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ORDINARY DIFFERENTIAL EQUATIONS

Lyapunov Irregularity Coefficient as a Function of the Parameter for Families of Linear Differential Systems Whose Dependence on the Parameter Is Continuous Uniformly on the Time Half-Line

E. A. Barabanov[1*] **and V. V. Bykov**[2**]

[1]*Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, 220072 Belarus*
[2]*Lomonosov Moscow State University, Moscow, 119991 Russia*
e-mail: *bar@im.bas-net.by, ***vvbykov@gmail.com*

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Abstract—We consider families of n-dimensional ($n \geq 2$) linear differential systems on the time half-line with parameter belonging to a metric space. We obtain a complete description of the Lyapunov irregularity coefficient as a function of the parameter for families whose dependence on the parameter is continuous in the sense of uniform convergence on the time half-line. As a corollary, we completely describe the parametric dependence of the Lyapunov irregularity coefficient of a regular linear system with a linear parametric perturbation decaying at infinity uniformly with respect to the parameter.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

For a positive integer n, let $\mathcal{M}_n$ be the class of linear differential systems

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

whose coefficient matrix $A(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is piecewise continuous and bounded on the time half-line $\mathbb{R}_+$. In what follows, we identify system (1) with its coefficient matrix and write $A \in \mathcal{M}_n$. For a system $A \in \mathcal{M}_n$, let $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$ be its Lyapunov exponents [1, p. 561], and let $\sigma_{\mathrm{L}}(A)$ be its Lyapunov irregularity coefficient [1, p. 563]; i.e.,

$$\sigma_{\mathrm{L}}(A) = \sum_{i=1}^{n} \lambda_i(A) - \varliminf_{t\to+\infty} \frac{1}{t}\int_0^t \operatorname{tr} A(\tau)\, d\tau,$$

where tr is the trace of a matrix. By virtue of the Lyapunov inequality [2, p. 72], the quantity $\sigma_{\mathrm{L}}(A)$ is nonnegative.

The Lyapunov irregularity coefficient is one of the most important asymptotic characteristics of systems in the class $\mathcal{M}_n$. The condition $\sigma_{\mathrm{L}}(A) = 0$ singles out the subclass $\mathcal{R}_n \subset \mathcal{M}_n$ of Lyapunov regular systems, historically the first class of systems for which the problem of conditional stability by the first approximation has been proved to be solved in the affirmative [1, pp. 563–566]. Moreover, this coefficient is used to state sufficient conditions characterizing the response of a system $A \in \mathcal{M}_n$ to exponentially decaying linear and higher-order infinitesimal nonlinear perturbations. For example, the Lyapunov exponents of a system A are preserved under linear exponentially decaying perturbations $Q(\cdot)$ satisfying the estimate [3] $\|Q(t)\| \leq C\exp(-\sigma t)$, $t \in \mathbb{R}_+$, with some positive constants C and σ provided that $\sigma > \sigma_{\mathrm{L}}(A)$. (In particular, linear exponentially decaying perturbations do not affect the Lyapunov exponents of regular systems.) If the order $m > 1$ of a higher-order infinitesimal perturbation $f(t,x)$ ($\|f(t,x)\| \leq \mathrm{const}\|x\|^m$, $t \in \mathbb{R}_+$) of a system A

satisfies the estimate $(m-1)\lambda_n(A)+\sigma_{\mathrm{L}}(A)<0$, then the trivial solution of the perturbed system is stable (the Lyapunov–Massera theorem [1, pp. 577–579; 4]).

Along with the Lyapunov irregularity coefficient, the Perron $\sigma_{\mathrm{P}}(A)$ [5] and Grobman $\sigma_{\mathrm{G}}(A)$ [6] irregularity coefficients are defined for a system $A\in\mathcal{M}_n$ as well (e.g., see [2, pp. 67, 73] or [7, pp. 9–10]). For these coefficients, we have the inequalities $0\le\sigma_{\mathrm{P}}(A)\le\sigma_{\mathrm{G}}(A)\le\sigma_{\mathrm{L}}(A)\le n\sigma_{\mathrm{P}}(A)$ [7, p. 13], which, as established in [8], determine all possible relations between these coefficients in the class $\mathcal{M}_n$. It follows from the above inequalities that the vanishing of at least one of these coefficients is equivalent to the Lyapunov regularity of the system A. The role of the Grobman irregularity coefficient is quite similar to that of the Lyapunov irregularity coefficient. (In particular, the above assertions remain valid if we replace the quantity $\sigma_{\mathrm{L}}(A)$ appearing in them with the quantity $\sigma_{\mathrm{G}}(A)$ not exceeding it.) This is far from being the case for the Perron irregularity coefficient, and its properties have been studied to a much lesser extent (see [7, pp. 303–307]).

Let M be a metric space. Consider a family

$$\dot{x}=A(t,\mu)x,\qquad x\in\mathbb{R}^n,\qquad t\in\mathbb{R}_+,\tag{2}$$

of linear differential systems depending on a parameter $\mu\in M$ such that the matrix-valued function $A(\cdot,\mu)\colon\mathbb{R}_+\to\mathbb{R}^{n\times n}$ on the time half-line $\mathbb{R}_+$ is continuous and bounded (by a constant depending on μ) for each $\mu\in M$. Therefore, for each $\mu\in M$ we obtain a linear differential system in the family (2) with continuous coefficients bounded on the half-line. We denote the Lyapunov exponents of this system by $\lambda_1(\mu;A)\le\ldots\le\lambda_n(\mu;A)$ and its Lyapunov irregularity coefficient by $\sigma_{\mathrm{L}}(\mu;A)$.

The family of matrix-valued mappings $A(\cdot,\mu)$, $\mu\in M$, is commonly considered under one of the following two natural assumptions: this family is continuous in either **(a)** the compact-open topology or **(b)** the uniform topology. Condition **(a)** is equivalent to saying that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points in M converges to a point μ_0, then the sequence of functions $A(\cdot,\mu_k)$ of the variable $t\ge0$ converges to the function $A(\cdot,\mu_0)$ as $k\to+\infty$ uniformly on each closed interval of the half-line $\mathbb{R}_+$, while condition **(b)** implies that the convergence is uniform on the entire time half-line $\mathbb{R}_+$. We denote the classes of families (2) continuous in the above sense in the compact-open topology and in the uniform topology by $\mathcal{C}^n(M)$ and $\mathcal{U}^n(M)$, respectively. We obviously have the inclusion $\mathcal{U}^n(M)\subset\mathcal{C}^n(M)$. In the sequel, we identify the family (2) with the matrix-valued function $A(\cdot,\cdot)$ defining it and hence write $A\in\mathcal{C}^n(M)$ or $A\in\mathcal{U}^n(M)$.

Along with the class $\mathcal{U}^n(M)$, consider the subclass $\mathcal{UZ}^n_{\mathcal{R}}(M)$ defined as follows. For a number $n\in\mathbb{N}$ and a metric space M, by $\mathcal{Z}_n(M)$ we denote the class of jointly continuous matrix-valued functions $Q(\cdot,\cdot)\colon\mathbb{R}_+\times M\to\mathbb{R}^{n\times n}$ decaying to zero uniformly with respect to $\mu\in M$ (i.e., satisfying the condition $\sup_{\mu\in M}\|Q(t,\mu)\|\to0$ as $t\to+\infty$). The class $\mathcal{UZ}^n_{\mathcal{R}}(M)$ consists of the families

$$\dot{x}=(B(t)+Q(t,\mu))x,\qquad x\in\mathbb{R}^n,\qquad t\in\mathbb{R}_+,\tag{3}$$

where $B\in\mathcal{R}_n$ and $Q\in\mathcal{Z}_n(M)$. We denote the coefficient matrix of the family (3) by $A(t,\mu)$ and, as above, identify it with the family itself and write $A\in\mathcal{UZ}^n_{\mathcal{R}}(M)$.

The aim of the present paper is to give a complete descriptive function-theoretic characterization of each of the function classes

$$\mathfrak{S}[\mathcal{U}^n(M)]=\{\sigma_{\mathrm{L}}(\cdot\,;A):A\in\mathcal{U}^n(M)\}\quad\text{and}\quad\mathfrak{S}[\mathcal{UZ}^n_{\mathcal{R}}(M)]=\{\sigma_{\mathrm{L}}(\cdot\,;A):A\in\mathcal{UZ}^n_{\mathcal{R}}(M)\}$$

for any $n\in\mathbb{N}$ and any metric space M.

2. PRELIMINARIES ON DESCRIPTIVE SET THEORY AND DESCRIPTIVE FUNCTION THEORY

To make the presentation self-contained, let us give the necessary definitions from descriptive set theory and descriptive function theory [9].

Let $f(\cdot)$ be a real-valued function defined on a set $\mathcal{M}$. For each $r\in\mathbb{R}$, the *Lebesgue set* $[f\ge r]$ of the function $f(\cdot)$ is defined to be the set $[f\ge r]=\{t\in\mathcal{M}\colon f(t)\ge r\}$, i.e., the preimage of the half-open interval $[r,+\infty)$ under the mapping f. A function $g\colon\mathcal{M}\to\mathbb{R}$ is called a *majorant* of a function $f\colon\mathcal{M}\to\mathbb{R}$ if the inequality $g(\mu)\ge f(\mu)$ holds for all $\mu\in\mathcal{M}$.

If $\mathcal{M}$ is a topological space, then its first three Borel classes of sets are known to be defined as follows [9, p. 203]. The zero class consists of closed and open sets (their classes being denoted by F and G, respectively). The first class consists of sets of the type G_δ and of the type F_σ (the G_δ- and F_σ-sets), i.e., the sets that can be represented in the form of countable intersections of open sets or countable unions of closed sets, respectively. The second class includes sets of the type $F_{\sigma\delta}$ and of the type $G_{\delta\sigma}$ (the $F_{\sigma\delta}$- and $G_{\delta\sigma}$-sets), i.e., the sets that can be represented in the form of countable intersections of F_σ-sets or countable unions of G_δ-sets, respectively. We do not need Borel classes with higher numbers in the present paper.

Let $\mathfrak{N}$ be a system of subsets of a set $\mathcal{M}$. We say [9, pp. 266–267] that a function $f(\cdot)\colon \mathcal{M} \to \mathbb{R}$ *belongs to the class* $(^*, \mathfrak{N})$, or *is a function of the class* $(^*, \mathfrak{N})$, if for each $r \in \mathbb{R}$ its Lebesgue set $[f \geq r]$ belongs to the system $\mathfrak{N}$. In what follows, we only need functions $M \to \mathbb{R}$ of the class $(^*, G_\delta)$, i.e., functions such that the preimage of the half-open interval $[r, +\infty)$ is a G_δ-set in the metric space M for each $r \in \mathbb{R}$.

3. SURVEY OF PREVIOUS RESULTS

The direction in the theory of Lyapunov exponents dealing with the dependence of asymptotic properties and characteristics of parametric differential systems on the parameter is due to V.M. Millionshchikov, who initiated systematic research in this direction with a series of papers, of which we only mention the paper [10]. We are also indebted to him for understanding that the language of the Baire theory of discontinuous functions is an adequate language for describing such a dependence. We point out that here one speaks of a complete description of all possible types of behavior of some properties or characteristics of a system under changes in the system parameters. Since then, quite a few results have been obtained, of which we will only mention those directly related to the problems considered in the present paper, i.e., concerning the dependence of the regularity property of parametric linear systems on the parameter.

The analysis of the parameter dependence of the regularity property of parametric linear systems started from Bogdanov's problem on the existence of a system $A \in \mathcal{R}_n$ such that the system

$$\dot{x} = \mu A(t)x, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}_+, \tag{4}$$

is irregular for some $\mu \in \mathbb{R}$. Although this problem is known as Bogdanov's problem, Bogdanov himself repeatedly stated that it had been posed as early as in the 1950s at Erugin's seminar at Leningrad State University. Bogdanov's problem was solved in the affirmative in [11]: it was established that for each number $n \geq 2$ there exists a system $A \in \mathcal{R}_n$ that becomes irregular if its coefficient matrix is multiplied by some real number. We assign the set $W_A = \{\mu \in \mathbb{R} : \mu A \notin \mathcal{R}_n\}$, called the *irregularity set* of the system A, to each system $A \in \mathcal{M}_n$. Following the result in [11], Izobov [12] posed the problem of finding a complete description of the class $\mathcal{W}^n = \{W_A : A \in \mathcal{M}_n\}$ of irregularity sets. Various examples of particular irregularity sets were constructed in the papers [13–17]. It was proved in [18] and [19] that an arbitrary open or closed, respectively, set on the real line not containing zero is the irregularity set of some linear system in $\mathcal{M}_n$. At the same time, it was shown in [15] that the irregularity sets are $G_{\delta\sigma}$-sets not containing zero. The converse is true as well [18] if the coefficient matrix of the system is allowed to be unbounded; this gives a complete description of the class of irregularity sets for systems (1) with, generally speaking, unbounded coefficients. Describing the irregularity sets of the systems in $\mathcal{M}_n$ is an open problem yet.

Irregularity sets can be considered not only for the families (4) but also for the families in the above-introduced classes $\mathcal{C}^n(M)$, $\mathcal{U}^n(M)$, and $\mathcal{UZ}^n_{\mathcal{R}}(M)$. For example, the irregularity set of a family $A \in \mathcal{C}^n(M)$ consists of $\mu \in M$ such that the system $A(\cdot, \mu)$ is irregular. In particular, let us introduce the compact-open and uniform topologies on the class $\mathcal{M}_n$ and denote the resulting metric spaces by $\mathcal{M}^{\mathrm{c}}_n$ and $\mathcal{M}^{\mathrm{u}}_n$. The spaces $\mathcal{M}^{\mathrm{c}}_n$ and $\mathcal{M}^{\mathrm{u}}_n$ can be viewed as elements of the classes $\mathcal{C}^n(M)$ and $\mathcal{U}^n(M)$, respectively, in which the parameter space M coincides with the respective space itself; i.e., $M = \mathcal{M}^{\mathrm{c}}_n$ and $M = \mathcal{M}^{\mathrm{u}}_n$, respectively. Vetokhin [20] generalized the result in [15] on the Borel type of the irregularity set of the families (4): if $n \geq 2$, then the set $\mathcal{R}_n$ of regular systems is an $F_{\sigma\delta}$-set and is not a $G_{\delta\sigma}$-set in the space $\mathcal{M}^{\mathrm{u}}_n$; in particular, it follows from this result that the irregularity sets of the families (4) must be $G_{\delta\sigma}$-sets. It was proved in [18] that a subset of a metric space M is an irregularity set for some family in $\mathcal{C}^n(M)$ if and only if it is a $G_{\delta\sigma}$-set in this space.

The fact that the points of discontinuity of the functionals $\lambda_k\colon \mathcal{M}_n^{\mathrm{u}} \to \mathbb{R}$, $k = 1, \ldots, n$, $n \geq 2$, include regular systems was established by Vinograd as early as in [21, 22]; we return to this important result in Section 4.2. This result was strengthened in the paper [23]: even the restrictions of these functionals to the class of regular systems are discontinuous. It was proved in [24] that these restrictions are neither lower nor upper semicontinuous, while the restrictions of the functionals $\lambda_k\colon \mathcal{M}_n^{\mathrm{c}} \to \mathbb{R}$, $k = 1, \ldots, n$, $n \geq 2$, to the class of regular systems are exactly of the first Baire class.

Various examples of the possible dependence of the Lyapunov exponents and the characteristic function of the set of regular systems on the parameter $\mu \in \mathbb{R}$ for families in $\mathcal{U}^n(\mathbb{R})$, $n \geq 2$, were constructed in [25, 26]. Namely, a family for which all these functions are everywhere discontinuous was constructed in [25], and a family for which these functions are step functions was constructed in [26]. (In the latter example, the system is regular and all of its Lyapunov exponents are zero for $\mu \leq 0$, while for $\mu > 0$ it is irregular and all of its Lyapunov exponents are equal to unity.) The latter result was strengthened in the papers [27, 28], where it was shown that among families in $\mathcal{U}^n(\mathbb{R})$, $n \geq 2$, with a step-like dependence there exist families that are infinitely differentiable with respect to the parameter and such that their matrix $A(\cdot\,;\mu)$ is obtained from the matrix $A(\cdot\,;0)$ by some (depending on μ) perturbation decaying at infinity.

4. MAIN THEOREM

For families in the class $\mathcal{U}^n(M)$, a complete description of their Lyapunov irregularity coefficient is given by the following assertion.

Theorem 1. *For each metric space M, a function $\sigma\colon M \to \mathbb{R}_+$ is the Lyapunov irregularity coefficient of some family in $\mathcal{U}^n(M)$ (i.e., belongs to the class $\mathfrak{S}[\mathcal{U}^n(M)]$) if and only if*

(a) *It is continuous for $n = 1$.*

(b) *It belongs to the class $({}^*, G_\delta)$ and has a continuous majorant for $n \geq 2$.*

Remark. The description of the class $\mathfrak{S}[\mathcal{C}^n(M)] = \{\sigma_{\mathrm{L}}(\cdot\,;A) : A \in \mathcal{C}^n(M)\}$ follows from the result in [29] and is as follows: for each $n \in \mathbb{N}$ and each metric space M, a function $\sigma\colon M \to \mathbb{R}_+$ is the Lyapunov irregularity coefficient of some family in $\mathcal{C}^n(M)$ if and only if it belongs to the class $({}^*, G_\delta)$. This description also obviously follows from the more general result in [30] providing a complete description of the class of vector functions $\{(\sigma_{\mathrm{L}}(\cdot\,;A), \sigma_{\mathrm{P}}(\cdot\,;A)) : A \in \mathcal{C}^n(M)\}$ composed of the Lyapunov and Perron irregularity coefficients of families in $\mathcal{C}^n(M)$: for each $n \in \mathbb{N}$ and each metric space M, a vector function $(\sigma_1, \sigma_2)\colon M \to \mathbb{R}_+^2$ belongs to that class if and only if the functions σ_1 and σ_2 are functions of the class $({}^*, G_\delta)$ and the inequalities

$$0 \leq \sigma_2(\mu) \leq \sigma_1(\mu) \leq n\sigma_2(\mu)$$

hold for each $\mu \in M$.

Since $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$, we have the inclusion $\mathfrak{S}[\mathcal{U}^n(M)] \subset \mathfrak{S}[\mathcal{C}^n(M)]$. Let us show that, generally speaking, the inclusion is proper. To this end, it suffices to give an example of a nonnegative function in the class $({}^*, G_\delta)$ that does not have a continuous majorant. Consider the function

$$\sigma(\mu) = \begin{cases} |\mu|^{-1} & \text{for} \quad \mu \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{for} \quad \mu = 0. \end{cases}$$

This function is obviously nonnegative and belongs to the class $({}^*, G_\delta)$. Indeed, its Lebesgue set $[\sigma \geq r]$ is the union of half-open intervals $[-r^{-1}, 0)$ and $(0, r^{-1}]$ if $r > 0$ and the real line $\mathbb{R}$ if $r \leq 0$. The first of these sets is a G_δ-set, because $[-r^{-1}, 0) \cup (0, r^{-1}] = \bigcap_{k\in\mathbb{N}}((-r^{-1} - k^{-1}, 0) \cup (0, r^{-1} + k^{-1}))$. However, this function does not have a continuous majorant, because it is unbounded in any neighborhood of the point $x = 0$.

5. PROOF OF THE THEOREM FOR $n = 1$

Necessity. Given a family $A \in \mathcal{U}^1(M)$, i.e., the equation

$$\dot{x} = A(t, \mu)x, \qquad x \in \mathbb{R}^1, \qquad t \in \mathbb{R}_+, \tag{5}$$

it is clear that each of the functions

$$\lambda(\mu) = \varlimsup_{t\to+\infty} \frac{1}{t}\int_0^t A(\tau,\mu)d\tau \quad\text{and}\quad s(\mu) = \varliminf_{t\to+\infty} \frac{1}{t}\int_0^t A(\tau,\mu)d\tau \tag{6}$$

is a continuous function $M \to \mathbb{R}$. Indeed, if $\mu_k \to \mu_0$ as $k \to +\infty$, then, according to the definition of the class $\mathcal{U}^1(M)$, for each $\varepsilon > 0$ there exists a $k(\varepsilon) \in \mathbb{N}$ such that $|A(t,\mu_k) - A(t,\mu_0)| \le \varepsilon$ for all $t \in \mathbb{R}_+$ and $k \ge k(\varepsilon)$. Hence for these k we have the inequalities $|\lambda(\mu_k) - \lambda(\mu_0)| \le \varepsilon$ and $|s(\mu_k) - s(\mu_0)| \le \varepsilon$; this implies the continuity of the functions (6). Therefore, the Lyapunov irregularity coefficient $\sigma(\mu) = \lambda(\mu) - s(\mu)$ of Eq. (5) is a continuous function, and it is obvious that it is nonnegative.

Sufficiency. Let $\sigma\colon M \to \mathbb{R}_+$ be a continuous function. Consider a sequence $(T_k)_{k\in\mathbb{N}}$ monotone increasing to $+\infty$ of points on the time half-line such that $T_1 = 0$ and $T_{k+1}/T_k \to +\infty$ as $k \to +\infty$. Define a function $\widetilde{A}(\cdot\,;\cdot)\colon \mathbb{R}_+ \times M \to \mathbb{R}_+$ by the relation

$$\widetilde{A}(t,\mu) = \begin{cases} \sigma(\mu) & \text{if } \ t \in [T_{2l-1}, T_{2l}), \\ 0 & \text{if } \ t \in [T_{2l}, T_{2l+1}), \end{cases} \qquad l \in \mathbb{N}.$$

Set $T'_k = T_{k+1} - (T_{k+1} - T_k)/k$, $k \ge 1$. Based on the function $\widetilde{A}(t,\mu)$, we construct the continuous function $A(\cdot,\cdot)\colon \mathbb{R}_+ \times M \to \mathbb{R}_+$ that coincides with $\widetilde{A}(t,\mu)$ for $(t,\mu) \in [T_k, T'_k] \times M$, $k \ge 1$, and is the linear function connecting the points $(T'_k, \widetilde{A}(T'_k,\mu))$ and $(T_{k+1}, \widetilde{A}(T_{k+1},\mu))$ on the interval $[T'_k, T_{k+1}]$ for each $\mu \in M$. The continuity of the function $A(\cdot,\cdot)$ follows from the continuity of the function $\sigma(\cdot)$ and the continuity of linear functions.

It is obvious that the functions (6) for the function $A(\cdot,\cdot)$ thus constructed satisfy the inequalities $\lambda(\mu) \le \sigma(\mu)$ and $s(\mu) \ge 0$ for all $\mu \in M$. On the other hand, by the definition of the function $A(\cdot,\cdot)$, we have the inequalities

$$\int_0^{T_{2l}} A(\tau,\mu)\,d\tau \ge \sigma(\mu)(T'_{2l-1} - T_{2l-1}) \quad\text{and}\quad \int_0^{T'_{2l}} A(\tau,\mu)\,d\tau \le \sigma(\mu)T_{2l},$$

which, together with the fact that $T'_k/T_{k+1} \to 1$ as $k \to \infty$, imply the opposite inequalities $\lambda(\mu) \ge \sigma(\mu)$ and $s(\mu) \le 0$. Therefore, the Lyapunov irregularity coefficient of Eq. (5) with such a function $A(\cdot,\cdot)$ coincides with the function σ. The proof of the theorem for $n = 1$ is complete.

6. PROOF OF THE THEOREM FOR $n \ge 2$

Necessity. Consider a family $A \in \mathcal{U}^n(M)$, $n \ge 2$. It was established in [31] that each of the functions $\lambda_i(\cdot\,;A)$, $i = 1,\dots,n$, belongs to the class $({}^*, G_\delta)$ and has a continuous majorant. In the same way as in the proof of necessity for $n = 1$, one can prove that the function

$$\mu \mapsto \varliminf_{t\to+\infty} t^{-1}\int_0^t \operatorname{tr} A(\tau,\mu)\,d\tau$$

is continuous. Hence the function $\sigma_{\mathrm{L}}(\cdot\,;A)$ has a continuous majorant. Since every continuous function belongs to the class $({}^*, G_\delta)$, it follows by virtue of [9, p. 267, II] that so does the function $\sigma_{\mathrm{L}}(\cdot\,;A)$. The nonnegativity of the function $\sigma_{\mathrm{L}}(\cdot\,;A)$ follows from the Lyapunov inequality [2, p. 72]. The proof of the necessity of the assumptions in the theorem for $n \ge 2$ is complete.

Sufficiency. Consider a function $\sigma\colon M \to \mathbb{R}_+$ belonging to the class $({}^*, G_\delta)$ and having a continuous majorant $b\colon M \to \mathbb{R}_+$. We will construct a family $A \in \mathcal{U}^n(M)$ for which $\sigma_{\mathrm{L}}(\mu; A) = \sigma(\mu)$ for all $\mu \in M$. The construction of the desired family splits into several steps.

6.1. Preliminary Conventions

The following three conventions–definitions make for a convenient language in the subsequent reasoning.

Given a differential system $\dot{x} = A(t)x$ (a system A) defined only for $t \in [0, \delta)$, the expression "repeat the system A k times" implies considering the system

$$\dot{x} = A_k(t)x, \qquad t \in [0, k\delta),$$

with the matrix $A_k(\cdot)$ defined by the formulas

$$A_k(t) \equiv A(t - i\delta) \quad \text{for} \quad t \in [i\delta, (i+1)\delta), \quad i = 0, \ldots, k-1.$$

The expression "repeat the system A k times starting from the point $t = T$" implies considering the system

$$\dot{x} = A_{k,T}(t)x, \qquad t \in [T, T + k\delta),$$

where $A_{k,T}(t) \equiv A_k(t - T)$. Thus, from now on a pair of subscripts on a system (matrix), e.g., $A_{k,T}$, indicates the system (matrix) associated with the already defined system (matrix) A by the above relations.

Given a linear differential system $\dot{x} = B(t)x$ (a system B) defined only for $t \in [\tau, T)$, the expression "extend the system B oddly beyond the point T" implies considering the system

$$\dot{x} = B^*(t)x, \qquad t \in [T, 2T - \tau)$$

on the half-open interval $[T, 2T - \tau)$, where $B^*(t) \equiv -B(2T - t)$ for $t \in [T, 2T - \tau)$. Thus, from now on a system (matrix) with a superscript $*$, e.g., B^*, implies the system (matrix) associated with the already defined system (matrix) B by the indicated relations. Obviously, the following assertion (let us call it property $\mathcal{A}$), which is important in the sequel, holds: if the coefficient matrix of a system B is piecewise constant, then the Cauchy matrix $X_{B^*}(\cdot, \cdot)$ of the odd extension system B^* satisfies the relation $X_{B^*}(T + t, T) = X_B(T - t, T)$ for all $t \in [0, T - \tau]$. In particular, $X_{B^*}(2T - \tau, T)X_B(T, \tau) = E$, where E is the identity matrix.

Let us also adopt the convention that if a pair of subscripts and a superscript $*$ occur in the notation of a matrix (system), then the operation of repetition comes first, followed by the operation of extension; i.e., the notation $A^*_{k,T}$ means that we first repeat the system A k times starting from the point $t = T$ and then extend the resulting system oddly.

In all subsequent constructions, we use the operations of (i) repeating some system the required number of times starting from a certain point and (ii) extending the respective system oddly, as well as an operation (to be introduced below) of (iii) a special perturbation of an appropriate system belonging to a class defined below. Operations (i) and (ii) have been defined above, and operation (iii) is described in the next subsection.

6.2. Preliminary Constructs

Let us proceed to operation (iii) and the respective class (family) of systems. Let us specify the class of systems to be perturbed.

6.2.1. Family of unperturbed systems defined on finite intervals. Let us define a family of linear two-dimensional systems depending on two parameters $m > 0$ and $b > 0$. Their coefficient matrix will be denoted by $B[m; b](t)$. The matrix $B[m; b](t)$ is defined only on the interval $\Delta(m) \overset{\text{def}}{=} [0, 4(m + 1)]$, and to define it, we will need notation for some points of this interval. We denote

$$T_0^m = 0, \qquad T_i^m = T_{i-1}^m + \begin{cases} 1 & \text{if} \ \ i = 2, 4, 5, 7, \\ m & \text{if} \ \ i = 1, 3, 6, 8, \end{cases} \qquad i = 1, \ldots, 8,$$

and $\Delta_i(m) = [T_{i-1}^m, T_i^m]$, $i = 1, \ldots, 8$; i.e., the length of the interval $\Delta_i(m)$ is m if $i = 1, 3, 6, 8$ and 1 if $i = 2, 4, 5, 7$. By $\Delta_i^\circ(m)$ we denote the interior of the interval $\Delta_i(m)$, $i = 1, \ldots, 8$. The matrix

$B[m;b](t)$ is given by the relation

$$B[m;b](t)=\begin{cases}\operatorname{diag}[b,-b] & \text{for } t\in\Delta_1(m)\bigsqcup\Delta_8(m),\\ \operatorname{diag}[-b,b] & \text{for } t\in\Delta_3(m)\bigsqcup\Delta_6(m),\\ 2^{-1}\pi J_2 & \text{for } t\in\Delta_4^\circ(m),\\ O_2 & \text{for other } t\in\Delta(m),\end{cases}$$

where

$$J_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\quad\text{and}\quad O_2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Obviously, the trace $\operatorname{tr}B[m;b](t)$ of the matrix $B[m;b](t)$ is zero for all $t\in\Delta(m)$. For the Cauchy matrix $X_{B[m;b]}(\cdot,\cdot)$ of the system

$$\dot x=B[m;b](t)x,\qquad x=(x_1,x_2)^{\mathrm T}\in\mathbb R^2,\qquad t\in\Delta(m),\tag{7}$$

the following relations are obvious:

$$X_{B[m;b]}(T_8,T_0)=J_2\quad\text{and}\quad \|X_{B[m;b]}(T'',T')\|\le\exp(bm)\quad\text{for any}\quad T'',T'\in\Delta(m).\tag{8}$$

6.2.2. Family of unperturbed systems defined on the half-line. Let us describe a family of Lyapunov regular two-dimensional systems, each to be used for constructing a system implementing a given Lyapunov irregularity coefficient.

Fix some bounded sequence $\beta=(b_q)_{q=1}^\infty$ of positive numbers. Let us select a sequence $\omega=(m_q)_{q=1}^\infty$ of numbers no less than unity and a sequence $\kappa=(k_q)_{q=1}^\infty$ of positive integers such that the following relations hold:

$$\lim_{q\to\infty}\frac{m_q}{\tau_{q-1}}=0,\qquad \lim_{q\to\infty}m_q=\infty,\qquad \lim_{q\to\infty}\frac{m_qk_q}{\tau_{q-1}}=\infty,\tag{9}$$

where the sequence $(\tau_q)_{q=0}^\infty$ is defined recursively by the formula $\tau_q=\tau_{q-1}+8(m_q+1)k_q$, $q\in\mathbb N$. For example, the sequences $m_q=q$ and $k_q=2^{2^q}$, $q\in\mathbb N$, will be suitable. We also set $\tau_0=0$ and $\tau'_q=(\tau_{q-1}+\tau_q)/2=\tau_{q-1}+4(m_q+1)k_q$.

We denote the coefficient matrix of the family of systems being defined by $P[\kappa;\omega;\beta](t)$ and specify it as follows:

$$P[\kappa;\omega;\beta](t)=\begin{cases}B_{k_q,\tau_{q-1}}[m_q;b_q](t) & \text{for}\quad t\in[\tau_{q-1},\tau'_q),\\ B^*_{k_q,\tau_{q-1}}[m_q;b_q](t) & \text{for}\quad t\in[\tau'_q,\tau_q),\end{cases}\qquad q\in\mathbb N.$$

Note that the trace of the matrix $P[\kappa;\omega;\beta](t)$ is identically zero. Let us prove that under the first condition in (9) the system

$$\dot x=P[\kappa;\omega;\beta](t)x,\qquad x\in\mathbb R^2,\qquad t\in\mathbb R_+,\tag{10}$$

is Lyapunov regular with zero Lyapunov exponents whatever the sequences β, ω, and κ might be.

Denote $t_q^l=\tau_{q-1}+4l(m_q+1)$, $l=0,\dots,2k_q$. Let $x(\cdot)$ be a solution of system (10) with the initial vector satisfying the condition $\|x(0)\|=1$. According to (8), we have $\|x(t_q^l)\|=1$ for any $q\in\mathbb N$ and $l=1,\dots,k_q$, and then, by virtue of property $\mathcal A$, we have the relation $\|x(t_q^l)\|=1$ for any $q\in\mathbb N$ and $l=k_q+1,\dots,2k_q$. Inside each interval $[t_q^l,t_q^{l+1}]$, $l=0,\dots,2k_q-1$, in view of inequality (8), the solution $x(\cdot)$ satisfies the estimate $\|x(t)\|\le\exp(b_qm_q)$, $t\in[t_q^l,t_q^{l+1}]$. Since

$t_q^l \geq \tau_{q-1}$, it follows from this estimate that $t^{-1}\ln\|x(t)\| \leq b_q m_q/\tau_{q-1}$. Since the sequence $(b_q)_{q\in\mathbb{N}}$ is bounded, we conclude by virtue of the first relation in (9) and the equalities established above that the Lyapunov exponents of system (10) are zero. Hence this system is regular, because it has zero trace.

6.2.3. Family of perturbed systems defined on finite intervals. The desired system will be constructed as a perturbation of system (10). The main element in the construction of the system is special perturbations of the matrix $B[m;b](t)$. Let us proceed to defining these perturbation matrices (i.e., perturbations of system (7)).

We define the perturbation matrix (denoted by $Q[m;b](t)$) as

$$Q[m;b](t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ e^{-bm} & 0 \end{pmatrix} & \text{if } \ t \in \Delta_2(m) \bigsqcup \Delta_5(m), \\ \begin{pmatrix} 0 & -e^{-bm} \\ 0 & 0 \end{pmatrix} & \text{if } \ t \in \Delta_7(m), \end{cases}$$

and $Q[m;b](t) \equiv O_2$ for the other $t \in \Delta(m)$.

Set $C[m;b](t) = B[m;b](t) + Q[m;b](t)$, $t \in \Delta(m)$. Obviously, the trace of the matrix $C[m;b](t)$ is identically zero. Consider the perturbed system

$$\dot{y} = C[m;b](t)y, \qquad y \in \mathbb{R}^2, \qquad t \in \Delta(m), \tag{11}$$

which will be called system $C[m;b]$. Let us calculate the Cauchy matrices $Y(\cdot,\cdot)$ of system (11) on the intervals $\Delta_i(m)$, $i = 1,\ldots,8$:

$$Y(T_1^m, T_0^m) = Y(T_8^m, T_7^m) = \operatorname{diag}[e^{mb}, e^{-mb}], \quad Y(T_3^m, T_2^m) = Y(T_6^m, T_5^m) = \operatorname{diag}[e^{-mb}, e^{mb}], \tag{12}$$

$$Y(T_2^m, T_1^m) = Y(T_5^m, T_4^m) = \begin{pmatrix} 1 & 0 \\ e^{-bm} & 1 \end{pmatrix}, \quad Y(T_7^m, T_6^m) = \begin{pmatrix} 1 & -e^{-bm} \\ 0 & 1 \end{pmatrix}, \quad Y(T_4^m, T_3^m) = J_2. \tag{13}$$

Let us determine the Cauchy matrix $Y(T_8^m, T_0^m)$ of the perturbed system. Since

$$Y(T_8^m, T_0^m) = Y(T_8^m, T_7^m)\cdots Y(T_1^m, T_0^m),$$

we multiply the matrices (12) and (13) in the order specified and arrive at

$$Y(T_8^m, T_0^m) = \operatorname{diag}[e^{2bm}, e^{-2bm}].$$

Set $e_1 = (1,0)^{\mathrm{T}}$ and $e_2 = (0,1)^{\mathrm{T}}$. Then

$$Y(T_8^m, T_0^m)e_1 = (e^{2bm}, 0)^{\mathrm{T}} \quad \text{and} \quad Y(T_8^m, T_0^m)e_2 = (0, e^{-2bm})^{\mathrm{T}}.$$

Thus, while the norm of the solution $x_i(\cdot)$ of the unperturbed system issuing at time $t = 0$ from the vector e_i does not change (remains unity) by time T_8^m, the norm of the solution $y_i(\cdot)$ of the perturbed system issuing at time $t = 0$ from the vector e_i increases by the factor e^{2bm} for $i = 1$ and decreases by the factor e^{2bm} for $i = 2$, with the vector $y_i(T_8^m)$ remaining collinear to the vector e_i, $i = 1, 2$.

6.2.4. Family of perturbed systems defined on the half-line. According to [32] (see also [29, Remark 2]), for the function $\sigma(\cdot)$ there exists a sequence of continuous functions $\sigma_q\colon M \to \mathbb{R}$, $q \in \mathbb{N}$, such that

$$\sigma(\mu) = \overline{\lim_{q\to\infty}}\, \sigma_q(\mu), \qquad \mu \in M.$$

Let us construct a family of 2×2 matrices $D(t, \mu)$, $(t, \mu) \in \mathbb{R}_+ \times M$, that are piecewise constant in t for each $\mu \in M$ and continuous in μ uniformly with respect to $t \in \mathbb{R}_+$ and satisfy the relations $\sigma_{\mathrm{L}}(\mu; D) = \sigma(\mu)$ for all $\mu \in M$.

Without loss of generality, we can assume that the inequalities $b(\mu) \geq 1$ and $0 \leq \sigma_q(\mu) \leq b(\mu)$ are satisfied for all $q \in \mathbb{N}$ and $\mu \in M$. (Otherwise, we replace the function $b(\cdot)$ with the function $b(\cdot) + 1$ and every function $\sigma_q(\cdot)$ with the function $\min\{\max\{\sigma_q(\cdot), 0\}, b(\cdot)\}$.)

Let us fix sequences $\omega = (m_q)_{q=1}^{\infty}$ and $\kappa = (k_q)_{q=1}^{\infty}$ satisfying the conditions indicated in Section 6.2.2, and let $(\tau_q)_{q=1}^{\infty}$ be the sequence determined by ω and κ (see Section 6.2.2). As above, let τ_q' be the midpoint of the interval $[\tau_{q-1}, \tau_q]$, i.e., $\tau_q' = (\tau_{q-1} + \tau_q)/2 = \tau_{q-1} + 4(m_q + 1)k_q$, and let the points $t_q^l = \tau_{q-1} + 4l(m_q + 1)$, $l = 0, \ldots, k_q$, divide the interval $[\tau_{q-1}, \tau_q']$ into k_q equal parts.

Taking $m = m_q$ in system (11) and replacing the number b with the function $4b(\mu)$, we repeat this system k_q times starting from the point τ_{q-1} to obtain the following family of systems parametrized by the number $q \in \mathbb{N}$:

$$\dot{y} = B_{k_q, \tau_{q-1}}[m_q; 4b(\mu)](t) + Q_{k_q, \tau_{q-1}}[m_q; 4b(\mu)](t), \qquad y \in \mathbb{R}^2, \qquad t \in [\tau_{q-1}, \tau_q']. \tag{14}$$

This family will play an auxiliary role in what follows. Let $y_q(\cdot)$ be the solution of system (14) with the initial vector $y(\tau_{q-1}) = (1, 0)^{\mathrm{T}}$. As follows from Section 6.2.3, the relations $\|y_q(t_q^l)\| = \exp\{8b(\mu) l m_q\}$ hold for the norm of the solution $y_q(\cdot)$ at times $t = t_q^l$, $l = 0, \ldots, k_q$.

For each $q \in \mathbb{N}$ and each $l = 0, \ldots, k_q$, we define the sets

$$F_q^l = \{\mu \in M : \ln \|y_q(t_q^l)\| \leq \sigma_q(\mu) t_q^l\} \quad \text{and} \quad \Phi_q^l = \{\mu \in M : \ln \|y_q(t_q^l)\| \geq (\sigma_q(\mu) + \varepsilon_q) t_q^l\},$$

where $\varepsilon_q = 1/\tau_q^2$. Let $q \geq 2$. Obviously, $F_q^0 = M$ and $\Phi_q^0 = \varnothing$, because $\|y_q(t_q^0)\| = 1$. Moreover, it can readily be seen that the sets F_q^l and Φ_q^l are disjoint, with the following chains of inclusions holding true:

$$F_q^0 \supset F_q^1 \supset \ldots \supset F_q^{k_q} \quad \text{and} \quad \Phi_q^0 \subset \Phi_q^1 \subset \ldots \subset \Phi_q^{k_q}.$$

Since

$$\frac{\ln \|y_q(t_q^{k_q - 1})\|}{t_q^{k_q - 1}} = \frac{8b(\mu)(k_q - 1)m_q}{\tau_{q-1} + 4(k_q - 1)(m_q + 1)} \underset{q \to +\infty}{\to} 2b(\mu)$$

uniformly with respect to μ, by virtue of the second and third relations in (9) and because $2b(\mu) > \sigma_q(\mu)$, for all q starting from some $q_0 \geq 2$, we have $\Phi_q^{k_q - 1} = M$, and hence $F_q^{k_q} = \varnothing$. It can readily be seen that for any $q \in \mathbb{N}$ and $l = 0, \ldots, k_q - 1$ we have the double inequality

$$\varepsilon_q \leq \frac{\ln \|y_q(t_q^{l+1})\|}{t_q^{l+1}} - \frac{\ln \|y_q(t_q^l)\|}{t_q^l} \leq \frac{8b(\mu) m_q}{\tau_{q-1}}. \tag{15}$$

Since $F_q^{k_q} = \varnothing$ if $q \geq q_0$ and $F_q^0 = M$, it follows that for each $q \geq q_0$ and any $\mu \in M$ there exists a greatest number j (which we denote by $j_q(\mu)$) such that the inclusion $\mu \in F_q^j$ holds, i.e., $\mu \in F_q^{j_q(\mu)}$; however, $\mu \notin F_q^{j_q(\mu)+1}$. Then the double inequality

$$\sigma_q(\mu) - \frac{8b(\mu) m_q}{\tau_{q-1}} < \frac{\ln \|y_q(t_q^{j_q(\mu)})\|}{t_q^{j_q(\mu)}} \leq \sigma_q(\mu) \tag{16}$$

holds. (The right inequality in (16) holds by virtue of the inclusion $\mu \in F_q^{j_q(\mu)}$, and the left one follows from the right inequality in (15) with $l = j_q(\mu)$ and the fact that $\mu \notin F_q^{j_q(\mu)+1}$.)

In view of the continuity of the functions $b(\cdot)$ and $\sigma_q(\cdot)$, $q \in \mathbb{N}$, the sets F_q^l and Φ_q^l are closed as the preimages of closed sets under a continuous mapping. Let $\varphi_q^l \colon M \to [0, 1]$ be a continuous function equal to unity on F_q^l and zero on Φ_q^l. It follows from the preceding that only one of the numbers $\varphi_q^l(\mu)$, $l = 1, \ldots, k_q$,—the number $\varphi_q^{j_q(\mu)+1}(\mu)$—can prove to be distinct from 0 and 1.

Let us define the matrix $D(t,\mu)$. Set $D(t,\mu)=O_2$ for $t\in[0,\tau_{q_0-1})$. On the interval $[\tau_{q-1},\tau_q)$, $q\ge q_0$, first we define the matrix $D(t,\mu)$ on its left half $[\tau_{q-1},\tau_q')$ using the relations

$$D(t,\mu)=B_{k_q,\tau_{q-1}}[m_q;4b(\mu)](t)+\varphi_q^l(\mu)Q_{k_q,\tau_{q-1}}[m_q;4b(\mu)](t),\qquad t\in[t_q^{l-1},t_q^l),\qquad l=1,\dots,k_q,$$

and then we oddly extend the matrix $D(t,\mu)$, already defined for $t\in[\tau_{q-1},\tau_q')$, to the right half $[\tau_q',\tau_q)$. As can be seen, on each half-interval $[t_q^{l-1},t_q^l)$, $l=1,\dots,k_q$, the matrix $D(t,\mu)$ is the matrix of system (14) modified using the function φ_q^l. Since $\tau_q\to\infty$ as $q\to\infty$, it follows that the system $D(\cdot,\mu)$ is defined on the entire half-line $\mathbb{R}_+$.

Let us clarify the above construction. For a fixed $\mu\in M$, the system constructed looks as follows. For each $q\ge q_0$, the perturbed system $C[m_q;4b(\mu)]=B[m_q;4b(\mu)]+Q[m_q;4b(\mu)]$ is repeated $j_q(\mu)$ times (for $j_q(\mu)>0$) on the interval $[t_q^0,t_q^{j_q(\mu)})$ starting from the point $t=t_q^0$. On the interval $[t_q^{j_q(\mu)+1},t_q^{k_q})$, the unperturbed system $B[m_q;4b(\mu)]$ is repeated $k_q-j_q(\mu)-1$ times starting from the point $t=t_q^{j_q(\mu)+1}$. On the interval $[t_q^{j_q(\mu)},t_q^{j_q(\mu)+1})$, the constructed system is a convex combination of the perturbed and unperturbed systems, $D(t,\mu)=(1-r_q(\mu))B[m_q;4b(\mu)](t)+r_q(\mu)C[m_q;4b(\mu)](t)$, where $r_q(\mu)=\varphi_q^{j_q(\mu)+1}(\mu)\in[0,1]$. Further, the system defined on the interval $[t_q^0,t_q^{k_q})$ is extended oddly beyond the point $t_q^{k_q}$.

Let us show that the system constructed possesses the desired properties. Note that the matrix $D(t,\mu)$ has zero trace for all $t\in\mathbb{R}_+$, because this property is possessed by the matrices of the families $B[m;b](t)$ and $Q[m;b](t)$, $t\in\Delta(m)$.

Let y_i be the solution of the constructed system issuing at time $t=0$ from the vector e_i, $i=1,2$. Define a function $p_i\colon\mathbb{R}_+\to\mathbb{R}$ by the relation $p_i(t)=t^{-1}\ln\|y_i(t)\|$, $t\in\mathbb{R}_+$, $i=1,2$. Denote $\delta_q(\mu)=16b(\mu)(m_q+1)/\tau_{q-1}$. Fix an arbitrary $q\ge q_0$. By construction, the following assertions hold true for $i=1,2$: (i) if $t\in[t_q^{l-1},t_q^l)$, $l=1,\dots,k_q$, then $p_i(t)\le\max\{0,p_i(t_q^{l-1})\}+\delta_q(\mu)$; (ii) if $t\in[t_q^{k_q},\tau_q)$, then $p_i(t)\le\max\{p_i(2t_q^{k_q}-t),0\}$; (iii) $p_i(\tau_{q-1})=p_i(\tau_q)=0$. Further, for $l=0,\dots,j_q(\mu)$, by virtue of the left inequality of the chain (15), we have $p_1(t_q^l)\le p_1(t_q^{j_q(\mu)})$. Finally, $p_1(t_q^l)\le p_1(t_q^{j_q(\mu)})+\delta_q(\mu)$ for $l=j_q(\mu)+1,\dots,k_q$. It follows from what has been said that

$$\lambda[y_1]=\varlimsup_{t\to+\infty}p_1(t)=\varlimsup_{q\to\infty}p_1(t_q^{j_q(\mu)}),$$

and, hence, using the estimates (16), we obtain

$$\lambda[y_1]=\varlimsup_{q\to\infty}\sigma_q(\mu)=\sigma(\mu).$$

By construction, we have the inequality $p_2(t_q^l)\le 0$ for $l=0,\dots,j_q(\mu)$ and the inequality $p_2(t_q^l)\le\delta_q(\mu)$ for $l=j_q(\mu)+1,\dots,k_q$. Taking into account the above assertions (i)–(iii), we conclude that $p_2(t)\le 2\delta_q(\mu)$ for all $t\in[\tau_{q-1},\tau_q)$, $q\in\mathbb{N}$. Therefore,

$$\lambda[y_2]=\varlimsup_{t\to+\infty}p_2(t)=0.$$

Note that the basis of solutions $(y_2(\cdot),y_1(\cdot))$ of the system $D(\cdot,\mu)$ is normal. Indeed, if $\sigma(\mu)>0$, then this follows from the fact that the Lyapunov exponents $\lambda[y_1]$ and $\lambda[y_2]$ of these solutions are distinct. For $\sigma(\mu)=0$, the normality of the indicated basis follows from the Lyapunov inequality, because

$$\lambda[y_1]+\lambda[y_2]=0=\varlimsup_{t\to+\infty}\frac{1}{t}\int_0^t\operatorname{tr}D(s,\mu)\,ds.$$

It has thus been established that $\lambda_1(\mu;D)=0$ and $\lambda_2(\mu;D)=\sigma(\mu)$.

Note that if $\sigma(\mu) \equiv 0$, $\mu \in M$, then, choosing identically zero functions as the functions σ_q, we obtain a family $D(\cdot,\cdot)$ coinciding with the unperturbed family $P[\kappa;\omega;4\beta(\mu)](t)$, $t \in \mathbb{R}_+$, $\mu \in M$, defined in Section 6.2.2.

Let us show that the family constructed is continuous in μ uniformly with respect to $t \in \mathbb{R}_+$; i.e., the relation

$$\lim_{\nu\to\mu} \sup_{t\in\mathbb{R}_+} \|D(t,\mu) - D(t,\nu)\| = 0, \quad \mu \in M,$$

holds. Fix an arbitrary $\mu \in M$ and an $\varepsilon > 0$. Let $t \in [t_q^{l-1}, t_q^l)$ for some $q \in \mathbb{N}$ and $l = 1,\dots,k_q$. Then for any $\nu \in M$ we have the chain of inequalities

$$\begin{aligned}\|D(t,\mu) - D(t,\nu)\| &\le \|B[m_q;4b(\mu)](t) - B[m_q;4b(\nu)](t)\| + |\varphi_q^l(\mu) - \varphi_q^l(\nu)|\|Q[m_q;4b(\mu)](t)\| \\ &\quad + \varphi_q^l(\nu)\|Q[m_q;4b(\mu)](t) - Q[m_q;4b(\nu)](t)\| \\ &\le 5|b(\mu) - b(\nu)| + |\varphi_q^l(\mu) - \varphi_q^l(\nu)|\exp(-m_q).\end{aligned}$$

According to the second relation in (9), there exists a $q_* \in \mathbb{N}$ such that $\exp(-m_{q_*}) < \varepsilon/2$ for all $q \ge q_*$. Further, using the continuity of the function $b(\cdot)$ and the functions $\varphi_q^l(\cdot)$, $q = 1,\dots,q_*$, $l = 1,\dots,k_q$, we choose a neighborhood U of the point μ such that the inequalities

$$|b(\mu) - b(\nu)| < \varepsilon/10, \qquad |\varphi_q^l(\mu) - \varphi_q^l(\nu)| < \varepsilon/2, \qquad q = 1,\dots,q_*, \qquad l = 1,\dots,k_q,$$

hold for all $\nu \in U$. By construction, for any $q \in \mathbb{N}$ we have the inequality

$$\sup_{t\in[\tau_{q-1},\tau_q)} \|D(t,\mu) - D(t,\nu)\| = \sup_{t\in[t_q^0,t_q^{k_q})} \|D(t,\mu) - D(t,\nu)\|;$$

by the preceding, it follows that

$$\sup_{t\in[\tau_{q-1},\tau_q)} \|D(t,\mu) - D(t,\nu)\| < \varepsilon, \qquad q \in \mathbb{N},$$

for each $\nu \in U$. The continuity in the uniform topology of the family constructed has thus been established.

6.2.5. Continuous family implementing the irregularity coefficient. Let us show that there exists a family of systems

$$\dot{x} = A(t,\mu)x, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}_+, \qquad \mu \in M,$$

belonging to the class $\mathcal{U}^n(M)$ such that its Lyapunov irregularity coefficient $\sigma_{\mathrm{L}}(\cdot;A)$ coincides with a given function $\sigma(\cdot)$.

By $(t_k)_{k=1}^\infty$ we denote the increasing sequence of points of discontinuity of the matrices $D(\cdot,\mu)$, $\mu \in M$ (one and the same for all $\mu \in M$). By construction, $t_k - t_{k-1} \ge 1$, $k \in \mathbb{N}$, and $t_0 \equiv 0$. Let δ_k be the closed interval of length $|\delta_k| = 2^{-k}\exp(-t_{k+1}^2)$ centered at the point t_k, $k \in \mathbb{N}$. Further, let $s\colon \mathbb{R}_+ \to [0,1]$ be a continuous (e.g., piecewise linear) function vanishing at the points of the sequence (t_k) and equal to unity on the set $\mathbb{R}_+ \setminus \bigsqcup_{k=1}^\infty \delta_k$. For each $\mu \in M$, the obvious chain of inequalities

$$I(\mu) \equiv \int_0^{+\infty} \|s(\tau)D(\tau,\mu) - D(\tau,\mu)\|\exp(\tau^2)\,d\tau \le \sum_{k=1}^{\infty} \int_{t_k-|\delta_k|/2}^{t_k+|\delta_k|/2} \|D(\tau,\mu)\|\exp(\tau^2)\,d\tau \le \sup_{t\in\mathbb{R}_+} \|D(\tau,\mu)\|$$

implies that the integral $I(\mu)$ converges. Therefore, according to the Bogdanov–Grobman theorem [3, 6], the Lyapunov exponents of the systems $s(\cdot)D(\cdot,\mu)$ and $D(\cdot,\mu)$ coincide. Finally, set

$$A(t,\mu) = \operatorname{diag}[s(t)D(t,\mu), \underbrace{0,\dots,0}_{n-2}], \qquad t \in \mathbb{R}_+, \qquad \mu \in M.$$

For each $\mu \in M$, the matrix $A(\cdot,\mu)$ has the identically zero trace, because so does the matrix $D(\cdot,\mu)$. It follows from the inequality $\|A(t,\mu)-A(t,\nu)\| \le \|D(t,\mu)-D(t,\nu)\|$, $t \in \mathbb{R}_+$, $\mu,\nu \in M$, that $A \in \mathcal{U}^n(M)$. For each $\mu \in M$, all but the leading Lyapunov exponents of the system $A(\cdot,\mu)$ are zero, while the leading exponent $\lambda_n(\mu;A)$ is the same as the leading exponent $\lambda_2(\mu;D)$ of the system $D(\cdot,\mu)$, with the latter, in turn, being equal to $\sigma(\mu)$.

Thus, for each $\mu \in M$ the Lyapunov irregularity coefficient $\sigma_{\mathrm{L}}(\mu;A)$ of the system $A(\cdot,\mu)$ is $\sigma(\mu)$. The proof of the theorem is complete.

7. COROLLARIES

Vinograd [21, 22] provided examples of systems $A \in \mathcal{R}_2$ such that their Lyapunov exponents change under the action of certain linear perturbations decaying at infinity. He therewith gave a negative answer to a conjecture that persisted for a long time and claimed that for regular systems such perturbations preserve the Lyapunov exponents of these systems and hence their regularity.

The following two theorems generalize Vinograd's examples, the first being a direct corollary of Theorem 1 and the second following quite easily from the proof of the main theorem and the paper [33].

Theorem 2. *For each $n \ge 2$ and each metric space M, the function $\sigma\colon M \to \mathbb{R}_+$ is the Lyapunov irregularity coefficient of some family in $\mathcal{UZ}^n_{\mathcal{R}}(M)$ if and only if it is bounded and belongs to the class $({}^*,G_\delta)$.*

Theorem 3. *For each $n \ge 2$ and each metric space M, the vector function $(f_1,\ldots,f_n)^{\mathrm{T}}\colon M \to \mathbb{R}^n$ is the n-tuple of Lyapunov exponents $(\lambda_1,\ldots,\lambda_n)$ of some family $A \in \mathcal{UZ}^n_{\mathcal{R}}(M)$ (i.e., $f_k(\mu)=\lambda_k(\mu;A)$ for all $\mu \in M$ and $k=1,\ldots,n$) if and only if it satisfies the following conditions: its components f_k, $k=1,\ldots,n$, are bounded, belong to the class $({}^*,G_\delta)$, and satisfy the inequalities $f_1(\mu) \le \ldots \le f_n(\mu)$ for each $\mu \in M$.*

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