar{u}'^2(x)}{|\lambda|} = A^2 - \frac{\alpha}{2\lambda^2}\bar{u}^4(x) - \bar{u}^2(x) - \frac{a(x)}{|\lambda|}\bar{u}^2(x) + \int\limits_0^x \frac{a'(s)}{|\lambda|}\bar{u}^2(s)\,ds.$$

The expression on the right-hand side in the resulting formula must be nonnegative. Then for sufficiently large $|\lambda|$ we have

$$A^2 + O(|\lambda|^{-1}) - \bar{u}^2(x) \ge 0. \tag{26}$$

The maximum positive solutions $\bar{u}^2$ of inequalities (25) and (26) give the following asymptotics:

$$\max_{x\in\bar{\mathrm{I}}} u^2 = \begin{cases} A^2|\lambda|^{-1} + O(\lambda^{-2}) & \text{as } \lambda \to -\infty, \\ 2\alpha^{-1}\lambda + O(1) & \text{as } \lambda \to +\infty. \end{cases} \tag{27}$$

The asymptotic formulas (27) show that the properties of the problem under study are essentially different for negative and positive values of λ.

Inequalities (17) readily follow from the estimates

$$\mu^2 - \lambda < \mu^2 + a - \lambda + \alpha\theta < \mu^2 + a_+ - \lambda + \alpha c_-,$$

in which it is assumed that $\lambda < -\delta$. Since the left inequality in this two-sided inequality holds for all $\lambda < 0$, it follows that the right inequality in (17) is satisfied for all $\lambda < 0$ as well.

To derive the estimate (18), we need subtler arguments. Consider relation (8) as a "quadratic" equation for θ. Since $\theta = v^2 \ge 0$, we find from (8) that

$$\theta = -\frac{1}{\alpha}(\mu^2 + a - \lambda) + \frac{1}{\alpha}\left((\mu^2 + a - \lambda)^2 + 2\alpha A^2 + 2\alpha\int\limits_0^x a'(s)\theta(s)\,ds\right)^{1/2}. \tag{28}$$

Substituting the right-hand side of relation (28) for θ into the right-hand side of Eq. (12), we obtain

$$\mu' = -w_i(\mu; \lambda),$$

where

$$w_i(\mu; \lambda) \equiv \left((\mu^2 + a(x) - \lambda)^2 + 2\alpha A^2 + 2\alpha \int\limits_0^x a'(s)\theta(s)\, ds\right)^{1/2}, \tag{29}$$

$x \equiv g_i(\mu)$, and $x \in \mathrm{I}_i$, $i = 1, \ldots, n' + 1$. It readily follows from identity (29) that $\min_{\mu,\lambda} w(\mu; \lambda) \geq \sqrt{2\alpha}A > 0$. Thus, we have

$$T(\lambda; \mathrm{I}_i) = \int\limits_{-\infty}^{+\infty} \frac{ds}{w_i(s;\lambda)} = \int\limits_{-\infty}^{0} \frac{ds}{w_i(s;\lambda)} + \int\limits_{0}^{+\infty} \frac{ds}{w_i(s;\lambda)} = T_i^{(1)}(\lambda) + T_i^{(2)}(\lambda), \tag{30}$$

where the function w_i is defined by relation (29).

It follows from formulas (27) that for sufficiently large $\lambda > 0$ we have the inequality

$$0 \leq \int\limits_0^x a'(s)\theta(s)\, ds \leq B\lambda + O(1) = A'^2,$$

where $B > 0$ is some constant.

Thus, we obtain

$$\int\limits_0^{+\infty} \frac{ds}{w_i^+(s;\lambda)} \leq T_i^{(2)}(\lambda) \leq \int\limits_0^{+\infty} \frac{ds}{w_i^-(s;\lambda)}, \tag{31}$$

where $w_i^+(s;\lambda) = ((s^2 + a(x) - \lambda)^2 + 2\alpha A'^2)^{1/2}$, $w_i^-(s;\lambda) = ((s^2 + a(x) - \lambda)^2 + 2\alpha A^2)^{1/2}$, and $x \equiv g_i(s)$. The estimates for $T_i^{(1)}(\lambda)$ can be obtained in a similar manner.

Using the elementary inequalities

$$\frac{1}{|a| + b} \leq \frac{1}{\sqrt{a^2 + b^2}} \leq \frac{\sqrt{2}}{|a| + b},$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$, we obtain the following estimates for the left- and right-hand sides of inequality (31):

$$\int\limits_0^{+\infty} \frac{ds}{\hat{w}_i^+(s;\lambda)} \leq \int\limits_0^{+\infty} \frac{ds}{w_i^+(s;\lambda)} \leq T_i^{(2)}(\lambda) \leq \int\limits_0^{+\infty} \frac{ds}{w_i^-(s;\lambda)} \leq \int\limits_0^{+\infty} \frac{\sqrt{2}\, ds}{\hat{w}_i^-(s;\lambda)}, \tag{32}$$

where $\hat{w}_i^+(s;\lambda) = |s^2 + a(x) - \lambda| + A'\sqrt{2\alpha}$ and $\hat{w}_i^-(s;\lambda) = |s^2 + a(x) - \lambda| + A\sqrt{2\alpha}$.

Then for sufficiently large λ we have

$$\int\limits_0^{+\infty} \frac{ds}{\hat{w}_i^+(s;\lambda)} = \int\limits_0^{\sqrt{\lambda - a_+}} \frac{ds}{\hat{w}_i^+(s;\lambda)} + \int\limits_{\sqrt{\lambda - a_+}}^{\sqrt{\lambda - a_-}} \frac{ds}{\hat{w}_i^+(s;\lambda)} + \int\limits_{\sqrt{\lambda - a_-}}^{+\infty} \frac{ds}{\hat{w}_i^+(s;\lambda)}$$

$$= \int\limits_0^{\sqrt{\lambda - a_+}} \frac{ds}{\lambda - a(x) + A'\sqrt{2\alpha} - s^2} + \int\limits_{\sqrt{\lambda - a_+}}^{\sqrt{\lambda - a_-}} \frac{ds}{\hat{w}_i^+(s;\lambda)} + \int\limits_{\sqrt{\lambda - a_-}}^{+\infty} \frac{ds}{s^2 - (\lambda - a(x) - A'\sqrt{2\alpha})}$$

$$\geq \int_0^{\sqrt{\lambda-a_+}} \frac{ds}{\lambda - a_- + A'\sqrt{2\alpha} - s^2} + \int_{\sqrt{\lambda-a_+}}^{\sqrt{\lambda-a_-}} \frac{ds}{a_+ - a_- + A'\sqrt{2\alpha}} + \int_{\sqrt{\lambda-a_-}}^{+\infty} \frac{ds}{s^2 - (\lambda - a_+ - A'\sqrt{2\alpha})}$$

$$= T'_{i,1} + T'_{i,2} + T'_{i,3}.$$

By successively calculating the resulting integrals on the right-hand side in this inequality, we arrive at the formulas

$$T'_{i,1} = \frac{\ln\lambda}{4\sqrt{\lambda}} + O(\lambda^{-1/2}), \quad T'_{i,2} = O(\lambda^{-1}), \quad T'_{i,3} = \frac{\ln\lambda}{4\sqrt{\lambda}} + O(\lambda^{-1/2}).$$

In a similar way, for the last integral in inequalities (32) we obtain

$$\int_0^{+\infty} \frac{\sqrt{2}\,ds}{\hat{w}_i^-(s;\lambda)} = \int_0^{\sqrt{\lambda-a_+}} \frac{\sqrt{2}\,ds}{\hat{w}_i^-(s;\lambda)} + \int_{\sqrt{\lambda-a_+}}^{\sqrt{\lambda-a_-}} \frac{\sqrt{2}\,ds}{\hat{w}_i^-(s;\lambda)} + \int_{\sqrt{\lambda-a_-}}^{+\infty} \frac{\sqrt{2}\,ds}{\hat{w}_i^-(s;\lambda)}$$

$$= \int_0^{\sqrt{\lambda-a_+}} \frac{\sqrt{2}\,ds}{\lambda - a(x) + A\sqrt{2\alpha} - s^2} + \int_{\sqrt{\lambda-a_+}}^{\sqrt{\lambda-a_-}} \frac{\sqrt{2}\,ds}{\hat{w}_i^-(s;\lambda)} + \int_{\sqrt{\lambda-a_-}}^{+\infty} \frac{\sqrt{2}\,ds}{s^2 - (\lambda - a(x) - A\sqrt{2\alpha})}$$

$$\leq \int_0^{\sqrt{\lambda-a_+}} \frac{\sqrt{2}\,ds}{\lambda - a_+ + A\sqrt{2\alpha} - s^2} + \int_{\sqrt{\lambda-a_+}}^{\sqrt{\lambda-a_-}} \frac{\sqrt{2}\,ds}{A\sqrt{2\alpha}} + \int_{\sqrt{\lambda-a_-}}^{+\infty} \frac{\sqrt{2}\,ds}{s^2 - (\lambda - a_- - A\sqrt{2\alpha})} = T''_{i,1} + T''_{i,2} + T''_{i,3}.$$

By successively calculating the resulting integrals on the right-hand side in this inequality for large λ, we arrive at the formulas

$$T''_{i,1} = \frac{\ln\lambda}{\sqrt{2\lambda}} + O(\lambda^{-1/2}), \quad T''_{i,2} = O(\lambda^{-1/2}), \quad T''_{i,3} = \frac{\ln\lambda}{\sqrt{2\lambda}} + O(\lambda^{-1/2}).$$

Taking into account the resulting estimates and the estimates (32), we have the inequalities

$$\frac{\ln\lambda}{2\sqrt{\lambda}} + O(\lambda^{-1/2}) \leq \int_0^{+\infty} \frac{ds}{w_i^+(s;\lambda)} \leq T_i^{(2)}(\lambda) \leq \int_0^{+\infty} \frac{ds}{w_i^-(s;\lambda)} \leq \frac{\sqrt{2}\ln\lambda}{\sqrt{\lambda}} + O(\lambda^{-1/2}),$$

which are satisfied for large $\lambda > 0$. In view of formula (30) and the last inequality, we obtain the estimate (18). The proof of Assertion 2 is complete.

Proof of Theorem 3. It follows from Assertion 2 that the function $\Phi(\lambda; n'{+}1)$ (see formula (16)) is positive. Hence, whatever $h > 0$, there exists a number $n' = n'_0 \geq 0$ such that $\max\limits_{\lambda\in\mathbb{R}} \Phi(\lambda; n') > h$. It follows from Corollary 1 that

$$\lim_{\lambda\to\pm\infty} \Phi(\lambda; n'+1) = 0.$$

Therefore, for each $n' = n'_0, n'_0{+}1, \ldots$ there exist values $\lambda = \lambda^+_{n'}$ and $\lambda = \lambda^+_{n'}$ such that $\Phi(\lambda^\pm_{n'}; n') < h$. Since $\Phi(\lambda; n')$ is a continuous function of the variable λ (see Assertion 2), it follows that for each $n' = n'_0, n'_0 + 1 \ldots$ there exists at least one negative value $\lambda = \widehat{\lambda}^-_{n'}$ and one positive value $\lambda = \widehat{\lambda}^+_{n'}$ such that $\Phi(\widehat{\lambda}^\pm_{n'}; n') = h$. By Theorem 2, $\lambda = \lambda^+_{n'}$ and $\lambda = \lambda^+_{n'}$ are eigenvalues of problem $\mathcal{P}$. It is obvious that there exist infinitely many negative as well as positive eigenvalues. Now it suffices to shift the numbering so that it starts from zero rather than n'_0.

The asymptotic estimates for the negative and positive eigenvalues follow from formulas (19).

The estimates for the maxima of the eigenfunctions for sufficiently large $|\widehat{\lambda}_k^-|$ and $\widehat{\lambda}_k^+$ have been derived in the proof of Assertion 2 (see formula (27)). The proof of Theorem 3 is complete.

Proof of Theorem 4. Consider Eq. (16). Using arguments almost identical to the proof of Theorem 2, one can readily show that Eq. (16) for $\alpha = 0$ is equivalent to problem $\mathcal{P}^0$. Thus, consider the equation

$$\widetilde{\Phi}(\lambda; n'+1) \equiv \sum_{i=1}^{n'+1} \widetilde{T}(\lambda; \widetilde{\mathrm{I}}_i) = h, \tag{33}$$

where

$$\widetilde{T}(\lambda; \widetilde{\mathrm{I}}_i) = \int_{-\infty}^{+\infty} \frac{ds}{\widetilde{w}_i(s;\lambda)},$$

while the function $\widetilde{w}_i(\mu;\lambda) \equiv \mu^2 + a(x) - \lambda$ is defined on each interval $\widetilde{\mathrm{I}}'_i$, $i = 1, \ldots, n'+1$, by means of the mapping $x \equiv \widetilde{g}_i(\mu)$. In the above formulas, the intervals $\widetilde{\mathrm{I}}'_i$ are defined by analogy with the intervals I'_i, and the functions $\widetilde{g}_i(\mu)$, by analogy with the functions $g_i(\mu)$ introduced in Sec. 2.2 (see formulas (9)–(14)) for the linear equation (4).

Let $\lambda \in (-\infty, a_-)$. In this case, even with no allowance for relation (8), it is obvious that the function $\widetilde{w}_i(\mu;\lambda)$ does not vanish. According to Theorem 1, all eigenvalues $\widetilde{\lambda}_k$ of problem $\mathcal{P}^0$ are of multiplicity 1. By virtue of the equivalence of Eq. (33) to problem $\mathcal{P}^0$, the solutions $\lambda = \widetilde{\lambda}$ of Eq. (33) are also simple roots of this equation.

Consider the relation

$$\Phi(\lambda; n'+1) - \widetilde{\Phi}(\lambda; n'+1) = h - \widetilde{\Phi}(\lambda; n'+1). \tag{34}$$

The zeros of the right-hand side of this relation are the eigenvalues $\widetilde{\lambda}$ of problem $\mathcal{P}^0$. Moreover, by virtue of the above argument, each eigenvalue $\widetilde{\lambda}$ can be included in some neighborhood (interval) $U_{\widetilde{\lambda},\delta}$ such that the right-hand side of the expression (34) takes opposite signs at the endpoints of the interval $\bar{U}_{\widetilde{\lambda},\delta}$.

Consider the left-hand side of the expression (34), which we will write as

$$\begin{aligned}\Phi(\lambda; n'+1) - \widetilde{\Phi}(\lambda; n'+1) &= \sum_{i=1}^{n'+1} T(\lambda; \mathrm{I}_i) - \sum_{i=1}^{n'+1} \widetilde{T}(\lambda; \mathrm{I}_i) = \sum_{i=1}^{n'+1} \int_{-\infty}^{+\infty} \left(\frac{1}{w_i(s;\lambda)} - \frac{1}{\widetilde{w}_i(s;\lambda)} \right) ds \\ &= \sum_{i=1}^{n'+1} \int_{-\infty}^{+\infty} \frac{a(\widetilde{g}_i(s)) - a(g_i(s)) - \alpha\theta(g_i(s;\alpha))}{w_i(s;\lambda)\widetilde{w}_i(s;\lambda)}\, ds.\end{aligned} \tag{35}$$

It follows from formula (27) that the functions u are bounded for λ bounded above, and hence the functions u are bounded for $\lambda \in (-\infty, a_-)$. It follows that the smaller $\alpha\,(>0)$, the smaller the difference between the functions $w_i(s;\lambda)$ and $\widetilde{w}_i(s;\lambda)$ is for all such λ. Hence, for a sufficiently small $\alpha\,(>0)$, the numerators of the above integrals will be small in absolute value; moreover, the denominators of these integrals are such that the integrals converge. Therefore, each term on the right-hand side in relation (35) can be made arbitrarily small given a sufficiently small α. Considering the above reasoning, we conclude that there exists an $\alpha\,(>0)$ such that the neighborhood $U_{\widetilde{\lambda},\delta}$ of each eigenvalue $\widetilde{\lambda}$ of problem $\mathcal{P}_0$ will contain at least one eigenvalue $\widehat{\lambda}$ of problem $\mathcal{P}$; moreover, $\widehat{\lambda} \in (-\infty, a_-)$.

Since the denominators of the integrands in (35) increase with $|\lambda|$, it follows that there exists a common α (>0) for all $\lambda < \lambda_0$, where λ_0 is a given negative number (possibly, sufficiently large in absolute value).

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