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ORDINARY DIFFERENTIAL EQUATIONS

Lebesgue Sets of Izobov Exponents of Linear Differential Systems. I

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Abstract—We give a complete description of the Lebesgue sets of upper Izobov σ-exponents of linear differential systems continuously depending on a parameter varying in a metric space. We prove the simultaneous attainability of the upper Izobov σ-exponents by the Lyapunov exponents and their upper semicontinuity as functions of the perturbation exponent $-\sigma$.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

For a given positive integer $n \geq 2$, by $\widetilde{\mathcal{M}}^n$ we denote the space of linear systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

with piecewise continuous matrix-valued functions A equipped with the operations of addition and multiplication by a real number, natural for functions. By $\mathcal{M}^n$ we denote the subspace of systems with coefficients bounded on the half-line $\mathbb{R}_+$. We endow the space $\widetilde{\mathcal{M}}^n$ with *uniform* and *compact-open* topologies, defined, respectively, by the metrics

$$\rho_U(A, B) = \sup_{t \in \mathbb{R}_+} \min\{|A(t) - B(t)|, 1\}, \quad \rho_C(A, B) = \sup_{t \in \mathbb{R}_+} \min\{|A(t) - B(t)|, e^{-t}\}, \quad A, B \in \widetilde{\mathcal{M}}^n,$$

where we have denoted $|Y| = \sup_{|x|=1} |Yx|$, $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$, and $x = (x_1, \ldots, x_n)^{\mathrm{T}}$. We agree to denote the resulting topological spaces by $\widetilde{\mathcal{M}}^n_U$ and $\widetilde{\mathcal{M}}^n_C$, similar notation being also used for their subspaces. Note that the uniform topology is also defined on $\mathcal{M}^n$ by the norm

$$\|A\| = \sup_{t \in \mathbb{R}_+} |A(t)|, \quad A \in \mathcal{M}^n.$$

In what follows, we identify system (1) and the matrix-valued function defining it and hence write $A \in \widetilde{\mathcal{M}}^n$ or $A \in \mathcal{M}^n$.

Definition 1. The *characteristic exponent* of a function $f : \mathbb{R}_+ \to \mathbb{R}^m$ ($m \in \mathbb{N}$) is the quantity (we take $\ln 0 = -\infty$)

$$\lambda[f] = \varlimsup_{t \to +\infty} \ln |f(t)|^{1/t}.$$

Definition 2. The *Lyapunov exponents* of a system $A \in \widetilde{\mathcal{M}}^n$ are the quantities [1]

$$\lambda_i(A) = \inf_{L \in G_i(S(A))} \sup_{x \in L} \lambda[x], \quad i = 1, \ldots, n,$$

where $S(A)$ is the solution space of the system A and $G_i(V)$ is the set of i-dimensional subspaces of the vector space V.

In our notation, the Lyapunov exponents are numbered, unlike [1], in nondescending order.

Perron [2] (see also [3, Sec. 1.4]) constructed an example of a system $A \in \mathcal{M}^n$ whose Lyapunov exponents are not invariant with respect to exponentially decaying perturbations of its coefficients. On the other hand, the Lyapunov exponents of any system in $\mathcal{M}^n$ remain invariant with respect to perturbations decaying more rapidly than some exponential (depending on the system) [4, 5]. Naturally, this gives rise to problems of deriving estimates and calculating precise mobility boundaries for the Lyapunov exponents under exponentially decaying perturbations, as well as describing the properties of such boundaries. There are quite a few papers dealing with these problems (see, e.g., [4–16]). It is the study of this class of perturbations that seems to be especially important owing to its close relation to the Lyapunov problem of stability by the first approximation [12].

The present paper investigates the mobility boundaries of Lyapunov exponents under exponentially decaying perturbations from the viewpoint of descriptive function theory, with no requirements imposed, unlike the above-cited publications, on the coefficients of the systems under consideration to be bounded on the half-line.

2. STATEMENT OF RESULTS

Definition 3. For $i = 1, \ldots, n$ and each $\sigma > 0$ we define the *upper Izobov σ-exponents* [11] of a system $A \in \widetilde{\mathcal{M}}^n$ by the relation

$$\nabla_{\sigma,i}(A) = \sup_{Q \in \hat{\mathcal{E}}_\sigma} \lambda_i(A + Q), \quad i = 1, \ldots, n, \tag{2}$$

where $\hat{\mathcal{E}}_\sigma = \{Q \in \mathcal{M}^n : \lambda[Q] \le -\sigma\}$.

Since we do not assume the coefficients of the systems under consideration to be bounded on the half-line, it follows that the values of the above-defined quantities belong to the extended real line $\overline{\mathbb{R}} \equiv \mathbb{R} \sqcup \{-\infty, +\infty\}$, which we take to be equipped with the standard order and the associated order topology. Notice that the resulting topological space is homeomorphic to the interval $[-1, 1]$.

The higher upper Izobov σ-exponent $\nabla_{\sigma,n}$ can be used to study stability by the first approximation, as shown by the following assertion.

Theorem 1. *Assume that the condition*

$$\nabla_{\sigma,n}(A) < -\sigma/(m-1)$$

is satisfied for a system $A \in \widetilde{\mathcal{M}}^n$ and numbers $m > 1$ and $\sigma > 0$ and a continuous function $\Psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ has a nonpositive characteristic exponent: $\lambda[\Psi] \le 0$. Then for each number $\alpha > \nabla_{\sigma,n}(A)$ there exist positive constants $\delta = \delta(A, m, \sigma, \Psi, \alpha)$ and $C = C(A, m, \sigma, \Psi, \alpha)$ such that, for any domain $U \subset \mathbb{R}^n$ containing the origin and any continuous function $f\colon \mathbb{R}_+ \times U \to \mathbb{R}^n$ satisfying the condition

$$|f(t,x)| \le \Psi(t)|x|^m, \quad (t,x) \in \mathbb{R}_+ \times U,$$

every solution of the system

$$\dot{x} = A(t)x + f(t,x), \quad x \in U, \quad t \in \mathbb{R}_+, \tag{3}$$

with the initial condition $|x(0)| < \delta$ satisfies the estimate

$$|x(t)| \le C|x(0)|e^{\alpha t}, \quad t \in \mathbb{R}_+. \tag{4}$$

In particular, the zero solution of system (3) *is exponentially stable.*

Izobov [11] established the following formula for the exponent $\nabla_{\sigma,n}$ in the case where all coefficients of system (1) are bounded:

$$\nabla_{\sigma,n}(A) = \varlimsup_{k \to \infty} \frac{1}{k}\xi_k(\sigma), \quad \xi_k(\sigma) = \max_{i<k}\{\ln|X_A(k,i)| + \xi_i(\sigma) - \sigma i\}, \quad k \in \mathbb{N}, \quad \xi_0(\sigma) \equiv 0, \tag{5}$$

where $X_A(\cdot,\cdot)$ is the Cauchy operator of the system A. It follows from the proof of this formula that the least upper bound is attained in relation (2) whenever $A \in \mathcal{M}^n$ and $i = n$. It turns out that this is also true without the assumption about the boundedness of the coefficients of system (1), and moreover, the least upper bounds are attained in relation (2) simultaneously for all $i = 1, \dots, n$; namely, the following assertion holds.

Theorem 2. *For any system $A \in \widetilde{\mathcal{M}}^n$ and number $\sigma > 0$, there exists a matrix-valued function $Q \in \hat{\mathcal{E}}_\sigma$ such that*

$$\lambda_i(A+Q) = \nabla_{\sigma,i}(A), \quad i = 1, \dots, n.$$

Remark 1. A similar assertion was established in [17] for the upper mobility boundaries of Lyapunov exponents under uniformly small perturbations.

The set of functions $\{\sigma \mapsto \nabla_{\sigma,n}(A) : A \in \mathcal{M}^n\}$ was completely described in [14]: it consists of all bounded concave functions that are constant for all σ starting from some (function-specific) value. In particular, all these functions are continuous. At the same time, for any $i = 1, \dots, n-1$ and $\sigma_0 > 0$ there exists a system $A \in \mathcal{M}^n$ such that the function $\sigma \mapsto \nabla_{\sigma,i}(A)$ is not lower semicontinuous at the point σ_0 [15]. The natural question arises as to whether this function is upper semicontinuous for each system $A \in \widetilde{\mathcal{M}}^n$.

Theorem 3. *For any $i = 1, \dots, n$ and $A \in \widetilde{\mathcal{M}}^n$, the function $(0, +\infty) \to \overline{\mathbb{R}}$ defined by the rule $\sigma \mapsto \nabla_{\sigma,i}(A)$ is upper semicontinuous.*

Remark 2. The present author does not know whether the function $\sigma \mapsto \nabla_{\sigma,n}(A)$ is lower semicontinuous at each point $A \in \widetilde{\mathcal{M}}^n \setminus \mathcal{M}^n$. (For $A \in \mathcal{M}^n$, the continuity of this function was established in [11].)

Let M be a metric space. Consider a family

$$\dot{x} = A(t,\mu)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{6}$$

of linear differential systems depending on a parameter $\mu \in M$ such that for each μ system (6) has piecewise continuous coefficients. Every functional $\Lambda : \widetilde{\mathcal{M}}^n \to \overline{\mathbb{R}}$ defines a function $\Lambda(\cdot\,; A) : M \to \overline{\mathbb{R}}$ of the parameter μ; this function takes each $\mu \in M$ to the value of this functional on system (6). Below we describe the Lebesgue sets of such functions defined by the upper Izobov σ-exponents.

We denote the set of all continuous mappings $A\colon \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ by $\mathcal{C}^n(M)$ and the set of bounded mappings $A \in \mathcal{C}^n(M)$ by $\mathcal{B}^n(M)$. Note that each mapping $A \in \mathcal{C}^n(M)$ corresponds to a family of systems (6) with continuous coefficients that continuously depend on the parameter μ in the sense of the compact-open topology on the space $\widetilde{\mathcal{M}}^n$. In what follows, $\mathcal{U}^n(M)$ stands for the set of mappings $A \in \mathcal{C}^n(M)$ continuous in μ uniformly with respect to $t \in \mathbb{R}_+$, i.e., satisfying the condition

$$\lim_{\nu\to\mu} \|A(\cdot,\nu) - A(\cdot,\mu)\| \equiv \lim_{\nu\to\mu} \sup_{t\in\mathbb{R}_+} |A(t,\nu) - A(t,\mu)| = 0, \quad \mu \in M.$$

In other words, each mapping $A \in \mathcal{U}^n(M)$ corresponds to a family of systems (6) with continuous coefficients that continuously depend on the parameter μ in the sense of the uniform topology on the space $\widetilde{\mathcal{M}}^n$. Finally, we write $\mathcal{BU}^n(M) = \mathcal{B}^n(M) \cap \mathcal{U}^n(M)$.

Let X be a metric space. Recall the definition [18, Sec. 41.1] of Lebesgue sets of a function $f : X \to \overline{\mathbb{R}}$. For each number $r \in \mathbb{R}$, the Lebesgue sets $[f > r]$, $[f \geq r]$, and $[f = r]$ of the function f are the preimages of the intervals $(r, +\infty]$ and $[r, +\infty]$ and the point $\{r\}$, respectively. By $G_\delta(X)$, $G_{\delta\sigma}(X)$, and $F_{\sigma\delta}(X)$ we denote the sets of subsets of X of the types G_δ, $G_{\delta\sigma}$, and $F_{\sigma\delta}$, respectively [18, Sec. 32].

The Lebesgue sets of the upper Izobov σ-exponents of families (6) continuously depending on the parameter in the sense of the compact-open or uniform topology on the space $\widetilde{\mathcal{M}}^n$ are completely described by the following assertion.

Theorem 4. *The following relations hold for any metric space M and numbers $i = 1, \dots, n$, $\sigma > 0$, and $r \in \mathbb{R}$:*

1. $\{[\nabla_{\sigma,i}(\cdot\,; A) \geq r] : A \in \mathcal{C}^n(M)\} = \{[\nabla_{\sigma,i}(\cdot\,; A) \geq r] : A \in \mathcal{BU}^n(M)\} = G_\delta(M)$.
2. $\{[\nabla_{\sigma,i}(\cdot\,; A) > r] : A \in \mathcal{C}^n(M)\} = \{[\nabla_{i,\sigma}(\cdot\,; A) > r] : A \in \mathcal{BU}^n(M)\} = G_{\delta\sigma}(M)$.
3. $\{[\nabla_{\sigma,i}(\cdot\,; A) = r] : A \in \mathcal{C}^n(M)\} = \{[\nabla_{\sigma,i}(\cdot\,; A) = r] : A \in \mathcal{BU}^n(M)\} = F_{\sigma\delta}(M)$.

Remark 3. A similar description of Lebesgue sets for the Lyapunov exponents was obtained in [19].

Let us denote the sets of points of upper and lower semicontinuity [19] of a function $f : M \to \overline{\mathbb{R}}$ by $US(f)$ and $LS(f)$, respectively. Further, let $\mathfrak{F}(M)$ be the set of $F_{\sigma\delta}$-subsets of M containing all isolated points of M, and let $\mathfrak{G}(M)$ be the set of all G_δ-subsets dense in M.

Vetokhin [16] used formula (5) to prove that, for any $\sigma > 0$ and any family $A \in \mathcal{C}^n(M)$ such that $A(\cdot, \mu) \in \mathcal{M}^n$ for each $\mu \in M$, the function $\nabla_{\sigma,n}(\cdot\,; A)$ belongs to Baire class 2 and, assuming that M is complete, the set $US(\nabla_{\sigma,n}(\cdot\,; A))$ contains a G_δ-set dense in M. The following assertion is a strengthening of this result.

Corollary. *For any metric space M, any numbers $i = 1, \ldots, n$ and $\sigma > 0$, and any $A \in \mathcal{C}^n(M)$, the function $\nabla_{\sigma,i}(\cdot\,; A)$ belongs to Baire class 2. Moreover,*

$$\{LS(\nabla_{\sigma,i}(\cdot\,; A)) : A \in \mathcal{C}^n(M)\} = \{LS(\nabla_{\sigma,i}(\cdot\,; A)) : A \in \mathcal{BU}^n(M)\} = \mathfrak{F}(M)$$

and

$$G_\delta(M) \supset \{US(\nabla_{\sigma,i}(\cdot\,; A)) : A \in \mathcal{C}^n(M)\} \supset \{US(\nabla_{\sigma,i}(\cdot\,; A)) : A \in \mathcal{BU}^n(M)\} \supset \mathfrak{G}(M),$$

the last two inclusions being equalities if M is complete.

3. PROOFS OF THE ASSERTIONS

Proof of Theorem 1. Without loss of generality, we can assume that the domain U is a ball of radius $\rho \in (0, 1)$ centered at the origin and the number $\alpha > \nabla_{\sigma,n}(A)$ is such that $(m-1)\alpha + \sigma < 0$. The substitution $x(t) = e^{\alpha t} y(t)$ reduces the original system to the system

$$\dot{y} = \tilde{A}(t)y + g(t, y), \quad \tilde{A}(t) = A(t) - \alpha E_n, \quad g(t, y) = e^{-\alpha t} f(t, e^{\alpha t} y), \quad (t, y) \in V, \tag{7}$$

where $V = \{(t, y) : t \in \mathbb{R}_+, e^{\alpha t} y \in U\}$ and E_n is the identity $n \times n$ matrix. In what follows, we consider the restriction of system (7) to the set $\tilde{V} \equiv \mathbb{R}_+ \times U \subset V$. For each point $(t, y) \in \tilde{V}$, we have the chain of inequalities

$$|g(t, y)| \le e^{\alpha(m-1)t}\Psi(t)|y|^m \le e^{((m-1)\alpha+\sigma)t}\Psi(t)e^{-\sigma t}|y| \le K e^{-\sigma t}|y|,$$

where $K = \sup\limits_{t \in \mathbb{R}_+} e^{((m-1)\alpha+\sigma)t}\Psi(t) < \infty$, because $\lambda[\Psi] \le 0$.

Let $\mathcal{Q}$ be the set of continuous matrix-valued functions $Q \in \mathcal{M}^n$ satisfying the condition

$$|Q(t)| \le K e^{-\sigma t} \tag{8}$$

for all $t \in \mathbb{R}_+$. Note that if $\nabla_{\sigma,n}(A) = -\infty$, then $\nabla_{\sigma,n}(\tilde{A}) = \nabla_{\sigma,n}(A) = -\infty$; otherwise, $\nabla_{\sigma,n}(\tilde{A}) = \nabla_{\sigma,n}(A) - \alpha$. Since $\mathcal{Q} \subset \hat{\mathcal{E}}_\sigma$, we have the inequality

$$\sup_{Q \in \mathcal{Q}} \lambda_n(\tilde{A} + Q) \le \nabla_{\sigma,n}(\tilde{A}) < 0.$$

Applying [17, Lemma 3] to the function $f(t) = K e^{-\sigma t}$, $t \in \mathbb{R}_+$, we obtain

$$\inf_{k \in \mathbb{N}} \sup_{Q \in \mathcal{Q}} \sup_{t \ge k} \frac{1}{t} \ln |X_{\tilde{A}+Q}(t, 0)| < 0.$$

Therefore, there exists a $k \in \mathbb{N}$ such that the inequality $t^{-1} \ln |X_{\tilde{A}+Q}(t, 0)| < 0$, or, equivalently, $|X_{\tilde{A}+Q}(t, 0)| < 1$, is satisfied for all $Q \in \mathcal{Q}$ and $t \ge k$. For $t \in [0, k]$, by virtue of the well-known estimate [3, Eq. (3.3)], we have

$$|X_{\tilde{A}+Q}(t, 0)| \le \exp((\sup_{0 \le \tau \le k} |\tilde{A}(\tau)| + K)k) \equiv C.$$

Then $|X_{\tilde{A}+Q}(t,0)| \le C$ for all $t \in \mathbb{R}_+$. Thus, each solution y of any linear system $\tilde{A}+Q$, where $Q \in \mathcal{Q}$, satisfies the estimate

$$|y(t)| \le C|y(0)| \tag{9}$$

for all $t \ge 0$.

Set $\delta = \rho(2C)^{-1}$. Let $y : [0,T) \to U$, $T \in (0,+\infty]$, be a nonextendable nontrivial solution of system (7) such that $|y(0)| < \delta$. Then, by the linear inclusion principle [20, Sec. 12.3], the function y is also a solution of the linear system $\tilde{A}+Q$, where the matrix-valued function $Q : [0,T) \to \mathbb{R}^{n\times n}$ is defined by the equations

$$q_{ij}(t) = \frac{y_j(t)}{|y(t)|^2}\, g_i(t,y(t)), \quad i,j = 1,\dots,n, \quad t \in [0,T),$$

is continuous, and satisfies condition (8) for all $t \in [0,T)$. Assume that $T < \infty$. Since the function g is bounded on the set $\tilde{V}$, it follows that the function y satisfies the Lipschitz condition. Therefore, there exists a limit $\lim\limits_{t\to T-0} y(t)$, and it is nonzero by the extendability theorem [21, Theorem 23]. Hence there also exists a limit $\lim\limits_{t\to T-0} Q(t)$. Let us continuously extend the matrix-valued function Q to the entire half-line $\mathbb{R}_+$ so that condition (8) be satisfied for all $t \in \mathbb{R}_+$ and retain the old name for the extended function. Then $Q \in \mathcal{Q}$ and (9) is satisfied for all $t \in [0,T)$; it follows that the graph of the function y does not leave the compact set $[0,T] \times \{\xi \in \mathbb{R}^n : |\xi| \le \rho/2\}$. This contradicts the above-mentioned extendability theorem for system (7). Hence $T = \infty$, and inequality (9) is satisfied for all $t \in \mathbb{R}_+$.

Returning to the original system (3), we conclude that each of its solutions x with the initial condition $|x(0)| < \delta$ can be extended to the entire half-line $\mathbb{R}_+$ and satisfies the estimate (4). The proof of the theorem is complete.

For any $i = 1,\dots,n$ and $m,q \in \mathbb{N}$, let us define a function $\varphi_i^{mq} : \widetilde{\mathcal{M}}^n \to \mathbb{R}$ by the formula

$$\varphi_i^{mq}(A) = \inf_{L\in G_i(\mathbb{R}^n)} \max_{t\in[m,m+q]} \frac{1}{t} \ln |X_A(t,0)|_L|, \quad A \in \widetilde{\mathcal{M}}^n, \tag{10}$$

where $X_A(\cdot,\cdot)|_L$ is the restriction of the Cauchy operator of system (1) to the subspace $L \subset \mathbb{R}^n$.

Millionshchikov [22] established the following assertion, which we will substantially use in the sequel.

Lemma 1. *The functions* $\varphi_i^{mq}\colon \widetilde{\mathcal{M}}_C^n \to \mathbb{R}$, $m,q \in \mathbb{N}$, $i = 1,\dots,n$, *are continuous, and the relations*

$$\lambda_i(A) = \inf_{m\in\mathbb{N}} \sup_{q\in\mathbb{N}} \varphi_i^{mq}(A), \quad i = 1,\dots,n, \tag{11}$$

hold for each system $A \in \widetilde{\mathcal{M}}^n$.

We introduce the following notation ($\sigma, C \in \mathbb{R}_+^* \equiv \mathbb{R}_+ \setminus \{0\}$):

$$\mathcal{E}_{\sigma,C} = \{Q \in \mathcal{M}^n : |Q(t)| \le Ce^{-\sigma t}, \ \ t \in \mathbb{R}_+\}, \quad \mathcal{E}_\sigma = \bigcup_{C>0} \mathcal{E}_{\sigma,C}.$$

Lemma 2. *For any system* $A \in \widetilde{\mathcal{M}}^n$ *and number* $\sigma > 0$, *there exists a system* $B \in \hat{\mathcal{E}}_\sigma$ *such that*

$$\lambda_i(A+B) \ge \inf_{\varepsilon\in(0,\sigma)} \sup_{Q\in\mathcal{E}_{\sigma-\varepsilon,1}} \lambda_i(A+Q) \equiv \tilde{\nabla}_{\sigma,i}(A), \quad i = 1,\dots,n.$$

Proof. Without loss of generality, we assume that all quantities occurring in the proof take finite values. The general case can be reduced to this one by applying the bounding order-preserving homeomorphism $\Phi\colon \overline{\mathbb{R}} \to [-1,1]$ defined by the formula

$$\Phi(x) = \begin{cases} x/(|x|+1) & \text{if } \ x \in \mathbb{R}, \\ \operatorname{sgn} x & \text{if } \ x = \pm\infty. \end{cases}$$

Let τ be some bijection of $\mathbb{N}$ onto $\{1,\ldots,n\}\times\mathbb{N}\times\mathbb{N}$, and let $p_i\colon \{1,\ldots,n\}\times\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $i=1,2,3$, be the projection of the product onto the ith factor. For each $k\in\mathbb{N}$, set $i_k=p_1(\tau(k))$ and $m_k=p_2(\tau(k))$.

Take an arbitrary sequence $(\varepsilon_k)_{k=0}^{\infty}$ of numbers in the interval $(0,\sigma)$, monotone decreasing to zero. We use induction to construct a strictly increasing sequence $(t_k)_{k=0}^{\infty}$ of points on the half-line $\mathbb{R}_+$, a sequence $(q_k)_{k=0}^{\infty}$ of positive integers, and a sequence $(B_k)_{k=0}^{\infty}$ of systems in $\mathcal{M}^n$ satisfying the conditions

$$t_j\ge t_{j-1}+1,\quad B_j(t)=B_{j-1}(t),\quad t\in[0,t_{j-1}],\tag{12}$$

$$|B_j(t)|\le\exp(-(\sigma-\varepsilon_j)t),\quad t\ge t_{j-1}+1,\quad |B_j(t)|\le\exp(-(\sigma-\varepsilon_{j-1})t),\quad t\in[t_{j-1},t_{j-1}+1],\tag{13}$$

$$\lambda_{i_j}(A+B_j)>\tilde{\nabla}_{\sigma,i_j}(A)-\varepsilon_j,\tag{14}$$

$$\varphi_{i_j}^{m_jq_j}(A+B_j)>\lambda_{i_j}(A+B_j)-\varepsilon_j\tag{15}$$

for each $j\in\mathbb{N}$, where the φ_i^{mq}, $m,q\in\mathbb{N}$, $i=1,\ldots,n$, are the functions defined in (10).

Set $t_0=0$, $q_0=1$, and $B_0=0$. Assume that for some $k\in\mathbb{N}$ we have already defined numbers t_j and q_j and systems $B_j\in\mathcal{M}^n$, $j=0,\ldots,k-1$, satisfying conditions (12)–(15) for each $j=1,\ldots,k-1$.

By the definition of $\tilde{\nabla}_{\sigma,i_k}(A)$, there exists a system $C_k\in\mathcal{E}_{\sigma-\varepsilon_k,1}$ such that $\lambda_{i_k}(A+C_k)>\tilde{\nabla}_{\sigma,i_k}(A)-\varepsilon_k$. Set

$$B_k(t)=\begin{cases}B_{k-1}(t) & \text{if } t\in[0,t_{k-1}],\\ (1-t+t_{k-1})B_{k-1}(t)+(t-t_{k-1})C_k(t) & \text{if } t\in[t_{k-1},t_{k-1}+1],\\ C_k(t) & \text{if } t\ge t_{k-1}+1.\end{cases}$$

Using the induction assumption (or the base case for $k=1$), we conclude that the system B_k has piecewise continuous coefficients and satisfies conditions (13) for $j=k$. Since the systems B_k and C_k coincide for all $t\ge t_{k-1}+1$, by virtue of the well-known residual property [23] of the Lyapunov exponents we have $\lambda_{i_k}(A+B_k)=\lambda_{i_k}(A+C_k)$, which implies inequality (14) for $j=k$. By (11), there exists a $q_k\in\mathbb{N}$ such that inequality (15) holds for $j=k$. Set $t_k=\max\{t_{k-1}+1,m_k+q_k\}$. This completes the induction step and hence the construction of the sequences (t_k), (q_k), and (B_k).

Now set $B(t_0)=0$ and $B(t)=B_k(t)$ for $t\in(t_{k-1},t_k]$, $k\in\mathbb{N}$. By the first condition in (12), we have $\mathbb{R}_+=\{t_0\}\cup\bigcup_{k=1}^{\infty}(t_{k-1},t_k]$, and hence the system B is defined on the entire half-line $\mathbb{R}_+$. Let us show that it has the desired properties.

By construction, for each $k\in\mathbb{N}$ the system B coincides with the system B_k on the interval $[0,t_k]$, and hence B has piecewise continuous coefficients and satisfies the inequality

$$|B(t)|\le\exp(-(\sigma-\varepsilon_{k-1})t),\quad t\ge t_{k-1};$$

it follows that $\lambda[B]\le-\sigma+\varepsilon_{k-1}$. Passing to the limit as $k\to\infty$ in the last inequality, we arrive at the inclusion $B\in\hat{\mathcal{E}}_\sigma$.

Fix arbitrary $i=1,\ldots,n$ and $\delta>0$. Choose a $k_0\in\mathbb{N}$ such that $2\varepsilon_{k_0}<\delta$. Since $\tau(\mathbb{N})=\{1,\ldots,n\}\times\mathbb{N}\times\mathbb{N}$, it follows that for each $m\in\mathbb{N}$ there exists a $k\ge k_0$ such that $i_k=i$ and $m_k=m$. Since the value of the function $\varphi_{i_k}^{m_kq_k}$ on a system depends only on its restriction to the interval $[0,m_k+q_k]$, on which the systems B and B_k coincide, we have the chain of relations

$$\varphi_i^{mq_k}(A+B)=\varphi_{i_k}^{m_kq_k}(A+B)=\varphi_{i_k}^{m_kq_k}(A+B_k)>\lambda_{i_k}(A+B_k)-\varepsilon_k>\tilde{\nabla}_{\sigma,i_k}(A)-2\varepsilon_k>\tilde{\nabla}_{\sigma,i_k}(A)-\delta,$$

which implies that

$$\lambda_i(A+B)=\inf_{m\in\mathbb{N}}\sup_{q\in\mathbb{N}}\varphi_i^{mq}(A+B)\ge\tilde{\nabla}_{\sigma,i}(A)-\delta.$$

Since the number δ is arbitrarily small, we see that the last inequality implies the desired result. The proof of the lemma is complete.

Lemma 3. *For each system* $A \in \widetilde{\mathcal{M}}^n$ *and each number* $\sigma > 0$, *one has*

$$\nabla_{\sigma,i}(A) = \tilde{\nabla}_{\sigma,i}(A) = \hat{\nabla}_{\sigma,i}(A) \equiv \inf_{\varepsilon \in (0,\sigma)} \sup_{Q \in \mathcal{E}_{\sigma-\varepsilon}} \lambda_i(A+Q), \quad i = 1, \dots, n.$$

Proof. Fix an $i = 1, \dots, n$. Let us show that for each $\varepsilon \in (0, \sigma/2)$ we have the chain of inclusions

$$\{\lambda_i(A+Q) : Q \in \hat{\mathcal{E}}_\sigma\} \subset \{\lambda_i(A+Q) : Q \in \mathcal{E}_{\sigma-\varepsilon}\} \subset \{\lambda_i(A+Q) : Q \in \mathcal{E}_{\sigma-2\varepsilon,1}\}. \tag{16}$$

Indeed, for each $Q \in \hat{\mathcal{E}}_\sigma$ we have $\overline{\lim\limits_{t\to\infty}}\, t^{-1} \ln |Q(t)| < -\sigma + \varepsilon$. By the definition of the upper limit, there exists a $T > 0$ such that the inequality $|Q(t)| \le e^{(-\sigma+\varepsilon)t}$ holds for all $t \ge T$. Then for each $t \in \mathbb{R}_+$ we have $|Q(t)| \le Ce^{-(\sigma-\varepsilon)t}$, where $C = \sup\limits_{t\in[0,T]} |Q(t)|e^{(\sigma-\varepsilon)t} + 1$. Thus, $\hat{\mathcal{E}}_\sigma \subset \mathcal{E}_{\sigma-\varepsilon}$, and the first inclusion in (16) has been established.

Let us establish the second inclusion. Let $Q \in \mathcal{E}_{\sigma-\varepsilon}$. Then for some $C > 0$ we have $|Q(t)| \le Ce^{-(\sigma-\varepsilon)t}$, $t \in \mathbb{R}_+$. Let us choose a $T > 0$ such that $Ce^{-\varepsilon T} \le 1$. Set

$$\tilde{Q}(t) = \begin{cases} 0 & \text{if } \ t \in [0, T], \\ (t - T)Q(t) & \text{if } \ t \in [T, T + 1], \\ Q(t) & \text{if } \ t \ge T + 1. \end{cases}$$

Then $\lambda_i(A + Q) = \lambda_i(A + \tilde{Q})$, and we have $\tilde{Q} \in \mathcal{E}_{\sigma-2\varepsilon,1}$ by virtue of the chain of inequalities

$$|\tilde{Q}(t)| \le |Q(t)| \le Ce^{-(\sigma-\varepsilon)t} = Ce^{-\varepsilon t}e^{-(\sigma-2\varepsilon)t} \le e^{-(\sigma-2\varepsilon)t}, \quad t \ge T.$$

By passing to the least upper bound in (16), we obtain

$$\sup\{\lambda_i(A+Q) : Q \in \hat{\mathcal{E}}_\sigma\} \le \sup\{\lambda_i(A+Q) : Q \in \mathcal{E}_{\sigma-\varepsilon}\} \le \sup\{\lambda_i(A+Q) : Q \in \mathcal{E}_{\sigma-2\varepsilon,1}\}.$$

Passing to the greatest lower bound with respect to $\varepsilon \in (0, \sigma)$ in the first inequality and with respect to $\varepsilon \in (0, \sigma/2)$ in the second, we obtain $\nabla_{\sigma,i}(A) \le \hat{\nabla}_{\sigma,i}(A) \le \tilde{\nabla}_{\sigma,i}(A)$. However, by Lemma 2 we have the inequality $\nabla_{\sigma,i}(A) \ge \tilde{\nabla}_{\sigma,i}(A)$, and thus we arrive at the desired result. The proof of the lemma is complete.

Proof of Theorem 2. By Lemma 2, there exists a system $Q \in \hat{\mathcal{E}}_\sigma$ such that $\lambda_i(A+Q) \ge \tilde{\nabla}_{\sigma,i}(A)$, $i = 1, \dots, n$. The assertion of the theorem now follows from the chain

$$\lambda_i(A + Q) \le \nabla_{\sigma,i}(A) = \tilde{\nabla}_{\sigma,i}(A), \quad i = 1, \dots, n,$$

where the first relation results from the definition of $\nabla_{\sigma,i}(A)$ and the second has been established in Lemma 3. The proof of the theorem is complete.

Proof of Theorem 3. Let us use Lemma 3 to establish the assertion for the function $\hat{\nabla}_{\sigma,i}$. Suppose that the inequality $\hat{\nabla}_{\sigma,i}(A) < \mu$ holds for some $\sigma > 0$ and $\mu \in \mathbb{R}$. Then there exists a $\delta \in (0, \sigma)$ such that $\sup\limits_{Q\in\mathcal{E}_{\sigma-\delta}} \lambda_i(A + Q) < \mu$. Now if $\sigma' > \sigma - \delta$, then there exists an $\eta > 0$ such that $\sigma' - \eta > \sigma - \delta$. Then $\mathcal{E}_{\sigma'-\eta} \subset \mathcal{E}_{\sigma-\delta}$, and the chain of inequalities

$$\hat{\nabla}_{\sigma',i}(A) \le \sup_{Q\in\mathcal{E}_{\sigma'-\eta}} \lambda_i(A + Q) \le \sup_{Q\in\mathcal{E}_{\sigma-\delta}} \lambda_i(A + Q) < \mu$$

holds. Thus, the preimage of each half-line $[-\infty, \mu)$, $\mu \in \mathbb{R}$, under the mapping $\sigma \mapsto \hat{\nabla}_{\sigma,i}(A)$ is an open set, because this preimage contains some neighborhood $(\sigma - \delta, +\infty)$ of every point σ of itself. The proof of the theorem is complete.

The next assertion was established in [19], where for each $i = 1, \dots, n$ the authors derived a description of the set of functions $\{\lambda_i(\cdot\,; A) : A \in \mathcal{BU}^n(M)\}$ and their Lebesgue sets. Assertion 3 in Lemma 4 below has not been stated explicitly in [19], but it readily follows from the proof of assertion 2 provided therein (i.e., in that paper).

First, recall the following definition [18, Sec. 41.1]. Let $\mathfrak{M}$ be some family of subsets of a metric space X. We say that a function $f : X \to \overline{\mathbb{R}}$ belongs to the class $(\mathfrak{M}, ^*)$ (respectively, the class $(^*, \mathfrak{M})$) if the inclusion $[f > r] \in \mathfrak{M}$ (respectively, the inclusion $[f \ge r] \in \mathfrak{M}$) holds for each $r \in \mathbb{R}$.

Lemma 4. *For any metric space M and numbers $n \ge 2$ and $i = 1, \dots, n$, the set of functions $\{\lambda_i(\cdot\,; A) : A \in \mathcal{BU}^n(M)\}$ coincides with the set of bounded functions in the class $(^*, G_\delta)$ and the following relations hold for each $r \in \mathbb{R}$:*

1. $\{[\lambda_i(\cdot\,; A) \ge r] : A \in \mathcal{BU}^n(M)\} = G_\delta(M)$.
2. $\{[\lambda_i(\cdot\,; A) > r] : A \in \mathcal{BU}^n(M)\} = G_{\delta\sigma}(M)$.
3. $\{[\lambda_i(\cdot\,; A) = r] : A \in \mathcal{BU}^n(M)\} = F_{\sigma\delta}(M)$.

The next lemma, to some extent, reduces the problem of realizing a given function by an Izobov σ-exponent to a similar problem for a Lyapunov exponent.

Lemma 5. *For any metric space M, any numbers $n \ge 2$, $i = 1, \dots, n$, and $\sigma > 0$, and any bounded function $f : M \to \mathbb{R}$ of the class $(^*, G_\delta)$, there exists a family $A \in \mathcal{BU}^n(M)$ and a number $\alpha > 0$ such that*

$$\inf_{Q \in \hat{\mathcal{E}}_\sigma} \lambda_i(A(\cdot, \mu) + Q) = \sup_{Q \in \hat{\mathcal{E}}_\sigma} \lambda_i(A(\cdot, \mu) + Q) = \lambda_i(A(\cdot, \mu)) = \alpha f(\mu), \quad \mu \in M.$$

Proof. By Lemma 4, there exists a family $B \in \mathcal{BU}^n(M)$ satisfying the condition $\lambda_i(\cdot\,; B) = f$. Set $L = \sup_{\mu \in M} \|B(\cdot, \mu)\| + 1$. Define a family $A \in \mathcal{BU}^n(M)$ by the formula

$$A(t, \mu) = \alpha B(\alpha t, \mu), \quad t \in \mathbb{R}_+, \quad \mu \in M,$$

where $\alpha = \sigma(3L)^{-1}$. Now if a vector function $x(\cdot)$ is a solution of the system $B(\cdot, \mu)$ for some $\mu \in M$, then the vector function $y(t) \equiv x(\alpha t)$, $t \in \mathbb{R}_+$, is a solution of the system $A(\cdot, \mu)$ with $x(0) = y(0)$. Consequently, $\lambda_i(\mu; A) = \alpha \lambda_i(\mu; B) = \alpha f(\mu)$, $\mu \in M$.

The Grobman irregularity coefficient [3, § 1.3] of systems in the family thus constructed obeys the estimate (following from [3, Eq. (3.3)])

$$\sigma_{\mathrm{g}}(A(\cdot, \mu)) \le 2\|A(\cdot, \mu)\| < \sigma, \quad \mu \in M.$$

Now the desired result follows from the Grobman theorem [3, Sec. 8.1]. The proof of the lemma is complete.

Proof of Theorem 4. 1. Let us show that the function $\widetilde{\mathcal{M}}^n_C \times \mathbb{R}^*_+ \to \overline{\mathbb{R}}$ defined by the rule $(A, \sigma) \mapsto \nabla_{\sigma,i}(A)$ belongs to the class $(^*, G_\delta)$.

(a) Fix $k, l \in \mathbb{N}$. For each $q \in \mathbb{N}$, set

$$\psi_q(B, \sigma) = \sup_{t \in [0,q]} |B(t)| e^{\sigma t}, \quad B \in \widetilde{\mathcal{M}}^n, \quad \sigma \in \mathbb{R}^*_+.$$

Let us prove that the function $F^{kl}_i : \widetilde{M}^n_C \times \mathbb{R}^*_+ \to \mathbb{R}$ defined by the relation

$$F^{kl}_i(A, \sigma) = \sup\{\varphi^{kl}_i(A + B) : \psi_{k+l}(B, \sigma) < 1\}, \quad A \in \widetilde{\mathcal{M}}^n, \quad \sigma \in \mathbb{R}^*_+,$$

where the function φ^{kl}_i is defined in relation (10), belongs to the class $(G, ^*)$; i.e., it is lower semicontinuous. Indeed, given an $r \in \mathbb{R}$, the set $\mathcal{A} = \{(A, B) \in \widetilde{\mathcal{M}}^n \times \widetilde{\mathcal{M}}^n : \varphi^{kl}_i(A + B) > r\}$ is open in the space $\widetilde{\mathcal{M}}^n_C \times \widetilde{\mathcal{M}}^n_C$ as the preimage of the open set $(r, +\infty)$ under the continuous (by Lemma 1) mapping $(A, B) \mapsto \varphi^{kl}_i(A + B)$, and the set $\mathcal{B} = \{(B, \sigma) \in \widetilde{\mathcal{M}}^n_C \times \mathbb{R}^*_+ : \psi_{k+l}(B, \sigma) < 1\}$ is open in the space $\widetilde{\mathcal{M}}^n_C \times \mathbb{R}^*_+$ as the preimage of the open set $(-\infty, 1)$ under the continuous mapping ψ_{k+l}. Then the set $\{(A,\sigma) : F^{kl}_i(A,\sigma) > r\}$ is open as the projection [24, Sec. 15.II, Theorem 1] of the open subset $(\mathcal{A} \times \mathbb{R}^*_+) \cap (\widetilde{\mathcal{M}}^n \times \mathcal{B})$ of the space $\widetilde{\mathcal{M}}^n_C \times \widetilde{\mathcal{M}}^n_C \times \mathbb{R}^*_+$ onto the product of the first and third factors.

(b) Take arbitrary $A \in \widetilde{\mathcal{M}}^n$ and $\sigma > 0$. Let us establish the relation

$$\xi \equiv \sup_{B\in\mathcal{E}_{\sigma,1}} \varphi^{kl}_i(A+B) = \sup\{\varphi^{kl}_i(A+B) : \psi_{k+l}(B,\sigma) \le 1\} \equiv \mu. \tag{17}$$

By virtue of the inclusion $\mathcal{E}_{\sigma,1} \subset \{B \in \widetilde{\mathcal{M}}^n : \psi_{k+l}(B,\sigma) \le 1\}$, we have $\xi \le \mu$. On the other hand, for each $B \in \widetilde{\mathcal{M}}^n$ satisfying the condition $\psi_{k+l}(B,\sigma) \le 1$ we set

$$\tilde{B}(t) = \begin{cases} B(t) & \text{if } t \in [0, k+l], \\ B(k+l)e^{-\sigma(t-(k+l))} & \text{if } t \ge k+l. \end{cases}$$

Then $\tilde{B} \in \mathcal{E}_{\sigma,1}$ and $\varphi^{kl}_i(A+\tilde{B}) = \varphi^{kl}_i(A+B)$, because the value of the function φ^{kl}_i on a system is determined by its restriction to the interval $[0, k+l]$, on which the systems B and $\tilde{B}$ coincide. Thus, $\xi \ge \mu$, and relation (17) has been established.

Now let us prove the relation

$$\sup\{\varphi^{kl}_i(A+B) : \psi_{k+l}(B,\sigma) \le 1\} = \sup\{\varphi^{kl}_i(A+B) : \psi_{k+l}(B,\sigma) < 1\}. \tag{18}$$

The set $\{B \in \widetilde{\mathcal{M}}^n : \psi_{k+l}(B,\sigma) \le 1\}$ is the closure of the set $\{B \in \widetilde{\mathcal{M}}^n : \psi_{k+l}(B,\sigma) < 1\}$ in the space $\widetilde{\mathcal{M}}^n_C$. In view of the continuity of the mapping $B \mapsto \varphi^{kl}_i(A+B)$ on the space $\widetilde{\mathcal{M}}^n_C$, we conclude that the first of the sets occurring in (18) under the least upper bound sign is the closure of the second. Hence their least upper bounds coincide.

(c) Now let us show that the function $\eta_i : \widetilde{\mathcal{M}}^n_C \times \mathbb{R}^*_+ \to \overline{\mathbb{R}}$ defined by the formula

$$\eta_i(A,\sigma) = \sup_{B\in\mathcal{E}_{\sigma,1}} \lambda_i(A+B), \quad A \in \widetilde{\mathcal{M}}^n, \quad \sigma \in \mathbb{R}^*_+,$$

belongs to the class $({}^*, G_\delta)$. Using [17, Lemma 3], we obtain

$$\eta_i(A,\sigma) = \inf_{k\in\mathbb{N}} \sup_{B\in\mathcal{E}_{\sigma,1}} \inf_{L\in G_i(\mathbb{R}^n)} \sup_{t\ge k} \frac{1}{t} \ln |X_{A+B}(t,0)|_L|, \quad A \in \widetilde{\mathcal{M}}^n, \quad \sigma \in \mathbb{R}^*_+.$$

For any $D \in \widetilde{\mathcal{M}}^n$, $t > 0$, and $L \in G_i(\mathbb{R}^n)$, set $\chi(D,t,L) = t^{-1} \ln |X_D(t,0)|_L|$. By [22, Lemma 2], for any $D \in \widetilde{\mathcal{M}}^n$ and $t > 0$ the function $L \mapsto \chi(D,t,L)$ is continuous. (The set $G_i(\mathbb{R}^n)$ is equipped with the standard topology, in which it is a compact space.) Applying [17, Lemma 2], for each $D \in \widetilde{\mathcal{M}}^n$ we obtain the chain of relations

$$\inf_{L\in G_i(\mathbb{R}^n)} \sup_{t\ge k} \chi(D,t,L) = \inf_{L\in G_i(\mathbb{R}^n)} \sup_{l\in\mathbb{N}} \max_{t\in[k,k+l]} \chi(D,t,L)$$

$$= \sup_{l\in\mathbb{N}} \inf_{L\in G_i(\mathbb{R}^n)} \max_{t\in[k,k+l]} \chi(D,t,L) = \sup_{l\in\mathbb{N}} \varphi^{kl}_i(D);$$

it follows that

$$\eta_i(A,\sigma) = \inf_{k\in\mathbb{N}} \sup_{l\in\mathbb{N}} \sup_{B\in\mathcal{E}_{\sigma,1}} \varphi^{kl}_i(A+B), \quad A \in \widetilde{\mathcal{M}}^n, \quad \sigma \in \mathbb{R}^*_+.$$

Finally, using relations (17) and (18), we obtain

$$\eta_i(A,\sigma) = \inf_{k\in\mathbb{N}} \sup_{l\in\mathbb{N}} F^{kl}_i(A,\sigma), \quad A \in \widetilde{\mathcal{M}}^n, \quad \sigma \in \mathbb{R}^*_+.$$

In part 1(a) of the proof, it was established that the functions $F_i^{kl} : \widetilde{\mathcal{M}}_C^n \times \mathbb{R}_+^* \to \mathbb{R}$, $k, l \in \mathbb{N}$, belong to the class $(G,^*)$. Then, according to [18, Sec. 41.1.I], the functions $(A,\sigma) \mapsto \sup_{l\in\mathbb{N}} F_i^{kl}(A,\sigma)$, $k \in \mathbb{N}$, belong to this class as well and hence also belong to the class $(^*, G_\delta)$ in view of the relation $[r,+\infty] = \bigcap_{k\in\mathbb{N}}(r-k^{-1},+\infty]$, $r \in \mathbb{R}$, and the properties of the preimage of a set. Therefore, according to [18, Sec. 41.1.II], the function η_i belongs to the class $(^*, G_\delta)$.

Let $\sigma'' > \sigma' > 0$. Then $\mathcal{E}_{\sigma'',1} \subset \mathcal{E}_{\sigma',1}$, which implies the inequality $\eta_i(A,\sigma'') \le \eta_i(A,\sigma')$ for each $A \in \widetilde{\mathcal{M}}^n$. Hence we have the chain of relations

$$\tilde{\nabla}_{\sigma,i}(A) = \inf_{\varepsilon\in(0,\sigma)} \eta_i(A,\sigma-\varepsilon) = \lim_{\varepsilon\to 0+} \eta_i(A,\sigma-\varepsilon) = \inf_{m\in\mathbb{N}} \eta_i(A,\sigma(1-2^{-m})), \quad A \in \widetilde{\mathcal{M}}^n, \quad \sigma \in \mathbb{R}_+^*.$$

Noting that $\tilde{\nabla}_{\sigma,i}(A) = \nabla_{\sigma,i}(A)$ (by Lemma 3) and applying the assertion in [18, Sec. 41.1.II] to the right-hand side of the last relation, we conclude that the function $(A,\sigma) \mapsto \nabla_{\sigma,i}(A)$ belongs to the class $(^*, G_\delta)$.

2. (a) Now let us fix arbitrary $\sigma > 0$ and $r \in \mathbb{R}$. By virtue of item 1, the set

$$\{(A,\varsigma) \in \widetilde{\mathcal{M}}^n \times \mathbb{R}_+^* : \nabla_{\varsigma,i}(A) \ge r\} \cap (\widetilde{\mathcal{M}}^n \times \{\sigma\})$$

is a G_δ-set in the product $\widetilde{\mathcal{M}}_C^n \times \{\sigma\}$. Consequently, the projection $\{A \in \widetilde{\mathcal{M}}^n : \nabla_{\sigma,i}(A) \ge r\}$ of this set onto the first factor is a G_δ-set in the space $\widetilde{\mathcal{M}}_C^n$, because the restriction of the projection of the product $\widetilde{\mathcal{M}}_C^n \times \mathbb{R}_+^*$ onto the first factor to the subspace $\widetilde{\mathcal{M}}_C^n \times \{\sigma\}$ is a homeomorphsim.

Now, given a family $A \in \mathcal{C}^n(M)$, the set $[\nabla_{\sigma,i}(\cdot\,;A) \ge r]$ is the preimage of the set $\{A \in \widetilde{\mathcal{M}}^n : \nabla_{\sigma,i}(A) \ge r\}$ under the continuous (see, e.g., [17, Lemma 4]) mapping $M \to \widetilde{\mathcal{M}}_C^n$ defined by the rule $\mu \mapsto A(\cdot,\mu)$ and hence is a G_δ-set in the space M. We have thus established the inclusion

$$\{[\nabla_{\sigma,i}(\cdot\,;A) \ge r] : A \in \mathcal{C}^n(M)\} \subset G_\delta(M). \tag{19}$$

(b) The inclusion (19) and the relation

$$[\nabla_{\sigma,i}(\cdot\,;A) > r] = \bigcup_{k\in\mathbb{N}} [\nabla_{\sigma,i}(\cdot\,;A) \ge r + k^{-1}]$$

imply the inclusion

$$\{[\nabla_{\sigma,i}(\cdot\,;A) > r] : A \in \mathcal{C}^n(M)\} \subset G_{\delta\sigma}(M). \tag{20}$$

Finally, from inclusions (19) and (20), the inclusion $G_\delta(M) \subset F_{\sigma\delta}(M)$, and the relation

$$[\nabla_{\sigma,i}(\cdot\,;A) = r] = [\nabla_{\sigma,i}(\cdot\,;A) \ge r] \cap (M \setminus [\nabla_{\sigma,i}(\cdot\,;A) > r])$$

we obtain the inclusion

$$\{[\nabla_{\sigma,i}(\cdot\,;A) = r] : A \in \mathcal{C}^n(M)\} \subset F_{\sigma\delta}(M). \tag{21}$$

3. Take an arbitrary $r \in \mathbb{R}$ and prove the inclusion

$$\{[\nabla_{\sigma,i}(\cdot\,;A) \ge r] : A \in \mathcal{BU}^n(M)\} \supset G_\delta(M). \tag{22}$$

Given a set $S \in G_\delta(M)$, it follows from Lemma 4 that $S = [f \ge r]$ for some bounded function $f : M \to \mathbb{R}$ of the class $(^*, G_\delta)$. According to Lemma 5, there exists a family $A \in \mathcal{BU}^n(M)$ and a number $\alpha > 0$ that satisfy the condition

$$\nabla_{\sigma,i}(\mu; A) = \alpha f(\mu), \quad \mu \in M.$$

Set $B(t,\mu) = A(t,\mu) + (1-\alpha) r E_n$, $t \in \mathbb{R}_+$, $\mu \in M$. Now if a vector function $x(\cdot)$ is a solution of the system $A(\cdot,\mu) + Q$ for some $\mu \in M$ and $Q \in \hat{\mathcal{E}}_\sigma$, then the vector function $y(t) \equiv x(t)\exp((1-\alpha)rt)$,

$t \in \mathbb{R}_+$, is a solution of the system $B(\cdot, \mu) + Q$, with $x(0) = y(0)$. Consequently, the following chain of relations holds:

$$\nabla_{\sigma,i}(\mu; B) = \nabla_{\sigma,i}(\mu; A) + (1-\alpha)r = \alpha f(\mu) + (1-\alpha)r, \quad \mu \in M.$$

It remains to note that $[\nabla_{\sigma,i}(\cdot\,; B) \ge r] = [f \ge r] = S$. The inclusion (22) has been established. The inclusions

$$\{[\nabla_{\sigma,i}(\cdot\,; A) > r] : A \in \mathcal{BU}^n(M)\} \supset G_{\delta\sigma}(M) \text{ and } \{[\nabla_{\sigma,i}(\cdot\,; A) = r] : A \in \mathcal{BU}^n(M)\} \supset F_{\sigma\delta}(M) \quad (23)$$

can be proved in a similar way. The assertion of the theorem follows from inclusions (19)–(23) and the inclusion $\mathcal{BU}^n(M) \subset \mathcal{C}^n(M)$. The proof of the theorem is complete.

Proof of the Corollary. 1. Given a family $A \in \mathcal{C}^n(M)$, the set $[\nabla_{\sigma,i}(\cdot\,; A) \ge r]$ is a set of the type G_δ for each $r \in \overline{\mathbb{R}}$ by Theorem 4. It follows from the assertions in [18, Sec. 43.1.I] that the function $\nabla_{\sigma,i}(\cdot\,; A) : M \to \overline{\mathbb{R}}$ is the limit of a decreasing sequence of functions of Baire class 1. Thus, the function $\nabla_{\sigma,i}(\cdot\,; A)$ belongs to Baire class 2.

2. Since, as shown above, the function $\nabla_{\sigma,i}(\cdot\,; A)$ belongs to the class $(^*, G_\delta)$, from [25, Lemma 2] we obtain the inclusions $LS(\nabla_{\sigma,i}(\cdot\,; A)) \in \mathfrak{F}(M)$ and $US(\nabla_{\sigma,i}(\cdot\,; A)) \in G_\delta(M)$. If M is complete, then it follows from [26, p. 106] that the set $US(\nabla_{\sigma,i}(\cdot\,; A))$ is dense in M.

3. Given an arbitrary set $S \in \mathfrak{G}(M)$, by the result in [27], there exists a family $B \in \mathcal{BU}^n(M)$ such that $US(\lambda_i(\cdot\,; B)) = S$. By Lemma 4, the function $\lambda_i(\cdot\,; B)$ is a bounded function of the class $(^*, G_\delta)$. Using Lemma 5, we construct a family $A \in \mathcal{BU}^n(M)$ such that $\nabla_{\sigma,i}(\cdot\,; A) = \alpha\lambda_i(\cdot\,; B)$ for some $\alpha > 0$. Then it is obvious that $US(\nabla_{\sigma,i}(\cdot\,; A)) = US(\lambda_i(\cdot\,; B)) = S$. Thus, we have established the inclusion $\{US(\nabla_{\sigma,i}(\cdot\,; A)) : A \in \mathcal{BU}^n(M)\} \supset \mathfrak{G}(M)$. The inclusion $\{LS(\nabla_{\sigma,i}(\cdot\,; A)) : A \in \mathcal{BU}^n(M)\} \supset \mathfrak{F}(M)$ can be established in a similar way. Taking into account the results in item 2 and the inclusion $\mathcal{BU}^n(M) \subset \mathcal{C}^n(M)$, we arrive at the desired assertion.The proof of the corollary is complete.

Remark 4. Using the same argument and the result in [28], one can readily show that we have the relations

$$\{(LS(\nabla_{\sigma,1}(\cdot\,; A)), \ldots, LS(\nabla_{\sigma,n}(\cdot\,; A))) : A \in \mathcal{BU}^n(M)\} = (\mathfrak{F}(M))^n,$$

$$\{(US(\nabla_{\sigma,1}(\cdot\,; A)), \ldots, US(\nabla_{\sigma,n}(\cdot\,; A))) : A \in \mathcal{BU}^n(M)\} \supset (\mathfrak{G}(M))^n$$

describing the possible tuples of sets of points of lower (upper) semicontinuity of the upper Izobov σ-exponents of systems (6).

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