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ORDINARY DIFFERENTIAL EQUATIONS

Lyapunov Vector Functions, Krasnosel'skii Canonical Domains, and Existence of Poisson Bounded Solutions

K. S. Lapin[1*]

[1] *Mordovian State Pedagogical Institute, Saransk, 430007 Russia*
e-mail: *klapin@mail.ru

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Abstract—We introduce the concepts of Poisson boundedness and partial Poisson boundedness for a solution of a differential system. These properties mean that the solution or, respectively, its projection onto a given subspace is contained in some ball for the values of an independent variable belonging to countably many intervals converging to infinity. Based on the method of Lyapunov vector functions and the Krasnosel'skii canonical domain method, sufficient conditions are obtained for the existence of such solutions.

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The foundations of the theory of stability of motion were laid in the classical monograph by A.M. Lyapunov [1], in which, in particular, the Lyapunov function method was developed for studying various types of stability. Numerous applications of the Lyapunov function method to problems of the qualitative theory of differential equations and mechanics are given in the classical papers [2–5] and many others.

We present results on the application and development of the method of Lyapunov functions that are directly related to the issues considered in this paper. Yoshizawa [5] considered applications of the Lyapunov function method to the study of the boundedness property of solutions of systems of differential equations. Rumyantsev and Oziraner [6, pp. 223–228] developed the theory of partial (with respect to part of the variables) boundedness of solutions on the basis of the method of Lyapunov functions. Matrosov [7, pp. 282–348] presented a method of Lyapunov vector functions that generalizes the method of Lyapunov functions and provides much greater opportunities for studying various types of boundedness of solutions in comparison with the latter. Krasnosel'skii [8] (see also [9]) developed the canonical domain method and the director function method and used them to derive sufficient conditions for the existence of solutions bounded on the entire real line for an arbitrary nonlinear system [8, pp. 178–206].

On the other hand, the paper [10] commenced a study of new types of boundedness of solutions and, in particular, their Poisson boundedness. This concept is characterized by the fact that the solution, together with solutions close to it at some point in time, may not be contained in some ball of the phase space for all values of the independent variable but possess the property of returning to this ball a countable number of times within the same time intervals. Note that the concept of uniform Poisson boundedness of a solution and the concept of positive Poisson stability of the trajectory of motion of a dynamical system (see, e.g., [11]) are dual to each other in the sense of rearranging some universality and existence quantifiers in the corresponding definitions. The study of the conditions for the existence of Poisson stable solutions of systems of differential equations was carried out in the monographs [12, pp. 199–224; 13, pp. 105–130]. In the cited paper [10], in particular, sufficient conditions were obtained under which all solutions of the system are uniformly bounded in the sense of Poisson.

In the present paper, we introduce the concept of Poisson boundedness of a solution of a system that, in contrast to the concept of uniform Poisson boundedness [10], does not impose any restrictions on the behavior of solutions close to the given solution. Sufficient conditions for the existence of Poisson bounded solutions are obtained on the basis of the method of Lyapunov vector functions and the Krasnosel'skii canonical domain method. The concept of partial Poisson boundedness of a solution of the system is also introduced. Based of the method of Lyapunov vector functions and the Krasnosel'skii canonical domain method, we establish conditions sufficient for the existence of solutions partially bounded in the sense of Poisson. Now let us proceed to rigorous definitions and statements.

Suppose we are given an arbitrary system of differential equations of n variables

$$\frac{dx}{dt} = F(t, x), \quad t \in \mathbb{R}^+ \equiv [0, +\infty), \quad x \in \mathbb{R}^n, \tag{1}$$

where the vector function $F(t, x) = (F_1(t, x), \ldots, F_n(t, x))^{\mathrm{T}}$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^n$. We also assume that $F(t, x)$ satisfies the Lipschitz condition with respect to variable x, and in addition, we require the extendability of solutions of system (1) to the entire time half-axis $\mathbb{R}^+$.

In what follows, $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{R}^n$. For a solution $x = x(t)$ of system (1) issuing from a point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, we use the notation $x = x(t, t_0, x_0)$. For each $t_0 \in \mathbb{R}^+$, by $\mathbb{R}^+(t_0)$ we denote the set $\{t \in \mathbb{R} : t \geq t_0\}$. In the sequel, any nonnegative monotone increasing numerical sequence $\tau = \{\tau_i\}_{i \geq 1}$ such that $\lim\limits_{i \to \infty} \tau_i = +\infty$ will be called a $\mathcal{P}$-sequence, and for each $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, by $M(\tau)$ we denote the set $\bigcup_{i=1}^{\infty} [\tau_{2i-1}, \tau_{2i}]$.

Recall [5] that a solution $x = x(t, t_0, x_0)$ of system (1) is said to be *bounded* if there exists a number $\beta > 0$ such that the condition $\|x(t, t_0, x_0)\| \leq \beta$ is satisfied for all $t \in \mathbb{R}^+(t_0)$.

Definition 1. A solution $x = x(t, t_0, x_0)$ of system (1) is said to be *Poisson bounded* if there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, where $t_0 \in M(\tau)$, and a number $\beta > 0$ such that the condition $\|x(t, t_0, x_0)\| \leq \beta$ is satisfied for all $t \in \mathbb{R}^+(t_0) \cap M(\tau)$.

Speaking the geometric language, Definition 1 implies that a solution starting at some point in time from within a ball of radius $\beta > 0$ with center at the origin will return to this ball countably many times. It is obvious that if a solution of system (1) is bounded, then this solution will also be bounded in the sense of Poisson.

A simple example of a system with solutions that are bounded in the sense of Poisson but are not, generally speaking, bounded is the system [14]

$$\dot{x}_1 = \frac{e^{-t} - \cos t - f(t, x_1, x_2)}{2(1 + \sin t + e^{-t})}(x_1 + x_2),$$
$$\dot{x}_2 = \frac{\cos t - e^{-t} + f(t, x_1, x_2)}{2(1 + \sin t + e^{-t})}(x_1 - x_2),$$

where $t \in \mathbb{R}^+$, $(x_1, x_2) \in \mathbb{R}^2$ and $f(t, x_1, x_2)$ is any nonnegative continuous function. For each solution of this system, the $\mathcal{P}$-sequence from Definition 1 is the sequence $\tau = \{\tau_i\}_{i \geq 1}$, where $\tau_1 = 0$ and $\tau_2 < \tau_3 < \ldots < \tau_i < \ldots$ is the sequence of all roots of the equation $\sin t + e^{-t} = 0$.

Following [6], recall some information on Lyapunov vector functions that will be needed below. Suppose that we are given a continuously differentiable vector function

$$V(t, x) = (V_1(t, x), \ldots, V_k(t, x))^{\mathrm{T}}, \quad k \geq 1, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

The derivative of this vector function according to system (1) is defined by the relation $\dot{V}(t, x) = (\dot{V}_1(t, x), \ldots, \dot{V}_k(t, x))^{\mathrm{T}}$, where $\dot{V}_i(t, x)$ is the derivative according to system (1) of the function $V_i(t, x)$, $1 \leq i \leq k$. In what follows, for vectors $\xi = (\xi_1, \ldots, \xi_k)^{\mathrm{T}}$ and $\eta = (\eta_1, \ldots, \eta_k)^{\mathrm{T}} \in \mathbb{R}^k$ we use the notation $\xi \leq \eta$ if $\xi_i \leq \eta_i$ for each $1 \leq i \leq k$. Suppose that we are also given a continuous vector function

$$f(t, \xi) = (f_1(t, \xi), \ldots, f_k(t, \xi))^{\mathrm{T}}, \quad (t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^k.$$

In what follows, we use the notation $f(t, \xi) \in W$ if $f(t, \xi)$ satisfies the *Ważewski condition*, which requires that for each $1 \leq s \leq k$ the function $f_s(t, \xi)$ does not decrease in the variables $\xi_1, \ldots, \xi_{s-1}, \xi_{s+1}, \ldots, \xi_k$; i.e., the fact that $\xi_i \leq \eta_i$, $1 \leq i \leq k$, $i \neq s$, and $\xi_s = \eta_s$ implies the inequality $f_s(t, \xi) \leq f_s(t, \eta)$. It is obvious that the condition $f(t, \xi) \in W$ is always satisfied for $k = 1$.

A continuously differentiable vector function $V(t, x) \geq 0$ $(0 \in \mathbb{R}^k)$ and a system

$$\frac{d\xi}{dt} = f(t, \xi), \quad f(t, \xi) \in W, \tag{2}$$

are called, respectively, a *Lyapunov vector function* and a *comparison system* for system (1) if the following condition is satisfied:

$$\dot{V}(t, x) \leq f(t, V(t, x)).$$

Hereinafter, we always assume that the right-hand side of system (2) satisfies the Lipschitz condition with respect to ξ. Since one has the uniqueness of solution of the Cauchy problem for system (2), it follows by the Ważewski theorem (see, e.g., [2, p. 236]) that for each point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ the solution $x(t, t_0, x_0)$ of system (1), the Lyapunov vector function $V(t, x)$, and the solution $\xi(t, t_0, V(t_0, x_0))$ of the comparison system (2) for system (1) are related, for all $t \geq t_0$, by the inequality

$$V(t, x(t, t_0, x_0)) \leq \xi(t, t_0, V(t_0, x_0)). \tag{3}$$

Following [8] (see also [9]), recall that a *Krasnosel'skii canonical domain* in $\mathbb{R}^k$ is any compact subset $\Omega \subset \mathbb{R}^k$ that has nonempty interior and satisfies the conditions

1. Ω is defined by finitely many inequalities

$$\Phi_i(\xi) \leq 0, \quad \xi \in \mathbb{R}^k, \quad 1 \leq i \leq r, \tag{4}$$

   where the functions $\Phi_i(\xi)$ are continuously differentiable.
2. If one has the equality $\Phi_{i_0}(\xi_0) = 0$ at a point ξ_0 of the boundary $\partial\Omega$ of the set Ω, then $\operatorname{grad} \Phi_{i_0}(\xi_0) \neq 0$.

It should be noted that, unlike [8] and [9], we do not presume the convexity of the domain Ω, because we do not consider here the issues of existence of periodic solutions.

In the sequel, for each Krasnosel'skii canonical domain Ω in $\mathbb{R}^k$ defined by inequalities (4) and for each point $\xi \in \partial\Omega$, by $\alpha(\xi)$ we will denote the set of those indices i for which the condition $\Phi_i(\xi) = 0$ is satisfied.

Let us state and prove a condition sufficient for the existence of Poisson bounded solutions of system (1) in terms of Krasnosel'skii canonical domains and Lyapunov vector functions.

Theorem 1. *Assume that for system (1) there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, a nonincreasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$ for which $b(r) \to +\infty$ as $r \to +\infty$, and a Lyapunov vector function $V(t, x) = (V_1(t, x), \ldots, V_k(t, x))^{\mathrm{T}}$ such that for all $(t, x) \in M(\tau) \times \mathbb{R}^n$ the condition*

$$b(\|x\|) \leq \sum_{i=1}^{k} V_i(t, x) \tag{5}$$

is satisfied. In addition, let Ω be a Krasnosel'skii canonical domain in $\mathbb{R}^k$ defined by inequalities (4) for which

$$\mathcal{D} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : \Phi_i(V(t, x)) \leq 0, \quad 1 \leq i \leq r\} \neq \varnothing,$$

and let the right-hand side $f(t, \xi)$ of the comparison system (2) for system (1) satisfy the inequality

$$(\operatorname{grad} \Phi_i(\xi), f(t, \xi)) \leq 0 \tag{6}$$

for any $(t, \xi) \in \mathbb{R}^+ \times \partial\Omega$ and $i \in \alpha(\xi)$. Then the solution $x(t, t_0, x_0)$ of system (1) is Poisson bounded for each point $(t_0, x_0) \in \mathcal{D}$.

Proof. First, let us show that each solution $\xi(t, t_0, \xi_0)$ of the comparison system (2) for system (1), where $(t_0, \xi_0) \in \mathbb{R}^+ \times \Omega$, is bounded. For system (2), consider the system

$$\frac{d\xi}{dt} = f(t, \xi) + \gamma(s_0 - \xi) \tag{7}$$

with a real parameter $\gamma > 0$, where s_0 is any fixed interior point in Ω. The geometrically obvious inequality $(\operatorname{grad} \Phi_i(\xi), s_0 - \xi) < 0$ for $\xi \in \partial\Omega$, $i \in \alpha(\xi)$, as well as condition (6), implies that the right-hand side of system (7) satisfies the inequality

$$(\operatorname{grad} \Phi_i(\xi), f(t, \xi) + \gamma(s_0 - \xi)) < 0 \tag{8}$$

for all $(t, \xi) \in \mathbb{R}^+ \times \partial\Omega$ and $i \in \alpha(\xi)$.

Choose any point $(t_0, \xi_0) \in \mathbb{R}^+ \times \Omega$ and for each fixed $\gamma > 0$ consider the solution $\xi_\gamma(t, t_0, \xi_0)$ of system (7). Let us show that for all $t \geq t_0$ one has the inclusion $\xi_\gamma(t, t_0, \xi_0) \in \Omega$. Assume the contrary: for the solution $\xi_\gamma(t, t_0, \xi_0)$ there exists a number $t'_\gamma > t_0$ such that $\xi_\gamma(t'_\gamma, t_0, \xi_0) \notin \Omega$. It follows from this assumption, with allowance for the continuity of the solution $\xi_\gamma(t, t_0, \xi_0)$ with respect to t and for the compactness of the set Ω, that there exists a $t_0 \leq \bar{t}_\gamma < t'_\gamma$ such that the relations $\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0) \in \Omega$ and $\xi_\gamma(t, t_0, \xi_0) \notin \Omega$ hold for the values of $t > \bar{t}_\gamma$ sufficiently close to $\bar{t}_\gamma$. It is obvious that $\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0) \in \partial\Omega$, and consequently, we have

$$\Phi_i(\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0)) = 0 \quad \text{for} \quad i \in \alpha(\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0)) \quad \text{and}$$
$$\Phi_i(\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0)) < 0 \quad \text{for} \quad i \notin \alpha(\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0)).$$

Now, using condition (8), we obtain the inequality

$$\left.\frac{\Phi_i(\xi_\gamma(t, t_0, \xi_0))}{dt}\right|_{t=\bar{t}_\gamma} < 0, \quad i \in \alpha(\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0)),$$

which implies that $\Phi_i(\xi_\gamma(t, t_0, \xi_0)) < 0$, $i \in \alpha(\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0))$, for the values of $t > \bar{t}_\gamma$ sufficiently close to $\bar{t}_\gamma$. For the same values of $t > \bar{t}_\gamma$, one can assume that $\Phi_i(\xi_\gamma(t, t_0, \xi_0)) < 0$ for $i \notin \alpha(\xi_\gamma(\bar{t}_\gamma, t_0, \xi_0))$, because the functions $\Phi_i(\xi)$ are continuous. Thus, for the values of $t > \bar{t}_\gamma$ sufficiently close to $\bar{t}_\gamma$ one has the inequalities

$$\Phi_i(\xi_\gamma(t, t_0, \xi_0)) < 0, \quad 1 \leq i \leq r;$$

i.e., $\xi_\gamma(t, t_0, \xi_0) \in \Omega$. We have arrived at a contradiction with the fact that $\xi_\gamma(t, t_0, \xi_0) \notin \Omega$ for the values of $t > \bar{t}_\gamma$ sufficiently close to $\bar{t}_\gamma$. The above assumption is, therefore, wrong, and consequently, the inclusion $\xi_\gamma(t, t_0, \xi_0) \in \Omega$ is satisfied with all $t \geq t_0$ for the above-indicated solution $\xi_\gamma(t, t_0, \xi_0)$ of system (7).

Now consider system (7) for $\gamma \geq 0$ and its solution $\xi_0(t, t_0, \xi_0)$ for $\gamma = 0$, i.e., the solution of system (2), which we will denote from now on by $\xi(t, t_0, \xi_0)$. Let us show that $\xi(t, t_0, \xi_0) \in \Omega$ for $t \geq t_0$. To this end, we choose any numerical sequence $\{\gamma_i\}_{i\geq 1}$, $\gamma_i > 0$, converging to zero. Since system (7), whose right-hand side is considered with the parameter $\gamma \geq 0$, satisfies the conditions of the theorem on the continuous dependence of solutions on a parameter (see, e.g., [15, pp. 109–110]), it follows that for each fixed number $t \geq t_0$ the sequence $(\xi_{\gamma_i}(t, t_0, \xi_0))_{i\geq 1}$ of points in $\mathbb{R}^k$ converges to the point $\xi(t, t_0, \xi_0) \in \mathbb{R}^k$. It follows that the inclusion $\xi(t, t_0, \xi_0) \in \Omega$ is satisfied for the solution $\xi(t, t_0, \xi_0)$ of system (2) with all $t \geq t_0$. Indeed, assume that one has the relation

$$\xi(\tau, t_0, \xi_0) \notin \Omega$$

for some $\tau > t_0$. Since the set Ω is closed and the sequence $(\xi_{\gamma_i}(\tau, t_0, \xi_0))_{i\geq 1}$ of points in $\mathbb{R}^k$ converges to the point $\xi(\tau, t_0, \xi_0) \notin \Omega$, we conclude that $\xi_{\gamma_i}(\tau, t_0, \xi_0) \notin \Omega$ for sufficiently large i. This contradicts the fact that $\xi_{\gamma_i}(\tau, t_0, \xi_0) \in \Omega$ for all $i \geq 1$. Therefore, the above assumption is wrong, and consequently, the considered solution $\xi(t, t_0, \xi_0)$ of system (2) satisfies the condition $\xi(t, t_0, \xi_0) \in \Omega$ for all $t \geq t_0$.

Since the set Ω is compact in $\mathbb{R}^k$, we conclude that there exists a ball of radius $\alpha > 0$ in $\mathbb{R}^k$ with center the origin that contains Ω, and consequently, for all $t \geq t_0$ one has the inequality $\|\xi(t, t_0, \xi_0)\| \leq \alpha$.

We have thus shown that for each point $(t_0, \xi_0) \in \mathbb{R}^+ \times \Omega$ the solution $\xi(t, t_0, \xi_0)$ of the comparison system (2) for system (1) is bounded. Using this result, we can now show that for each point $(t_0, x_0) \in \mathcal{D}$ the solution $x(t, t_0, x_0)$ of system (1) is bounded in the sense of Poisson. Using condition (5) and inequality (3), for the solution $x(t, t_0, x_0)$ of system (1) and the solution $\xi(t, t_0, V(t_0, x_0))$ of the comparison system (2) we obtain the inequalities

$$b(\|x(t, t_0, x_0)\|) \leq \sum_{i=1}^{k} V_i(t, x(t, t_0, x_0)) \leq \sum_{i=1}^{k} \xi_i(t, t_0, V(t_0, x_0)),$$

holding for all $t \in M(\tau)$. Moreover, for each $t \geq 0$ we have the obvious inequalities

$$\sum_{i=1}^{k} \xi_i(t, t_0, V(t_0, x_0)) \leq \sum_{i=1}^{k} |\xi_i(t, t_0, V(t_0, x_0))| \leq k\|\xi(t, t_0, V(t_0, x_0))\|.$$

Since $(t_0, x_0) \in \mathcal{D}$, we have $V(t_0, x_0) \in \Omega$, and consequently, $\|\xi(t, t_0, V(t_0, x_0))\| \leq \alpha$ for all $t \geq t_0$. Based on this, as well as the above inequalities, we conclude that the inequality $b(\|x(t, t_0, x_0)\|) \leq k\alpha$ holds for all $t \in \mathbb{R}^+(t_0) \cap M(\tau)$. Using the condition $b(r) \to \infty$ as $r \to \infty$ and the fact that the number $k\alpha$ is fixed, we choose a number $\beta > 0$ such that $k\alpha \leq b(\beta)$. In view of this, for all $t \in \mathbb{R}^+(t_0) \cap M(\tau)$ we obtain the inequality $b(\|x(t, x_0, t_0)\|) \leq b(\beta)$. Since the function $b(r)$ is nondecreasing, we see that the last inequality implies that $\|x(t, x_0, t_0)\| \leq \beta$ for all $t \in \mathbb{R}^+(t_0) \cap M(\tau)$. We have thus shown that the solution $x(t, t_0, x_0)$ of system (1) is bounded in the sense of Poisson for each point $(t_0, x_0) \in \mathcal{D}$. The proof of the theorem is complete.

In what follows, for each $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $n \geq 2$, and any fixed $1 \leq m < n$ we use the notation $x = (y, z)$, where $y = (x_1, \ldots, x_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $z = (x_{m+1}, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^{n-m}$.

Recall [6] that a solution$x(t, t_0, x_0)$ of system (1) is said to be *y-bounded* if there exists a number $\beta > 0$ such that the condition $\|y(t, t_0, x_0)\| \leq \beta$ is satisfied for all $t \in \mathbb{R}^+(t_0)$.

Definition 2. A solution $x = x(t, t_0, x_0)$ to system (1) is said to be *y-bounded in the sense of Poisson* if there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, where $t_0 \in M(\tau)$, and a number $\beta > 0$ such that the condition $\|y(t, t_0, x_0)\| \leq \beta$ is satisfied for all $t \in \mathbb{R}^+(t_0) \cap M(\tau)$.

It can readily be seen that if a solution of system (1) is y-bounded, then this solution is Poisson y-bounded.

In what follows, for each $\xi = (\xi_1, \ldots, \xi_k)^{\mathrm{T}} \in \mathbb{R}^k$ and any fixed number $1 \leq p \leq k$ we will use the notation $\xi = (\mu, \vartheta)$, where $\mu = (\xi_1, \ldots, \xi_p)^{\mathrm{T}} \in \mathbb{R}^p$ and $\vartheta = (\xi_{p+1}, \ldots, \xi_k)^{\mathrm{T}} \in \mathbb{R}^{k-p}$.

The following assertion, which can be proved by analogy with Theorem 1, is a sufficient condition for the existence of Poisson y-bounded solutions of system (1).

Theorem 2. *Suppose that for system (1) there exists a $\mathcal{P}$-sequence $\tau = \{\tau_i\}_{i \geq 1}$, a nonincreasing function $b : \mathbb{R}^+ \to \mathbb{R}^+$ for which $b(r) \to +\infty$ as $r \to +\infty$, a Lyapunov vector function $V(t, x) = (V_1(t, x), \ldots, V_k(t, x))^{\mathrm{T}}$, and a number $1 \leq p \leq k$ such that the condition*

$$b(\|y\|) \leq \sum_{i=1}^{p} V_i(t, x)$$

is satisfied for all $(t, x) \in M(\tau) \times \mathbb{R}^n$. In addition, assume that there exists a Krasnosel'skii canonical domain Ω in $\mathbb{R}^p$ defined by inequalities (4) in which $\xi \in \mathbb{R}^k$ has been replaced by $\mu \in \mathbb{R}^p$, for which

$$\mathcal{D}^\mu = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : \Phi_i(V^\mu(t, x)) \leq 0, \quad 1 \leq i \leq r\} \neq \varnothing,$$

where $V^\mu(t, x) = (V_1(t, x), \ldots, V_p(t, x))^{\mathrm{T}}$. Finally, let all solutions of the comparison system (2) for system (1) be extendible to the half-axis $\mathbb{R}^+$, and let the right-hand side $f(t, \xi)$ of the comparison system satisfy the inequality

$$(\operatorname{grad} \Phi_i(\mu), f^\mu(t, \xi)) \leq 0$$

for any $(t, \xi) \in \mathbb{R}^+ \times (\partial\Omega \times \mathbb{R}^{k-p})$ and $i \in \alpha(\mu)$, where $f^\mu(t, \xi) = (f_1(t, \xi), \ldots, f_p(t, \xi))^{\mathrm{T}}$. Then the solution $x(t, t_0, x_0)$ of system (1) is y-bounded in the sense of Poisson for each point $(t_0, x_0) \in \mathcal{D}^\mu$.

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