ISSN 0012-2661, Differential Equations, 2020, Vol. 56, No. 10, pp. 1298–1305. © Pleiades Publishing, Ltd., 2020.
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ORDINARY DIFFERENTIAL EQUATIONS

Integral Dispersion Equation Method for Nonlinear Eigenvalue Problems

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Received March 28, 2020; revised March 28, 2020; accepted May 14, 2020

Abstract—An application of the integral dispersion equation method to the solution of two nonlinear eigenvalue problems is considered. One of these problems arises when studying the propagation of TE-waves in a dielectric shielded nonlinear layer with Kerr nonlinearity, while the other, a more complex one, generalizes the first and contains, in particular, a nonlinearity multiplying the highest derivative in the differential operator. The existence of an infinite set of eigenvalues of the problems is proved, and their asymptotics is found.

DOI: 10.1134/S0012266120100067

INTRODUCTION

Nonlinear eigenvalue problems are encountered, for example, when studying the propagation of polarized electromagnetic waves in nonlinear media [1–3]. In these problems, the differential operator depends nonlinearly on the solution function. In most cases of practical importance, it is not possible to obtain an explicit solution to the corresponding differential equation. However, there are a number of approaches that allow one to establish some properties of the eigenvalues of the problem, such as existence, asymptotic behavior, etc.

The integral dispersion equation method is one efficient technique for studying nonlinear eigenvalue problems. This method was proposed in [4–6] and then applied in [7, 8] to the solution of a number of specific physical problems. Its main idea is to produce some equation not containing the unknown function for the spectral parameter and then study this equation and establish the properties of its eigenvalues. Once the eigenvalues are known, it becomes easy to determine the eigenfunctions of the operator.

In the present paper, the integral dispersion equation method is presented in its simplest form, which differs from the one used in the aforementioned papers. We consider both the well-known eigenvalue problem that arises when studying the propagation of TE-waves in a dielectric shielded nonlinear layer with Kerr nonlinearity and a more complicated problem in which the nonlinearity multiplies the highest derivative in the differential operator.

1. INTEGRAL DISPERSION EQUATION METHOD

Consider the following nonlinear boundary eigenvalue problem for a second-order differential operator:

$$u'' - \lambda u + \alpha u^3 = 0, \quad x \in (0,h), \tag{1}$$

with the homogeneous boundary conditions

$$u(0) = u(h) = 0 \tag{2}$$

and the additional condition

$$u'(0) = A, \quad A > 0. \tag{3}$$

Here $\lambda \in \mathbb{R}$ is the spectral parameter and $\alpha > 0$ is the nonlinearity coefficient. We seek smooth real-valued solutions $u = u(x)$, $u \in C^2(0,h) \cap C^1[0,h]$.

The eigenvalue problem (1)–(3) is to find a $\lambda \in \mathbb{R}$ such that the problem has a nontrivial solution in the above-indicated class of functions.

Having multiplied Eq. (1) by $u'(x)$, we obtain $u'u'' - \lambda uu' + \alpha u'u^3 = 0$, and hence we have $((u')^2)' - \lambda(u^2)' + (u^4)'/2 = 0$, which implies the identity $(u')^2 - \lambda u^2 + \alpha u^4/2 = C_0$, where C_0 is a constant. In view of the continuous differentiability of the function $u(x)$, by virtue of conditions (2) and (3) we obtain $(u'(0))^2 = A^2 > 0$, $A^2 = C_0$. Then we have

$$(u')^2 - \lambda u^2 + \alpha u^4/2 = A^2. \tag{4}$$

From this equation, we find

$$u' = \pm\sqrt{A^2 + \lambda u^2 - \alpha u^4/2}. \tag{5}$$

Since the function $u'(x)$ can change sign on the interval $[0, h]$, we see that the choice of the sign of the radical in (5) remains as yet undefined.

Let $D := \lambda^2 + 2\alpha A^2 (> 0)$. Denote

$$z_1 := \frac{\lambda + \sqrt{D}}{\alpha}\ (> 0), \quad z_2 := \frac{\lambda - \sqrt{D}}{\alpha}\ (< 0).$$

Then

$$A^2 + \lambda u^2 - \alpha u^4/2 = -\alpha(u^2 - z_1)(u^2 - z_2)/2. \tag{6}$$

It follows from identity (4) that the left-hand side of this relation is nonnegative; therefore, considering the inequality $z_2 < 0$, we have $u^2 - z_1 \le 0$, and hence $|u| \le z_1$.

Let $x_i \in (0, h)$, $i = 1, \dots, N$, be the points of extremum of the function $u(x)$. By virtue of the differentiability of $u(x)$, it is necessary that $u'(x_i) = 0$. It is obvious from identity (4) and condition (2) that the points 0 and h are not points of extremum. It follows from (4) and (6) that the relation $|u(x_i)| = z_1$ holds at the points x_i of extremum of the function $u(x)$. Moreover, in view of condition (3), the points x_{2j+1} are the maxima and the points x_{2j} are the minima of the function $u(x)$; i.e., $u(x_{2j+1}) = \sqrt{z_1}$ and $u(x_{2j}) = -\sqrt{z_1}$. Relation (4) also implies that if $u(x) = 0$, then $u'(x) \ne 0$.

Thus, the intervals of increase for the function $u(x)$ are $(0, x_1)$, (x_2, x_3), and so on, while the intervals of decrease are, accordingly, (x_1, x_2), (x_3, x_4), and so on. The overall number of intervals of increase and decrease for the function $u(x)$ equals $N + 1$. Knowing the intervals of increase and decrease for the function $u(x)$ makes it possible to determine the sign in Eq. (5) on these intervals.

Let us integrate Eq. (5) over the interval $(0, x_1)$ taking into account the choice of the "+" sign in front of the radical. Set

$$w := \frac{1}{\sqrt{A^2 + \lambda u^2 - \alpha u^4/2}}\ (> 0).$$

We have

$$\int_0^{u(x)} w\,du = x + C_1, \quad x \in (0, x_1),$$

with some constant C_1. Substituting $x = 0$ into this formula (calculating the limit), we conclude that $C_1 = 0$. Therefore, we have

$$\int_0^{u(x)} w\,du = x, \quad x \in (0, x_1). \tag{7}$$

Further, substituting $x = x_1$ into identity (7), we obtain the relation

$$\int_0^{\sqrt{z_1}} w\,du = x_1. \tag{8}$$

Further, we repeat the same actions on all intervals (x_i, x_{i+1}), choosing the sign of the derivative of the function $u(x)$ depending on whether this interval is an interval of increase or an interval of decrease for the function $u(x)$. Repeating the procedure for the intervals (x_{2j-1}, x_{2j}) $(j \geq 1)$, we have

$$-\int\limits_{\sqrt{z_1}}^{u(x)} w\,du = x + C_{2j}, \quad x \in (x_{2j-1}, x_{2j}),$$

with some constants C_{2j}. Substituting $x = x_{2j-1}$ into this formula, we conclude that $C_{2j} = -x_{2j-1}$. Therefore,

$$-\int\limits_{\sqrt{z_1}}^{u(x)} w\,du = x - x_{2j-1}, \quad x \in (x_{2j-1}, x_{2j}). \tag{9}$$

Then, substituting $x = x_{2j}$ into identity (9), we arrive at the relation

$$\int\limits_{-\sqrt{z_1}}^{\sqrt{z_1}} w\,du = x_{2j} - x_{2j-1}. \tag{10}$$

Repeating the above procedure for the intervals (x_{2j}, x_{2j+1}) $(j \geq 1)$, we have

$$\int\limits_{-\sqrt{z_1}}^{u(x)} w\,du = x + C_{2j+1}, \quad x \in (x_{2j}, x_{2j+1}),$$

with some constants C_{2j+1}. Substituting $x = x_{2j}$ into this formula, we conclude that $C_{2j+1} = -x_{2j}$. Then we have

$$\int\limits_{-\sqrt{z_1}}^{u(x)} w\,du = x - x_{2j}, \quad x \in (x_{2j}, x_{2j+1}). \tag{11}$$

Further, substituting $x = x_{2j+1}$ into identity (11), we arrive at the relation

$$\int\limits_{-\sqrt{z_1}}^{\sqrt{z_1}} w\,du = x_{2j+1} - x_{2j}. \tag{12}$$

In the case of even N, on the last interval (x_N, h) we have

$$\int\limits_{-\sqrt{z_1}}^{u(x)} w\,du = x + C_{N+1}, \quad x \in (x_N, h),$$

with some constant C_{N+1}. Substituting $x = x_N$ into this formula, we conclude that $C_{N+1} = -\,x_N$. Therefore,

$$\int\limits_{-\sqrt{z_1}}^{u(x)} w\,du = x - x_N, \quad x \in (x_N, h). \tag{13}$$

Further, substituting $x = h$ into identity (13), for the case of even N we obtain the relation

$$\int\limits_{-\sqrt{z_1}}^{0} w\,du = h - x_N. \tag{14}$$

In the case of odd N, on the last interval (x_N, h) we have

$$-\int\limits_{\sqrt{z_1}}^{u(x)} w\,du = x + C_{N+1}, \quad x \in (x_N, h),$$

with some constant C_{N+1}. Substituting $x = x_N$ into this formula, we conclude that $C_{N+1} = -x_N$. Then

$$-\int\limits_{\sqrt{z_1}}^{u(x)} w\,du = x - x_N, \quad x \in (x_N, h). \tag{15}$$

Further, substituting $x = h$ into identity (15), in the case of odd N we obtain the relation

$$\int\limits_{0}^{\sqrt{z_1}} w\,du = h - x_N. \tag{16}$$

Now, summing relations (8), (10), (12), and (14) (or (16)), we arrive at the integral dispersion equation

$$NT(\lambda) = h \quad (N \geq 1), \tag{17}$$

where

$$T = T(\lambda) := \int\limits_{-\sqrt{z_1}}^{\sqrt{z_1}} w\,du. \tag{18}$$

Note that the improper integral (18) (similar to all the improper integrals above) is absolutely convergent by virtue of (6).

It is obvious from the boundary conditions (2) that there exists at least one point of extremum, and therefore, $N \geq 1$. The dispersion equations (17) must be solved for all $N \geq 1$; i.e., we have a set of dispersion equations, each determining nonlinear eigenvalues of problem (1)–(3).

Let us prove the converse. If a solution λ_0 to Eq. (17) has been found for some $N \geq 1$, then one can readily determine all the values x_i and accordingly C_i from the above relations. Further, the above integral relations for the function $u(x)$ can be used to determine the function $u(x)$ on each interval (x_i, x_{i+1}). By differentiating these relations, we conclude that relations (5) and further (4) hold. It follows that Eq. (1) is satisfied.

Since the function w is positive, we find from the equation $\int\limits_0^{u(x)} w\,du = x$, $x \in (0, x_1)$, at $x = 0$ that $u(0) = 0$. In a similar way, from the equations on the last interval we obtain (for both even and odd N) $|\int\limits_0^{u(x)} w\,du| = h - x$, $x \in (x_N, h)$, and hence $u(h) = 0$; i.e., the boundary conditions (2) are satisfied. Now condition (3) follows from Eq. (4) and from the fact that the function $u(x)$ is increasing on the interval $(0, x_1)$. The smoothness of the function $u(x)$ is verified using the formulas that explicitly define it on the intervals (x_i, x_{i+1}). We have thus proved the following assertion.

Theorem 1. *If a number $\lambda = \lambda_0$ is an eigenvalue of problem* (1)–(3), *then it is a solution of the dispersion equation* (17) *for some $N \geq 1$. Conversely, if λ_0 is a solution of the dispersion equation* (17) *for some $N \geq 1$, then it is an eigenvalue of problem* (1)–(3).

Theorem 1 reduces solving the nonlinear eigenvalue problem (1)–(3) to solving the dispersion equation (17). It is this fact that constitutes the main idea of the integral dispersion equation method.

2. ANALYSIS OF THE INTEGRAL DISPERSION EQUATION

Since $z_1 z_2 = -2A^2/\alpha$, we can use a change of variables in the integral (18) to obtain

$$T = \int\limits_{-\sqrt{z_1}}^{\sqrt{z_1}} w\,du = 2\sqrt{2}\int\limits_0^{\pi/2} \frac{dt}{\sqrt{\alpha z_1 \sin^2 t + 2A^2/z_1}}.$$

The last integral can be estimated as

$$\int\limits_0^{\pi/2} \frac{dt}{\sqrt{\alpha z_1 \sin^2 t + 2A^2/z_1}} \le \int\limits_0^{\pi/2} \frac{dt}{\sqrt{2A\sqrt{2\alpha}\sin t}} = M,$$

where M is a constant independent of z_1. It follows that there exist limits

$$\lim_{z_1\to 0} T(z_1) = 0, \quad \lim_{z_1\to +\infty} T(z_1) = 0.$$

Since

$$z_1 = z_1(\lambda) = \frac{\lambda + \sqrt{\lambda^2 + 2\alpha A^2}}{\alpha} = \frac{2A^2}{-\lambda + \sqrt{\lambda^2 + 2\alpha A^2}},$$

we have $z_1(\lambda) \to +\infty$ as $\lambda \to +\infty$ and $z_1(\lambda) \to 0$ as $\lambda \to -\infty$. Therefore, there exist limits

$$\lim_{\lambda\to +\infty} T(\lambda) = 0, \quad \lim_{\lambda\to -\infty} T(\lambda) = 0. \tag{19}$$

It is obvious that the function $T(\lambda)$ is positive and continuous for $\lambda \in (-\infty, +\infty)$. By virtue of (19), there exists

$$H = \max_{\lambda\in(-\infty,+\infty)} T(\lambda).$$

Thus, if $h/N < H$, then there exist at least two solutions of Eq. (17). Consequently, the following assertion holds.

Theorem 2. *There exists a number N^* (≥ 1) such that for each $N \ge N^*$ Eq. (17) has at least two solutions, with the solutions being distinct for distinct N. There exist infinitely many solutions $\lambda_N^{(+)}$ of Eq. (17) such that $\lambda_N^{(+)} > 0$ and $\lambda_N^{(+)} \to +\infty$ as $N \to \infty$. Further, there exist infinitely many solutions $\lambda_N^{(-)}$ of Eq. (17) such that $\lambda_N^{(-)} < 0$ and $\lambda_N^{(-)} \to -\infty$ as $N \to \infty$.*

Proof. For sufficiently large N we have the inequality $h/N < H$; therefore, it suffices to take $N^* > h/H$. The assumption about coinciding solutions for distinct N_1 and N_2 leads immediately to a contradiction, because then we have $T(\lambda) = h/N_1 = h/N_2$.

Further, by virtue of (19), $\lambda_N^{(+)}$ for large N is positive and tends to $+\infty$. Accordingly, $\lambda_N^{(-)}$ for large N is negative and tends to $-\infty$. The proof of the theorem is complete.

By means of a more detailed analysis of the function $T(\lambda)$, we can obtain a more accurate eigenvalue asymptotics, but we omit results on asymptotics from this paper.

3. NONLINEAR BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL OPERATOR

Consider the following nonlinear boundary eigenvalue problem for a more sophisticated differential operator of the second order:

$$(u')^m u'' - \lambda u + \alpha u^3 = 0, \quad x \in (0,h) \quad (m \ge 0), \tag{20}$$

with the homogeneous boundary conditions

$$u(0) = u(h) = 0 \tag{21}$$

and the additional condition

$$u'(0) = A, \quad A > 0. \tag{22}$$

We will seek smooth real-valued solutions $u = u(x)$, $u \in C^2(0, h) \cap C^1[0, h]$. The nonlinearity coefficient is $\alpha > 0$.

The eigenvalue problem (20)–(22) is to find $\lambda \in \mathbb{R}$ such that the problem has a nontrivial solution in the above-indicated function class.

Having multiplied Eq. (20) by $u'(x)$, we obtain $(u')^{m+1}u'' - \lambda uu' + \alpha u'u^3 = 0$, and hence we have $2((u')^{m+2})'/(m+2) - \lambda(u^2)' + (u^4)'/2 = 0$. The latter relation implies the identity $2(u')^{m+2}/(m+2) - \lambda u^2 + \alpha u^4/2 = C_0$, where C_0 is some constant. In view of the continuous differentiability of the function $u(x)$, by virtue of conditions (21) and (22) we conclude that $(u'(0))^{m+2}/(m+2) = C_0$ and $A^{m+2}/(m+2) = C_0 > 0$. Then

$$\frac{2}{m+2}(u')^{m+2} - \lambda u^2 + \alpha\frac{u^4}{2} = \frac{A^{m+2}}{m+2}. \tag{23}$$

Set

$$A_0 := \sqrt[m+2]{\frac{m+2}{2}}, \quad A_m^2 := \frac{2}{m+2}A^{m+2}.$$

For even m, from (23) we obtain the equation

$$u' = \pm A_0\left(A_m^2 + \lambda u^2 - \frac{\alpha}{2}u^4\right)^{1/(m+2)}, \tag{24}$$

and for odd m, the equation

$$u' = A_0\left(A_m^2 + \lambda u^2 - \frac{\alpha}{2}u^4\right)^{1/(m+2)}. \tag{25}$$

Let

$$D_m := \lambda^2 + 2\alpha A_m^2 \ (>0), \quad z_{1,m} := \frac{\lambda + \sqrt{D_m}}{\alpha} \ (>0), \quad z_{2,m} := \frac{\lambda - \sqrt{D_m}}{\alpha} \ (<0).$$

Then

$$A_m^2 + \lambda u^2 - \alpha u^4/2 = -\alpha(u^2 - z_{1,m})(u^2 - z_{2,m})/2. \tag{26}$$

Let us separately consider the cases of even and odd m. For an odd m, it follows from (25) and (26) that $u'(x) > 0$ for $|u(x)| < \sqrt{z_{1,m}}$ and $u'(x) < 0$ for $|u(x)| > \sqrt{z_{1,m}}$. An extremum can be reached only for $|u(x)| = \sqrt{z_{1,m}}$. Since $u'(0) > 0$ and $u(0) = 0$, we assume that the function $u(x)$ has a positive maximum at a point $x_0 \in (0, h)$, i.e., $u(x_0) = \sqrt{z_{1,m}}$. Then, in some right half-neighborhood of the point x_0, for $x > x_0$, either the function decreases and $u(x) < \sqrt{z_{1,m}}$, which contradicts the fact that $u'(x) > 0$, or the function takes constant values $u(x) = \sqrt{z_{1,m}} > 0$ for all $x > x_0, x \in (0, h)$, which is also impossible, because the function $u(x)$ is continuous and satisfies $u(h) = 0$. Thus, problem (20)–(22) has no solutions for odd m.

The case (24) of even m is similar to the case of $m = 0$, treated in Sec. 2. Denote

$$w_m := A_0{}^{-1}\left(A_m^2 + \lambda u^2 - \frac{\alpha}{2}u^4\right)^{-1/(m+2)}.$$

Applying the above-described method leads to the integral dispersion equations

$$NT_m(\lambda) = h \quad (N \geq 1), \tag{27}$$

where

$$T_m = T_m(\lambda) := \int\limits_{-\sqrt{z_{1,m}}}^{\sqrt{z_{1,m}}} w_m \, du. \tag{28}$$

Note that by virtue of (26), the improper integral (28) (similar to all the improper integrals above) is absolutely convergent. The following assertion holds.

Theorem 3. *If a number $\lambda = \lambda_0$ is an eigenvalue of problem (20)–(22), then it is a solution of the dispersion equation (27) for some $N \geq 1$. Conversely, if λ_0 is a solution of the dispersion equation (27) for some $N \geq 1$, then it is an eigenvalue of problem (20)–(22).*

The **proof** reproduces that of Theorem 1 conducted above for the case of $m = 0$.

Since $z_{1,m} z_{2,m} = -2A_m^2/\alpha$, we can make a change of variables in the integral (28) to obtain

$$T_m = \int\limits_{-\sqrt{z_{1,m}}}^{\sqrt{z_{1,m}}} w_m \, du = 2^{(m+3)/(m+2)} z_{1,m}^{m/(2m+4)} A_0^{-1} \int\limits_0^{\pi/2} \frac{\cos^{m/(m+2)} t \, dt}{(\alpha z_{1,m} \sin^2 t + 2A_m^2/z_{1,m})^{1/(m+2)}}.$$

The last integral can be estimated as

$$\int\limits_0^{\pi/2} \frac{\cos^{m/(m+2)} t \, dt}{(\alpha z_{1,m} \sin^2 t + 2A_m^2/z_{1,m})^{1/(m+2)}} \leq \int\limits_0^{\pi/2} \frac{dt}{(2A_m \sqrt{2\alpha} \sin t)^{1/(m+2)}} = M_m,$$

where M_m is a constant independent of $z_{1,m}$.

It follows that there exist limits

$$\lim_{z_1 \to 0} T_m(z_1) = 0, \quad \lim_{z_1 \to +\infty} T_m(z_1) = p_0 > 0, \quad m = 2; \quad \lim_{z_1 \to +\infty} T_m(z_1) = +\infty, \quad m > 2,$$

where

$$p_0 := 2^{5/4} A_0^{-1} \int\limits_0^{\pi/2} \frac{\cos^{1/2} t \, dt}{(\alpha \sin^2 t)^{1/4}}.$$

The case of $m = 0$ has been considered above.

Since

$$z_1 = z_1(\lambda) = \frac{\lambda + \sqrt{\lambda^2 + 2\alpha A^2}}{\alpha} = \frac{2A^2}{-\lambda + \sqrt{\lambda^2 + 2\alpha A^2}},$$

we have $z_1(\lambda) \to +\infty$ as $\lambda \to +\infty$ and $z_1(\lambda) \to 0$ as $\lambda \to -\infty$. Therefore, there exist limits

$$\lim_{\lambda \to +\infty} T_m(\lambda) = p_0 > 0, \quad m = 2; \quad \lim_{\lambda \to +\infty} T_m(\lambda) = +\infty, \quad m > 2; \quad \lim_{\lambda \to -\infty} T_m(\lambda) = 0. \qquad (29)$$

It is obvious that the function $T_m(\lambda)$ is positive and continuous for $\lambda \in (-\infty, +\infty)$. The following assertion holds by virtue of relations (29).

Theorem 4. *Let $m \geq 2$ be even. For $m = 2$, under the condition $p_0 > h/N$, Eq. (27) has at least one solution, the solutions being distinct for distinct N. For $m > 2$ Eq. (27) has at least one solution, the solutions being distinct for distinct N. Moreover, for $m \geq 2$ there exist infinitely many solutions $\lambda_N^{(-)}$ of Eq. (27) such that $\lambda_N^{(-)} < 0$ and $\lambda_N^{(-)} \to -\infty$ as $N \to \infty$.*

Proof. The condition $p_0 > h/N$ ensures at least one intersection of the graphs of the functions $y = T_m(\lambda)$ and $y = h/N$. To prove the remaining assertions, it suffices to reproduce reasoning similar to the proof of Theorem 2. The proof of the theorem is complete.

By means of a more detailed analysis of the function $T_m(\lambda)$, one can produce a more accurate eigenvalue asymptotics, but we omit results on asymptotics from this paper.

CONCLUSIONS

We have considered the integral dispersion equation method for solving two nonlinear eigenvalue problems arising when studying the propagation of TE-waves in a dielectric shielded nonlinear layer with a Kerr nonlinearity and a more complicated problem with a nonlinearity multiplying the highest derivative in the differential operator. The existence of an infinite set of eigenvalues of the problems is proved; their asymptotics are derived.

FUNDING

This work was supported by the Russian Foundation for Basic Research, project no. 19-31-51004.

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