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PARTIAL DIFFERENTIAL EQUATIONS

Initial–Boundary Value Problem for the Beam Vibration Equation in the Multidimensional Case

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Abstract—In the multidimensional case, we study the problem with initial and boundary conditions for the equation of vibrations of a beam with one end clamped and the other hinged. An existence and uniqueness theorem is proved for the posed problem in Sobolev classes. A solution of the problem under consideration is constructed as the sum of a series in the system of eigenfunctions of a multidimensional spectral problem for which the eigenvalues are determined as the roots of a transcendental equation and the system of eigenfunctions is constructed. It is shown that this system of eigenfunctions is complete and forms a Riesz basis in Sobolev spaces. Based on the completeness of the system of eigenfunctions, a theorem about the uniqueness of a solution to the posed initial–boundary value problem is stated.

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1. STATEMENT OF THE PROBLEM

Many problems about vibrations of rods, beams, and plates are of great importance in structural mechanics and lead to higher-order differential equations [1, pp. 141–143; 2, pp. 278–280 of the Russian translation; 3, Ch. 3 of the Russian translation; 4, p. 45; 5, p. 35 of the Russian translation; 6, Ch. 4 of the Russian translation]. The beam vibration equation also arises when calculating the stability of rotating shafts and studying ship vibration [7, Ch. 2].

In the present paper, in a domain $\Pi \times (0,T)$, where $\Pi = (0,l) \times \cdots \times (0,l)$ and l, T are given positive numbers, we consider the more general equation

$$D_{0t}^{\alpha} u(y,t) + a^2 \sum_{j=1}^{N} \frac{\partial^{4m} u(y,t)}{\partial y_j^{4m}} = f(y,t), \quad (y,t) \in \Pi \times (0,T), \quad p-1 < \alpha \le p, \quad m,p \in \mathbb{N}, \tag{1}$$

with the initial conditions

$$\lim_{t\to 0} D_{0t}^{\alpha-i} u(y,t) = \varphi_i(y), \qquad i = 1,\dots,p, \tag{2}$$

and the boundary conditions

$$\begin{aligned} &\left.\frac{\partial^{4k} u(y,t)}{\partial y_j^{4k}}\right|_{y_j=0} = 0, \qquad \left.\frac{\partial^{4k+1} u(y,t)}{\partial y_j^{4k+1}}\right|_{y_j=0} = 0, \\ &\left.\frac{\partial^{4k} u(y,t)}{\partial y_j^{4k}}\right|_{y_j=l} = 0, \qquad \left.\frac{\partial^{4k+2} u(y,t)}{\partial y_j^{4k+2}}\right|_{y_j=l} = 0, \qquad k = 0,\dots,m-1, \qquad j = 1,\dots,N. \end{aligned} \tag{3}$$

Here $(y,t) = (y_1,\dots,y_N,t) \in \Pi \times (0,T)$, the number $a > 0$ is fixed, and $f(y,t)$ and $\varphi_i(y)$, $i = 1,\dots,p$, are given functions. The Riemann–Liouville integro-differentiation operator D^{α} of order α with origin at a point $s \in \mathbb{R}$ is defined as follows:

$$D_{st}^{\alpha} u(y,t) = \frac{\operatorname{sgn}(t-s)}{\Gamma(-\alpha)} \int_s^t \frac{u(y,\tau)\,d\tau}{|t-\tau|^{\alpha+1}}$$

if $\alpha < 0$; $D_{st}^{\alpha}u(y,t) = u(y,t)$ if $\alpha = 0$; and

$$D_{st}^{\alpha}u(y,t) = \operatorname{sgn}^{p}(t-s)\frac{d^{p}}{dt^{p}}D_{st}^{\alpha-p}u(y,t) = \frac{\operatorname{sgn}^{p+1}(t-s)}{\Gamma(l-\alpha)}\frac{d^{p}}{dt^{p}}\int\limits_{s}^{t}\frac{u(y,\tau)\,d\tau}{|t-\tau|^{\alpha-p+1}}$$

if $p-1 < \alpha \le p$, $p \in \mathbb{N}$.

Note that separation of variables was used in the above papers to determine the fundamental frequencies (eigenvalues) for the simplest beam vibration equation; however, the issues of justifying the well-posedness of initial–boundary value problems have been left unexamined. In the papers [8–10], initial–boundary value problems were studied for the beam vibration equation, i.e., for Eq. (1) with $\alpha = 2$, $m = 1$, $N = 1$. In the present paper, based on the papers [8, 11, 12], we state an existence and uniqueness theorem for problem (1)–(3) in the class of generalized Sobolev functions. The solution is constructed in the form of a series in the system of eigenfunctions of a multidimensional problem.

2. COMPLETENESS OF SYSTEM OF EIGENFUNCTIONS IN SOBOLEV CLASSES

We will seek a solution $u(y,t)$ of problem (1)–(3) in the form of a Fourier series expansion

$$u(y,t) = \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}T_{n_1\cdots n_N}(t)v_{n_1\cdots n_N}(y),$$

where $T_{n_1\cdots n_N}(t) = (u(y,t), v_{n_1\cdots n_N}(y))$ are the series coefficients, $\{v_n(y) : n \in \mathbb{Z}_+^N\}$ is the system of eigenfunctions for the multidimensional spectral problem

$$\sum_{j=1}^{N}\frac{\partial^{4m}v(y)}{\partial y_j^{4m}} - \lambda v(y) = 0, \tag{4}$$

$$\left.\frac{\partial^{4k}v(y)}{\partial y_j^{4k}}\right|_{y_j=0} = 0, \qquad \left.\frac{\partial^{4k+1}v(y)}{\partial y_j^{4k+1}}\right|_{y_j=0} = 0,$$
$$\left.\frac{\partial^{4k}v(y)}{\partial y_j^{4k}}\right|_{y_j=l} = 0, \qquad \left.\frac{\partial^{4k+2}v(y)}{\partial y_j^{4k+2}}\right|_{y_j=l} = 0, \qquad k = 0,\dots,m-1, \qquad j = 1,\dots,N, \tag{5}$$

and λ is the variable separation constant.

We seek the eigenfunctions of problem (4), (5) in the form of the product

$$v(y) = X_1(y_1)\cdots X_N(y_N).$$

Then, to determine each of the functions $X_i(y_i)$, $i = 1,\dots,N$, instead of the spectral problem in (4), (5), we arrive at one and the same one-dimensional spectral problem

$$X^{(4m)}(x) - \lambda X(x) = 0, \qquad 0 < x < l, \tag{6}$$
$$X^{(4k)}(0) = X^{(4k+1)}(0) = X^{(4k)}(l) = X^{(4k+2)}(l) = 0, \qquad k = 0,\dots,m-1. \tag{7}$$

Here the $X_i(y_i)$ are denoted by $X(x)$ for simplicity.

By L we denote the differential operator generated by the differential expression $\ell(X) \equiv X^{(4m)}(x)$ on the set $W_2^{4m}(0,l) \cap C^{4m-1}[0,l]$, of, generally speaking, complex-valued functions satisfying the boundary conditions in (7).

The following assertion holds.

Lemma 1. *The operator* $LX \equiv X^{(4m)}(x)$ *with domain*

$$D(L) = \{X(x) : X(x) \in W_2^{4m}(0,l) \cap C^{4m-1}[0,l],$$
$$X^{(4k)}(0) = X^{(4k+1)}(0) = X^{(4k)}(l) = X^{(4k+2)}(l) = 0, \quad k = 0,\dots,m-1\},$$

is a positive symmetric operator in the space $L_2(0,l)$.

Proof. The positiveness of the operator L in the space $L_2(0,l)$ follows from the relations

$$\begin{aligned}(LX,X) &= \int_0^l LX\cdot\overline{X(x)}\,dx = \int_0^l X^{(4m)}(x)\overline{X(x)}\,dx \\ &= X^{(4m-1)}(x)\overline{X(x)}|_0^l - \int_0^l X^{(4m-1)}(x)\overline{X'(x)}\,dx \\ &= X^{(4m-1)}(x)\overline{X(x)}|_0^l - X^{(4m-2)}(x)\overline{X'(x)}|_0^l + \int_0^l X^{(4m-2)}(x)\overline{X''(x)}\,dx = \dots \\ &= X^{(4m-1)}(x)\overline{X(x)}|_0^l - X^{(4m-2)}(x)\overline{X'(x)}|_0^l + \dots + X^{(2m+1)}(x)\overline{X^{(2m-2)}(x)}|_0^l \\ &\quad - X^{(2m)}(x)\overline{X^{(2m-1)}(x)}|_0^l + \int_0^l X^{(2m)}(x)\overline{X^{(2m)}(x)}\,dx = \int_0^l \left|X^{(2m)}(x)\right|^2 dx \ge 0.\end{aligned}$$

Consequently, each eigenvalue of the operator L is nonnegative.

Let us prove that the operator L is symmetric in the space $L_2(0,l)$. Indeed, since the functions f and $\bar g$ belong to the domain $D(L)$, we have $Lf\in L_2(0,l)$ and $L\bar g = \overline{Lg}\in L_2(0,l)$. Further, the functions f and $\bar g$ satisfy the boundary conditions

$$f^{(4k)}(0) = f^{(4k+1)}(0) = f^{(4k)}(l) = f^{(4k+2)}(l) = 0$$

and

$$\overline{g^{(4k)}(0)} = \overline{g^{(4k+1)}(0)} = \overline{g^{(4k)}(l)} = \overline{g^{(4k+2)}(l)} = 0$$

for $k=0,\dots,m-1$. Then

$$\begin{aligned}(Lf,g) &= \int_0^l Lf\cdot\overline{g(x)}\,dx = \int_0^l f^{(4m)}(x)\overline{g(x)}\,dx = f^{(4m-1)}(x)\overline{g(x)}|_0^l - \int_0^l f^{(4m-1)}(x)\overline{g'(x)}\,dx \\ &= f^{(4m-1)}(x)\overline{g(x)}|_0^l - f^{(4m-2)}(x)\overline{g'(x)}|_0^l + \int_0^l f^{(4m-2)}(x)\overline{g''(x)}\,dx = \dots = f^{(4m-1)}(x)\overline{g(x)}|_0^l \\ &\quad - f^{(4m-2)}(x)\overline{g'(x)}|_0^l + \dots + f'(x)\overline{g^{(4m-2)}(x)}|_0^l - f(x)\overline{g^{(4m-1)}(x)}|_0^l + \int_0^l f(x)\overline{g^{(4m)}(x)}\,dx = (f,Lg)\end{aligned}$$

for $k=0,\dots,m-1$. Thus, $(Lf,g)=(f,Lg)$ for any $f,g\in D(L)$. The proof of the lemma is complete.

It is easily seen that $\lambda=0$ is not an eigenvalue of problem (6), (7). Indeed, for $\lambda=0$ the general solution of Eq. (6) has the form

$$X(x) = C_1\frac{x^{4m-1}}{(4m-1)!} + C_2\frac{x^{4m-2}}{(4m-2)!} + \dots + C_{4k}\frac{x^{4m-4k}}{(4m-4k)!} + \dots + C_{4m}, \tag{8}$$

where the C_i are arbitrary constants. The function (8) satisfies the first two conditions in (7) once the relations

$$X^{(4k)}(0) = C_{4(m-k)} = 0,\qquad X^{(4k+1)}(0) = C_{4(m-k)-1} = 0,\qquad k=0,\dots,m-1,$$

hold and the last two conditions in (7) once for each $k = 0, \ldots, m-1$ we have the system of equations

$$C_1 \frac{l^{4(m-k)-1}}{(4(m-k)-1)!} + C_2 \frac{l^{4(m-k)-2}}{(4(m-k)-2)!} + \cdots + C_{4(m-k)-3} \frac{l^3}{3!} + C_{4(m-k)-2} \frac{l^2}{2!} = 0,$$
$$C_1 \frac{l^{4(m-k)-3}}{(4(m-k)-3)!} + C_2 \frac{l^{4(m-k)-4}}{(4(m-k)-4)!} + \cdots + C_{4(m-k)-3} \frac{l}{1!} + C_{4(m-k)-2} = 0. \tag{9}$$

If $k = m-1$, then from (9) we obtain

$$C_1 \frac{l^3}{3!} + C_2 \frac{l^2}{2!} = 0, \qquad C_1 \frac{l}{1!} + C_2 = 0.$$

Since the determinant of this linear system is distinct from zero, the system has only the zero solution $C_1 = C_2 = 0$. Taking this into account, it follows from (9) for $k = m-2$ that

$$C_5 \frac{l^3}{3!} + C_6 \frac{l^2}{2!} = 0, \qquad C_5 \frac{l}{1!} + C_6 = 0.$$

Hence, as above, $C_5 = 0$, $C_6 = 0$. In a similar way, continuing this process, we will successively obtain $C_9 = 0$, $C_{10} = 0$; $C_{13} = 0$, $C_{14} = 0$; $\ldots$; $C_{4m-3} = 0$, $C_{4m-2} = 0$. Thus, $C_{4(m-k)-3} = 0$, $C_{4(m-k)-2} = 0$ for all $k = 0, \ldots, m-1$. Hence $X(x) \equiv 0$; i.e., $\lambda = 0$ is not an eigenvalue of problem (6), (7).

Let $\lambda = b^{4m}$, $b > 0$. Then the set of roots of the characteristic equation $\mu^{4m} - b^{4m} = 0$ for Eq. (6) consists of the numbers

$$\mu_j = b e^{ij\pi/(2m)}, \qquad j = 0, \ldots, 4m-1.$$

The following factorization identity holds for the operator L:

$$L - b^{4m} I = \frac{d^{4m}}{dx^{4m}} - b^{4m} I = \prod_{j=0}^{4m-1} \left(\frac{d}{dx} - \mu_j I \right) = \prod_{j=0}^{2m-1} \left(\frac{d^2}{dx^2} - \mu_j^2 I \right)$$
$$= \prod_{j=0}^{m-1} \left(\frac{d^4}{dx^4} - \mu_j^4 I \right) = \left(\frac{d^4}{dx^4} - b^4 I \right) \prod_{j=1}^{m-1} \left(\frac{d^4}{dx^4} - \mu_j^4 \right). \tag{10}$$

Here I is the identity operator and the $\mu_j^4 = b^4 e^{i2j\pi/m}$, $j = 1, \ldots, m-1$, are nonpositive numbers. It follows from identity (10) that the operator $LX \equiv X^{(4m)}(x)$ with domain $D(L)$ has an eigenfunction $X = X(x)$ if and only if the function $X(x)$ is a nontrivial solution of the problem

$$X^{(4)}(x) = b^4 X(x), \qquad 0 < x < l, \tag{11}$$
$$X(0) = X'(0) = X(l) = X''(l) = 0. \tag{12}$$

Indeed, let $X^{(4m)}(x) = b^{4m} X(x)$, $X(x) \in D(L)$ and suppose that $X(x) \not\equiv 0$ is not a solution to problem (11), (12). By virtue of (10), we have

$$\prod_{j=1}^{m-1} \left(\frac{d^4}{dx^4} - \mu_j^4 I \right) \left(\frac{d^4}{dx^4} - b^4 I \right) X \equiv 0, \qquad X(x) \in D(L), \qquad X(x) \not\equiv 0.$$

Since the spectral problem (11), (12) has only positive eigenvalues, this identity amounts to the identity

$$\prod_{j=2}^{m-1} \left(\frac{d^4}{dx^4} - \mu_j^4 I \right) \left(\frac{d^4}{dx^4} - b^4 I \right) X \equiv 0, \qquad X(x) \in D(L), \qquad X(x) \not\equiv 0.$$

In a similar way, we obtain

$$\prod_{j=3}^{m-1}\left(\frac{d^4}{dx^4}-\mu_j^4 I\right)\left(\frac{d^4}{dx^4}-b^4 I\right)X\equiv 0,\qquad X(x)\in D(L),\qquad X(x)\not\equiv 0.$$

Continuing this process, we arrive at the identity

$$\left(\frac{d^4}{dx^4}-b^4 I\right)X\equiv 0,\qquad X(x)\in D(L),\qquad X(x)\not\equiv 0.$$

However, this is in contradiction with the assumption made.

Thus, the operator $LX\equiv X^{(4m)}(x)$ with domain $D(L)$ has an eigenfunction $X(x)$ if and only if the function $X(x)$ is a nontrivial solution of problem (11), (12). Consequently, instead of the spectral problem (6), (7) we obtain the spectral problem (11), (12).

Let us find the eigenvalues and the corresponding eigenfunctions of the spectral problem (11), (12). We write the general solution of Eq. (11) in the form

$$X(x)=A\cos(bx)+B\sin(bx)+C\cosh(bx)+D\sinh(bx),\tag{13}$$

where A, B, C, and D are arbitrary constants. The function (13) satisfies the first two conditions in (12) once $C=-A$ and $D=-B$. Then the function (13) acquires the form

$$X(x)=A(\cos(bx)-\cosh(bx))+B(\sin(bx)-\sinh(bx)).\tag{14}$$

The function (14) satisfies the last two boundary conditions in (12) once the system of equations

$$\begin{aligned}A(\cos(bl)-\cosh(bl))+B(\sin(bl)-\sinh(bl))=0,\\ A(\cos(bl)+\cosh(bl))+B(\sin(bl)+\sinh(bl))=0\end{aligned}\tag{15}$$

takes place. Equating the determinant of this system with zero, i.e.,

$$\Delta=\begin{vmatrix}\cos(bl)-\cosh(bl) & \sin(bl)-\sinh(bl)\\ \cos(bl)+\cosh(bl) & \sin(bl)+\sinh(bl)\end{vmatrix}=2(\sinh(bl)\cos(bl)-\cosh(bl)\sin(bl))=0,$$

we arrive at the transcendental equation

$$\tan(lb)=\tanh(lb)\tag{16}$$

for the eigenvalues. It follows from the graphs of the functions $\tan(lb)$ and $\tanh(lb)$ than within each of the intervals $(\pi n/l,\pi n/l+\pi/(4l))$, $n\in\mathbb{Z}_+$, there is exactly one root b_n of Eq. (16), with $\pi n/l+\pi/(4l)-b_n\to 0$ as $n\to\infty$. Therefore, this equation has a countable set of roots (eigenvalues)

$$b_0<b_1<\cdots<b_n<\ldots,$$

and in this case, as $n\to\infty$, we have the asymptotic formula

$$b_n=\frac{\pi n}{l}+\frac{\pi}{4l}+O(e^{-2\pi n}).$$

From system (15), with allowance for Eq. (16), we express B via A and substitute the resulting expression into (14). As a result, we find the corresponding system of eigenfunctions

$$\overline{X}_n(x)=\frac{1}{\sqrt{l}}\left(\frac{\sin(b_n(l-x))}{\sin(b_n l)}-\frac{\sinh(b_n(l-x))}{\sinh(b_n l)}\right),\qquad n\in\mathbb{Z}_+.\tag{17}$$

Thus, the eigenvalues of problem (6), (7) are determined using the formula $\lambda_n=b_n^{4m}$, $n\in\mathbb{Z}_+$, where b_n is the root of Eq. (16), and the eigenfunctions, by formula (17).

The norm on the space $W_2^s(0,l)$, where $s \in \mathbb{N}$, is defined by the relation

$$\|f\|^2_{W_2^s(0,l)} = \|f\|^2_{L_2(0,l)} + \|D^s f\|^2_{L_2(0,l)}.$$

Let

$$X_n(x) = \frac{1}{\sqrt{1+b_n^{4s}}}\frac{1}{\sqrt{l}}\left(\frac{\sin(b_n(l-x))}{\sin(b_n l)} - \frac{\sinh(b_n(l-x))}{\sinh(b_n l)}\right), \qquad n \in \mathbb{Z}_+, \tag{18}$$

be the corresponding system of eigenfunctions of problem (6), (7).

The eigenfunctions $X_n(x)$ and $X_k(x)$ of the symmetric operator L corresponding to distinct eigenvalues λ_n and λ_k are known to be orthogonal in the space $L_2(0,l)$.

The following assertion holds.

Lemma 2. *The eigenfunctions $X_n(x)$ of the operator L that correspond to distinct eigenvalues $\lambda_n = b_n^{4m}$, $n \in \mathbb{Z}_+$, are orthonormal in the class $W_2^{2s}(0,l)$.*

Proof. Let $X_n(x)$ and $X_k(x)$ be the eigenfunctions of the operator L corresponding to the eigenvalues λ_n and λ_k, respectively. This implies that

$$LX_n = \lambda_n X_n, \qquad LX_k = \lambda_k X_k.$$

By virtue of the relation $X_n^{(4)} = b_n^4 X_n$, this means that

$$\begin{aligned}(X_n''(x), X_k''(x)) &= X_n''(x)X_k'(x)|_0^l - X_n'''(x)X_k(x)|_0^l + b_n^4(X_n(x), X_k(x)) \\ &= b_n^4(X_n(x), X_k(x)) = 0.\end{aligned}$$

Further, we have the following relations

$$(X_n^{(4)}, X_k^{(4)}) = b_n^4 b_k^4 (X_n, X_k) = 0, \qquad (X_n^{(6)}, X_k^{(6)}) = b_n^4 b_k^4 (X_n'', X_k'') = 0.$$

In a similar way, $(X_n^{(2s)}, X_k^{(2s)}) = 0$. Indeed, if s is even, $s = 2q$, then

$$(X_n^{(2s)}, X_k^{(2s)}) = b_n^{4q} b_k^{4q} (X_n, X_k) = 0.$$

If s is odd, $s = 2q+1$, then

$$(X_n^{(2s)}, X_k^{(2s)}) = b_n^{4q} b_k^{4q} (X_n'', X_k'') = 0.$$

Thus, $(X_n, X_k)_{W_2^{2s}(0,l)} = 0$ for $n \neq k$.

Calculating the norm of the function $X_n(x)$ in the Sobolev space $W_2^{2s}(0,l)$, we arrive at the relation

$$\|X_n(x)\|^2_{W_2^{2s}(0,l)} = \|X_n(x)\|^2_{L_2(0,l)} + \|D^{2s}X_n(x)\|^2_{L_2(0,l)} = \frac{1}{1+b_n^{4s}} + \frac{b_n^{4s}}{1+b_n^{4s}} = 1.$$

The proof of the lemma is complete.

Theorem 1. *The system of eigenfunctions* (18) *of the spectral problem* (11), (12) *is a complete orthonormal system in the Sobolev class $W_2^{2s}(0,l)$.*

Proof. As is known from the theory of differential operators [13, p. 91], the system of eigenfunctions $X_n(x)$ of a self-adjoint operator is a complete orthogonal system in the space $L_2(0,l)$. It follows from the relation $X_n^{(4s)}(x) = b_n^{4s} X_n(x)$ that

$$(X_n^{(4s)}, X_k^{(4s)}) = 0.$$

Consequently, the system of functions (18) is a complete orthonormal system in the space $W_2^{4s}(0,l)$.

If $f(x) \in W_2^{2s}(0,l)$, then there exists a sequence of functions $g_k(x) \in W_2^{4s}(0,l)$, $k \in \mathbb{N}$, such that

$$\|g_k(x) - f(x)\|_{W_2^{2s}(0,l)} \to 0 \quad \text{as} \quad k \to \infty,$$

where the series $g_k(x) = \sum_{n=0}^{\infty} b_n^{(k)} X_n(x)$ converges in the norm of the space $W_2^{4s}(0,l)$. This implies that for each function $f(x) \in W_2^{2s}(0,l)$ and for each $\varepsilon > 0$ there exists a sequence of functions $g_k(x) \in W_2^{4s}(0,l)$, $k \in \mathbb{N}$, for which we have the inequality

$$\|g_k(x) - f(x)\|_{W_2^{2s}(0,l)} < \varepsilon/2.$$

Further, for each $\varepsilon > 0$ there exists a number $N_k(\varepsilon)$ such that for all $N \geq N_k(\varepsilon)$ we have the inequality

$$\left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{W_2^{4s}(0,l)} < \frac{\varepsilon}{2}.$$

Let us introduce a Sobolev space $H^s(0,l)$ with the norm

$$\|f(x)\|^2_{H^s(0,l)} = \sum_{0 \leq k \leq s} \|D^k f(x)\|^2_{L_2(0,l)},$$

where $s \in \mathbb{N}$. In view of the equivalence of norms in the spaces $H^s(0,l)$ and $W_2^s(0,l)$, the inequalities

$$c_0 \|f(x)\|_{W_2^s(0,l)} \leq \|f(x)\|_{H^s(0,l)} \leq c_1 \|f(x)\|_{W_2^s(0,l)} \tag{19}$$

hold, where c_0 and c_1 are some positive constants. This implies that

$$\left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{4s}(0,l)} \leq c_1 \left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{W_2^{4s}(0,l)} < c_1 \frac{\varepsilon}{2}.$$

Using the embedding of the space $H^{4s}(0,l)$ in the space $H^{2s}(0,l)$, we have the estimate

$$\left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{2s}(0,l)} \leq \left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{4s}(0,l)} < c_1 \frac{\varepsilon}{2}.$$

Since

$$\|g_k(x) - f(x)\|_{H^{2s}(0,l)} \leq c_1 \|g_k(x) - f(x)\|_{W_2^{2s}(0,l)} < c_1 \frac{\varepsilon}{2},$$

by using the triangle inequality, we arrive at

$$\begin{aligned} \left\| f(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{2s}(0,l)} &\leq \|f(x) - g_k(x)\|_{H^{2s}(0,l)} + \left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{2s}(0,l)} \\ &< c_1 \frac{\varepsilon}{2} + c_1 \frac{\varepsilon}{2} = c_1 \varepsilon. \end{aligned}$$

It follows that

$$\|f(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x)\|_{W_2^{2s}(0,l)} < \frac{1}{c_0} \|f(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x)\|_{H^{2s}(0,l)} < \frac{c_1}{c_0} \varepsilon.$$

Hence the system of eigenfunctions $\{X_n(x)\}_{n=0}^{\infty}$ is complete in the space $W_2^{2s}(0,l)$ and, by Lemma 2, orthonormal in this space. The proof of the theorem is complete.

Theorem 2. *The system of eigenfunctions* (18) *of the spectral problem* (11), (12) *is a Riesz basis in the Sobolev space* $H^s(0,l)$.

Proof. By Theorem 1, the system of eigenfunctions $X_n(x)$ of the self-adjoint operator L is a complete orthonormal system in the space $W_2^{2s}(0,l)$. Hence if $f(x) \in H^s(0,l)$, then there exists a sequence of functions $g_k(x) \in W_2^{2s}(0,l)$ such that the following relation holds:

$$\|g_k(x) - f(x)\|_{H^s(0,l)} \to 0 \quad \text{as} \quad k \to \infty,$$

where $g_k(x) = \sum_{n=0}^{\infty} b_n^{(k)} X_n(x)$ converges in the norm of the space $W_2^{2s}(0,l)$, i.e., for each $\varepsilon > 0$ there exits a positive integer $M(\varepsilon)$ such that for all $k \ge M(\varepsilon)$ we have the inequality

$$\|g_k(x) - f(x)\|_{H^s(0,l)} < \varepsilon/2.$$

Further, for a given $\varepsilon > 0$ there exists a number $N_k(\varepsilon)$ such that for all $N \ge N_k(\varepsilon)$ we have the inequality

$$\left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{W_2^{2s}(0,l)} < \frac{\varepsilon}{2}.$$

Taking into account the latter inequality, by virtue of the right estimate in (19), we conclude that

$$\left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{2s}(0,l)} \le c_1 \left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{W_2^{2s}(0,l)} < c_1 \frac{\varepsilon}{2}.$$

Using the embedding of the space $H^{2s}(0,l)$ in the space $H^s(0,l)$, we will have

$$\left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{s}(0,l)} \le \left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{2s}(0,l)} < c_1 \frac{\varepsilon}{2}.$$

Using this fact and the triangle inequality, we obtain

$$\left\| f(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{s}(0,l)} \le \|f(x) - g_k(x)\|_{H^s(0,l)} + \left\| g_k(x) - \sum_{n=0}^{N} b_n^{(k)} X_n(x) \right\|_{H^{s}(0,l)} < (1 + c_1) \frac{\varepsilon}{2},$$

with this inequality holding for each permutation of functions in the system $\{X_n(x)\}_{n=0}^{\infty}$; i.e., the system of eigenfunctions $\{X_n(x)\}_{n=0}^{\infty}$ is a permutation basis in the class $H^s(0,l)$. It follows that the system of eigenfunctions $\{X_n(x)\}_{n=0}^{\infty}$ forms a Riesz basis in the Sobolev space $H^s(0,l)$. The proof of the theorem is complete.

The inner product is introduced in the space $W_2^{s_1,s_2}((0,l)\times(0,l))$ as follows:

$$\begin{aligned}
&(f(x,y), g(x,y))_{W_2^{s_1,s_2}((0,l)\times(0,l))} = (f(x,y), g(x,y))_{L_2((0,l)\times(0,l))} \\
&\quad + (D_x^{s_1} f(x,y), D_x^{s_1} g(x,y))_{L_2((0,l)\times(0,l))} + (D_y^{s_2} f(x,y), D_y^{s_2} g(x,y))_{L_2((0,l)\times(0,l))} \\
&\quad + (D_{x,y}^{s_1,s_2} f(x,y), D_{x,y}^{s_1,s_2} g(x,y))_{L_2((0,l)\times(0,l))}.
\end{aligned} \tag{20}$$

Accordingly, the norm in this space is given by the relation

$$\begin{aligned}
\|f(x,y)\|^2_{W_2^{s_1,s_2}((0,l)\times(0,l))} &= \|f(x,y)\|^2_{L_2((0,l)\times(0,l))} + \|D_x^{s_1} f(x,y)\|^2_{L_2((0,l)\times(0,l))} \\
&\quad + \|D_y^{s_2} f(x,y)\|^2_{L_2((0,l)\times(0,l))} + \|D_{x,y}^{s_1,s_2} f(x,y)\|^2_{L_2((0,l)\times(0,l))}.
\end{aligned}$$

Lemma 3. *If* $\{\varphi_k(x)\}$ *and* $\{\psi_n(z)\}$ *are complete orthonormal systems in the spaces* $W_2^{s_1}(0,l)$ *and* $W_2^{s_2}(0,l)$, *respectively, then the system of all products* $f_{kn}(x,z) = \varphi_k(x)\psi_n(z)$ *is a complete orthonormal system in the space* $W_2^{s_1,s_2}((0,l)\times(0,l))$.

Proof. By the Fubini theorem, we have

$$
\begin{aligned}
&\|f_{kn}(x,z)\|^2_{W_2^{s_1,s_2}((0,l)\times(0,l))} = \|\varphi_k(x)\|^2_{L_2(0,l)}\|\psi_n(z)\|^2_{L_2(0,l)} + \|D_x^{s_1}\varphi_k(x)\|^2_{L_2(0,l)}\|\psi_n(z)\|^2_{L_2(0,l)}\\
&\quad + \|\varphi_k(x)\|^2_{L_2(0,l)}\|D_z^{s_2}\psi_n(z)\|^2_{L_2(0,l)} + \|D_x^{s_1}\varphi_k(x)\|^2_{L_2(0,l)}\|D_z^{s_2}\psi_n(z)\|^2_{L_2(0,l)}\\
&= (\|\varphi_k(x)\|^2_{L_2(0,l)} + \|D_x^{s_1}\varphi_k(x)\|^2_{L_2(0,l)})\|\psi_n(z)\|^2_{L_2(0,l)} + (\|\varphi_k(x)\|^2_{L_2(0,l)} + \|D_x^{s_1}\varphi_k(x)\|^2_{L_2(0,l)})\\
&\quad \times \|D_z^{s_2}\psi_n(z)\|^2_{L_2(0,l)} = (\|\varphi_k(x)\|^2_{L_2(0,l)} + \|D_x^{s_1}\varphi_k(x)\|^2_{L_2(0,l)})(\|\psi_n(z)\|^2_{L_2(0,l)} + \|D_z^{s_2}\psi_n(z)\|^2_{L_2(0,l)}) = 1.
\end{aligned}
$$

If $k \neq k_1$ or $n \neq n_1$ then, by virtue of the same theorem, we derive

$$
\begin{aligned}
&(f_{kn}(x,z), f_{k_1n_1}(x,z))_{W_2^{s_1,s_2}((0,l)\times(0,l))}\\
&= (f_{kn}(x,z), f_{k_1n_1}(x,z))_{L_2((0,l)\times(0,l))} + (D_x^{s_1}f_{kn}(x,z), D_x^{s_1}f_{k_1n_1}(x,z))_{L_2((0,l)\times(0,l))}\\
&\quad + (D_z^{s_2}f_{kn}(x,z), D_z^{s_2}f_{k_1n_1}(x,z))_{L_2((0,l)\times(0,l))} + (D_{x,z}^{s_1,s_2}f_{kn}(x,z), D_{x,z}^{s_1,s_2}f_{k_1n_1}(x,z))_{L_2((0,l)\times(0,l))}\\
&= (\varphi_k(x), \varphi_{k_1}(x))_{L_2(0,l)}(\psi_n(z), \psi_{n_1}(z))_{L_2(0,l)} + (D_x^{s_1}\varphi_k(x), D_x^{s_1}\varphi_{k_1}(x))_{L_2(0,l)}(\psi_n(z), \psi_{n_1}(z))_{L_2(0,l)}\\
&\quad + (\varphi_k(x), \varphi_{k_1}(x))_{L_2(0,l)}(D_z^{s_2}\psi_n(z), D_z^{s_2}\psi_{n_1}(z))_{L_2(0,l)}\\
&\quad + (D_x^{s_1}\varphi_k(x), D_x^{s_1}\varphi_{k_1}(x))_{L_2(0,l)}(D_z^{s_2}\psi_n(z), D_z^{s_2}\psi_{n_1}(z))_{L_2(0,l)}\\
&= ((\varphi_k(x), \varphi_{k_1}(x))_{L_2(0,l)} + (D_x^{s_1}\varphi_k(x), D_x^{s_1}\varphi_{k_1}(x))_{L_2(0,l)})(\psi_n(z), \psi_{n_1}(z))_{L_2(0,l)}\\
&\quad + (((\varphi_k(x), \varphi_{k_1}(x))_{L_2(0,l)} + (D_x^{s_1}\varphi_k(x), D_x^{s_1}\varphi_{k_1}(x))_{L_2(0,l)})(D_z^{s_2}\psi_n(z), D_z^{s_2}\psi_{n_1}(z))_{L_2(0,l)}\\
&= ((\varphi_k(x), \varphi_{k_1}(x))_{L_2(0,l)} + (D_x^{s_1}\varphi_k(x), D_x^{s_1}\varphi_{k_1}(x))_{L_2(0,l)})\\
&\quad \times ((\psi_n(z), \psi_{n_1}(z))_{L_2(0,l)} + (D_z^{s_2}\psi_n(z), D_z^{s_2}\psi_{n_1}(z))_{L_2(0,l)}) = 0.
\end{aligned}
$$

Since the inner product (20) is defined for functions given on $\Pi = (0,l)\times(0,l)$, we will prove the completeness of the system $\{f_{kn}(x,z)\}$. Suppose that in $W_2^{s_1,s_2}(\Pi)$ there exists a function $f(x,z)$ orthogonal to all functions $f_{kn}(x,z)$. We set

$$F_k(z) = (f(x,z), \varphi_k(x))_{W_2^{s_1}(0,l)}.$$

It can be easily seen that the function $F_k(z)$ belongs to the class $W_2^{s_2}(0,l)$. Then we have

$$(F_k(z), \psi_n(z))_{W_2^{s_2}(0,l)} = (f(x,z), f_{kn}(x,z))_{W_2^{s_1,s_2}((0,l)\times(0,l))} = 0.$$

In view of the completeness of the system $\{\psi_n(z)\}$, it follows that for almost all z and each k the relations $F_k(z) = 0$ hold. However, then for almost each z we have the relations

$$(f(x,z), \varphi_k(x))_{W_2^{s_1}(0,l)} = 0$$

for all k. In view of the completeness of the system $\{\varphi_k(x)\}$, this implies that for almost each z the set of those x for which $f(x,z) \neq 0$ has the measure zero. By the Fubini theorem, this implies that on $\Pi = (0,l)\times(0,l)$ the function $f(x,z)$ is zero almost everywhere. The proof of the lemma is complete.

The inner product is introduced in the space $W_2^{s_1,s_2,\ldots,s_N}(\Pi)$ as follows:

$$
\begin{aligned}
&(f(x), g(x))_{W_2^{s_1,s_2,\ldots,s_N}(\Pi)} = (f(x), g(x))_{L_2(\Pi)}\\
&\quad + \sum_{j_1=1}^{N}(D_{x_{j_1}}^{s_{j_1}}f(x), D_{x_{j_1}}^{s_{j_1}}g(x))_{L_2(\Pi)} + \sum_{1\le j_1<j_2\le N}(D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}f(x), D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}g(x))_{L_2(\Pi)} + \ldots\\
&\quad + \sum_{1\le j_1<j_2<\cdots<j_N\le N}(D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}\cdots D_{x_{j_N}}^{s_{j_N}}f(x), D_{x_{j_1}}^{s_{j_1}}D_{x_{j_2}}^{s_{j_2}}\cdots D_{x_{j_N}}^{s_{j_N}}g(x))_{L_2(\Pi)}.
\end{aligned}
$$

Accordingly, the norm in this space is given by the relation

$$\|f(x)\|^2_{W_2^{s_1,s_2,\ldots,s_N}(\Pi)} = \|f(x)\|^2_{L_2(\Pi)} + \sum_{j_1=1}^{N} \|D^{s_{j_1}}_{x_{j_1}} f(x)\|^2_{L_2(\Pi)}$$
$$+ \sum_{1\le j_1<j_2\le N} \|D^{s_{j_1}}_{x_{j_1}} D^{s_{j_2}}_{x_{j_2}} f(x)\|^2_{L_2(\Pi)} + \cdots + \sum_{1\le j_1<j_2<\cdots<j_N\le N} \|D^{s_{j_1}}_{x_{j_1}} D^{s_{j_2}}_{x_{j_2}} \cdots D^{s_{j_N}}_{x_{j_N}} f(x)\|^2_{L_2(\Pi)}.$$

By induction, using the assertion in Lemma 3 as the base case, we arrive at the following assertion.

Lemma 4. *If $\{\varphi_{n_1}(x_1)\},\ldots,\{\varphi_{n_N}(x_N)\}$ are complete orthonormal systems in the spaces $W_2^{2s_1}(0,l),\ldots,W_2^{2s_N}(0,l)$, respectively, then the system of all products*

$$f_n(x) = f_{n_1\cdots n_N}(x_1,\ldots,x_N) = \varphi_{n_1}(x_1)\cdots\varphi_{n_N}(x_N)$$

is a complete orthonormal system in the space $W_2^{2s_1,2s_2,\ldots,2s_N}(\Pi)$.

Let us apply Lemma 4 to our orthonormal systems. In the space $W_2^{2s_1,2s_2,\ldots,2s_N}(\Pi)$ of functions of N variables $f(x) = f(x_1,\ldots,x_N)$, a complete orthonormal system is formed by all possible products

$$v_{n_1\cdots n_N}(x_1,\ldots,x_N) = X_{n_1}(x_1)\cdots X_{n_N}(x_N),$$

where

$$X_{n_j}(x) = \frac{1}{\sqrt{1+b^{4s}_{n_j}}}\frac{1}{\sqrt{l}}\left(\frac{\sin(b_{n_j}(l-x))}{\sin(b_{n_j}l)} - \frac{\sinh(b_{n_j}(l-x))}{\sinh(b_{n_j}l)}\right), \qquad n_j\in\mathbb{Z}_+,$$

b_{n_j} is the root of Eq. (16).

Thus, the following assertion holds.

Theorem 3. *The system of eigenfunctions*

$$\{v_{n_1\cdots n_N}(x_1,\ldots,x_N)\}_{(n_1,\ldots,n_N)\in\mathbb{Z}_+^N} = \left\{\prod_{j=1}^{N} X_{n_j}(x_j)\right\}_{(n_1,\ldots,n_N)\in\mathbb{Z}_+^N} \tag{21}$$

of the spectral problem in (4), (5) *is a complete orthonormal system in the Sobolev class $W_2^{2s_1,2s_2,\ldots,2s_N}(\Pi)$.*

The following assertion can be proved by analogy with Theorem 2.

Theorem 4. *The system of eigenfunctions* (21) *of the spectral problem in* (4), (5) *is a Riesz basis in the Sobolev space $H^{s_1,s_2,\ldots,s_N}(\Pi)$.*

Corollary 1. *If $s_j > k + N/2$, $k\in\mathbb{Z}_+$ then the Fourier series of the function*

$$f(x)\in H^{s_1,s_2,\ldots,s_N}(\Pi)\cap C^k(\Pi)$$

in the system of eigenfunctions (21) *of the spectral problem* (4), (5) *converges in the norm of the space $C^k(\Pi)$ to the function $f(x)$.*

Proof. Let $s_j > k+N/2$, $k\in\mathbb{Z}_+$. Then, by the Sobolev theorem, the embedding of the space $H^{s_1,s_2,\ldots,s_N}(\Pi)$ in the space $C^k(\Pi)$ takes place and the following estimate holds:

$$\|f(x)\|_{C^k(\Pi)} \le c\|f(x)\|_{H^{s_1,s_2,\ldots,s_N}(\Pi)}, \qquad c = \text{ const } > 0. \tag{22}$$

According to Theorem 4, each sequence of partial sums $S_n(x)\in H^{s_1,s_2,\ldots,s_N}(\Pi)$ of the Fourier series for the function $f(x)\in H^{s_1,s_2,\ldots,s_N}(\Pi)$ converges to the function $f(x)\in H^{s_1,s_2,\ldots,s_N}(\Pi)$ in the norm of the space $H^{s_1,s_2,\ldots,s_N}(\Pi)$; i.e., we have the relation

$$\|S_n(x) - f(x)\|_{H^{s_1,s_2,\ldots,s_N}(\Pi)} \to 0 \quad \text{as} \quad n\to\infty.$$

Therefore, making use of the estimate in (22), we conclude that for each function $f(x) \in H^{s_1,s_2,\ldots,s_N}(\Pi) \cap C^k(\Pi)$ we have the convergence

$$\|S_n(x) - f(x)\|_{C^k(\Pi)} \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that the Fourier series of the function $f(x) \in H^{s_1,s_2,\ldots,s_N}(\Pi) \cap C^k(\Pi)$ in the system of eigenfunctions (21) of the spectral problem in (4), (5) converges in the norm of the space $C^k(\Pi)$ to the function $f(x)$. The proof of Corollary 1 is complete.

3. EXISTENCE AND UNIQUENESS OF SOLUTION OF THE INITIAL–BOUNDARY VALUE PROBLEM

Since the system of eigenfunctions (21) of the spectral problem in (4), (5) is a Riesz basis in the Sobolev space $H^{s_1,s_2,\ldots,s_N}(\Pi)$, it follows that each function in this class can be represented in the form of a convergent Fourier series in this system. For each $t > 0$ we expand the solution $u(y,t)$ of problem (1), (3) in a Fourier series in the system of eigenfunctions (21) of the spectral problem (4), (5),

$$u(y,t) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} T_{n_1\cdots n_N}(t)\overline{v}_{n_1\cdots n_N}(y), \tag{23}$$

where $T_{n_1\cdots n_N}(t) = (u(y,t), \overline{v}_{n_1\cdots n_N}(y))$, $\overline{v}_{n_1\cdots n_N}(y)) = \prod_{j=1}^{N} \overline{X}_{n_j}(y_j)$, and b_{n_j} is the root of Eq. (16). By virtue of (1), (2), the unknown functions $T_n(t) = T_{n_1\cdots n_N}(t)$ satisfy the equations

$$D_{0t}^{\alpha} T_{n_1\cdots n_N}(t) + \lambda_{n_1\cdots n_N} T_{n_1\cdots n_N}(t) = f_{n_1\ldots n_N}(t), \qquad p-1 < \alpha \le p, \qquad p \in \mathbb{N}, \tag{24}$$

with the initial conditions

$$\lim_{t\to 0} D_{0t}^{\alpha-i} T_{n_1\cdots n_N}(t) = \varphi_{i,n_1\cdots n_N}, \qquad i = 1,\ldots,p, \qquad n_j \in \mathbb{N}. \tag{25}$$

The solution of the Cauchy problem in (24), (25) is known (e.g., see [14, pp. 601–602; 15, pp. 221–223; 16, pp. 16–17]) and has the form

$$\begin{aligned} T_{n_1\ldots n_N}(t) &= \sum_{i=1}^{p} \varphi_{i,(n_1\cdots n_N)} t^{\alpha-i} E_{\alpha,\alpha-i+1}(\mu_{n_1\cdots n_N} t^{\alpha}) \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[\mu_{n_1\cdots n_N}(t-\tau)^{\alpha}] f_{n_1\cdots n_N}(\tau)\, d\tau, \end{aligned} \tag{26}$$

where the coefficients are determined as follows:

$$\mu_{n_1\cdots n_N} = -\lambda_{n_1\cdots n_N} = -a^2 \sum_{j=1}^{N} \lambda_{n_j} = -a^2 \sum_{j=1}^{N} b_{n_j}^{4n}, \tag{27}$$

$$E_{\alpha,\alpha-i+1}(\mu_{n_1\cdots n_N} t^{\alpha}) = \sum_{q=0}^{\infty} \frac{(\mu_{n_1\cdots n_N} t^{\alpha})^q}{\Gamma(\alpha q + \alpha - i + 1)}, \tag{28}$$

$$E_{\alpha,\alpha}(\mu_{n_1\cdots n_N}(t-\tau)^{\alpha}) = \sum_{q=1}^{\infty} \frac{(\mu_{n_1\cdots n_N})^{q-1}(t-\tau)^{\alpha(q-1)}}{\Gamma(\alpha q)}, \tag{29}$$

$$f(y,t) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} f_{n_1\cdots n_N}(t)\overline{v}_{n_1\cdots n_N}(y_1,\ldots,y_N), \tag{30}$$

$$\varphi_i(y) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \varphi_{i,(n_1\cdots n_N)}\overline{v}_{n_1\cdots n_N}(y_1,\ldots,y_N), \quad i = 1,\ldots,p. \tag{31}$$

After substituting the solution (26) into the expansion (23), we arrive at the unique solution of problem (1)–(3) in the form of the series

$$u(y,t)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\Bigg[\sum_{i=1}^{p}\varphi_{i,(n_1\cdots n_N)}t^{\alpha-i}E_{\alpha,\alpha-i+1}(\mu_{n_1\cdots n_N}t^{\alpha})$$
$$+\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}[\mu_{n_1\cdots n_N}(t-\tau)^{\alpha}]f_{n_1\cdots n_N}(\tau)\,d\tau\Bigg]\overline{v}_{n_1\cdots n_N}(y_1,\dots,y_N). \qquad (32)$$

We have thus proved the following assertion, central to this paper.

Theorem 5. *There exists a unique solution of problem* (1)–(3), *which can be represented in the form of the series* (32) *with coefficients determined using formulas* (27)–(31).

In structural mechanics, of most interest are the cases where $\alpha=2$, $n=1$, $N=2,3$, i.e., respectively, the equations

$$u_{tt}+a^2(u_{y_1y_1y_1y_1}+u_{y_2y_2y_2y_2})=f(y_1,y_2,t),$$
$$u_{tt}+a^2(u_{y_1y_1y_1y_1}+u_{y_2y_2y_2y_2}+u_{y_3y_3y_3y_3})=f(y_1,y_2,y_2,t).$$

If $\alpha=2$, $n=1$, and $N=2$, then the solution (32) has the form

$$u(y_1,y_2,t)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Bigg[\varphi_{1,n_1n_2}\frac{\sin\big(a\sqrt{b_{n_1}^4+b_{n_2}^4}\,t\big)}{a\sqrt{b_{n_1}^4+b_{n_2}^4}}+\varphi_{2,n_1n_2}\cos\big(a\sqrt{b_{n_1}^4+b_{n_2}^4}\,t\big)$$
$$+\int\limits_0^t\frac{\sin\big(a\sqrt{b_{n_1}^4+b_{n_2}^4}(t-\tau)\big)}{a\sqrt{b_{n_1}^4+b_{n_2}^4}}f_{n_1n_2}(\tau)\,d\tau\Bigg]\overline{v}_{n_1n_2}(y_1,y_2),$$

where the coefficients are determined using the formulas

$$\varphi_i(y_1,y_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\varphi_{i,n_1n_2}\overline{v}_{n_1n_2}(y_1,y_2),\qquad \varphi_{i,n_1n_2}=(\varphi_i(y_1,y_2),\overline{v}_{n_1n_2}(y_1,y_2)),\qquad i=1,2,$$

$$f_{n_1n_2}(t)=(f(y_1,y_2,t),\overline{v}_{n_1n_2}(y_1,y_2)),\qquad \overline{v}_{n_1n_2}(y_1,y_2))=\prod_{j=1}^{2}\overline{X}_{n_j}(y_j).$$

In the case where $\alpha=2$, $n=1$, and $N=3$, the solution of problem (1)–(3) based on (32) is determined by the formula

$$u(y_1,y_2,y_3,t)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\Bigg[\varphi_{1,n_1n_2n_3}\frac{\sin\big(a\sqrt{b_{n_1}^4+b_{n_2}^4+b_{n_3}^4}\,t\big)}{a\sqrt{b_{n_1}^4+b_{n_2}^4+b_{n_3}^4}}$$
$$+\varphi_{2,n_1n_2n_3}\cos\big(a\sqrt{b_{n_1}^4+b_{n_2}^4+b_{n_3}^4}\,t\big)$$
$$+\int\limits_0^t\frac{\sin\big(a\sqrt{b_{n_1}^4+b_{n_2}^4+b_{n_3}^4}(t-\tau)\big)}{a\sqrt{b_{n_1}^4+b_{n_2}^4+b_{n_3}^4}}f_{n_1n_2n_3}(\tau)\,d\tau\Bigg]\overline{v}_{n_1n_2n_2}(y_1,y_2,y_3),$$

where the coefficients are found using the formulas

$$\varphi_i(y_1,y_2,y_3)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\varphi_{i,n_1,n_2,n_3}\overline{v}_{n_1n_2n_3}(y_1,y_2,y_3),$$
$$\varphi_{i,n_1,n_2,n_3}=(\varphi_i(y_1,y_2,y_3),\overline{v}_{n_1n_2n_3}(y_1,y_2,y_3)),\qquad i=1,2,$$
$$f_{n_1,n_2,n_3}(t)=(f(y_1,y_2,y_3,t),\overline{v}_{n_1n_2n_3}(y_1,y_2,y_3)),\qquad \overline{v}_{n_1n_2n_3}(y_1,y_2,y_3))=\prod_{j=1}^{3}\overline{X}_{n_j}(y_j).$$

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