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NUMERICAL METHODS

Numerical Method for Some Singular Integro-Differential Equations

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Abstract—Numerical solution schemes are constructed and justified for two singular integro-differential equations containing an integral, understood in the sense of the Cauchy principal value, over an interval of the real axis. An integral equation with logarithmic kernel of a special form is studied and approximately solved. Uniform error estimates for the approximate solutions are obtained.

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The present paper proposes a computational scheme for the numerical solution of the singular integro-differential equation

$$u'(x) + \gamma(x)\frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1}\frac{u(t)}{\sqrt{1-t^2}}\frac{dt}{t-x} = f(x), \qquad -1 \le x \le 1, \tag{1}$$

$$u(x_0) = 0, \tag{2}$$

where $x_0 \in [-1,1]$, $\gamma(x)$ and $f(x)$ are given functions of the class $C[-1,1]$, and $u(x)$ is the unknown function of the class $C^1[-1,1]$. This equation arises in some hydrodynamic problems [1, 2].

In addition to the numerical solution of Eq. (1), the paper presents a numerical solution of the Prandtl integro-differential equation, which plays an important role in problems of continuum mechanics. The boundary value problem for the Prandtl equation has the form [3]

$$\frac{\Gamma(x)}{B(x)} - \frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma'(t)}{t-x}\,dt = f(x), \qquad -1 < x < 1, \tag{3}$$

$$\Gamma(-1) = \Gamma(+1) = 0. \tag{4}$$

Here $B(x)$ and $f(x)$ are given functions of the class $C[-1,1]$, $\Gamma(x)$ is the unknown function of the class $C^1[-1,1]$, and $\Gamma'(x)$ is a function of the Hölder class. The integrals in Eqs. (1) and (3) are understood in the sense of the Cauchy principal value.

There are a tremendous number of papers where Eq. (3) is studied. It is only in rare special cases that there is a known exact solution of this equation (e.g., see [4]). Hence a large part of these papers deal with the development and justification of approximate solution methods, the most common being the Multhopp method [5].

When constructing approximate solutions of Eqs. (1) and (3), a key role is played by the *spectral relations*

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(t)}{\sqrt{1-t^2}}\frac{dt}{t-x} = U_{n-1}(x), \qquad -1 < x < 1,$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,U_{n-1}(t)\frac{dt}{t-x} = -T_n(x), \qquad -1 < x < 1, \qquad n \in \mathbb{N},$$

for singular integrals, where $T_n(x)$ and $U_{n-1}(x)$ are the Chebyshev polynomials of the first and second kind, respectively.

We will also use the interpolation polynomial on the Chebyshev nodes of the first kind for the function $f(x)$ [6],

$$f(x) \approx f_n(x) = \frac{1}{2}C_0^n T_0(t) + \sum_{j=1}^{n} C_j^n T_j(x), \tag{5}$$

where

$$C_j^n = \frac{2}{n+1}\sum_{k=0}^{n} f(x_k)T_j(x_k), \qquad j = 0,\dots,n, \qquad x_k = \cos\frac{2k+1}{2n+2}\pi, \qquad k = 0,\dots,n.$$

Using the well-known identities

$$T_0(x) = U_0(x), \qquad 2T_1(x) = U_1(x), \qquad 2T_j(x) = U_j(x) - U_{j-2}(x), \qquad j \ge 2,$$

we write the interpolation polynomial (5) in the form

$$f_n(x) = \sum_{j=0}^{n} c_j^n U_j(x), \tag{6}$$

where $c_j^n = G_j^n - G_{j+2}^n$, $j = 0,\dots,n-2$, $c_{n-1}^n = G_{n-1}^n$, $c_n^n = G_n^n$, and

$$G_j^n = \frac{1}{n+1}\sum_{k=0}^{n} T_j(x_k)f(x_k), \qquad j = 0,\dots,n, \qquad x_k = \cos\frac{2k+1}{2n+2}\pi, \qquad k = 0,\dots,n.$$

The following lemmas hold for these interpolation polynomials [7].

Lemma 1. *If the zeros of the Chebyshev polynomial of the first kind, i.e., the points* $x_k = \cos((2k+1)\pi/(2n+2))$, $k = 0,\dots,n$, *are taken as interpolation nodes, then the Lebesgue constants* λ_n *satisfy the estimate* $\lambda_n = O(\ln n)$, $n = 2,3,\dots$

Lemma 2. *If a function* $f(x)$, $x \in [a,b]$, *belongs to the Hölder class* $H(\alpha)$, *i.e., satisfies the Hölder condition with exponent* α, *then the best approximation polynomial* $E_n(f)$ *satisfies the estimate* $E_n(f) \le M/n^{\alpha}$, $0 < \alpha \le 1$, *where* M *is an absolute constant.*

Proceeding to the construction of an approximate solution of Eq. (1), we integrate by parts in the singular integral in this equation and obtain

$$\frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1}\frac{u(t)}{\sqrt{1-t^2}}\frac{dt}{t-x} = \frac{1}{\pi}\int_{-1}^{1} H(x,t)u'(t)\,dt, \tag{7}$$

where

$$H(x,t) = \ln\frac{1 - xt + \sqrt{1-x^2}\sqrt{1-t^2}}{|t-x|}, \qquad -1 \le x \le 1. \tag{8}$$

Owing to identity (7), Eq. (1) is equivalent to the integral equation

$$u'(x) + \gamma(x)\frac{1}{\pi}\int_{-1}^{1} H(x,t)u'(t)\,dt = f(x), \qquad -1 \le x \le 1, \tag{9}$$

with logarithmic kernel. Note that Fredholm theory applies to Eq. (9). Obviously, the function $H(x,t)$ is symmetric. Let us prove that it is nonnegative. Indeed,

$$\begin{aligned} H(x,t) = H(\cos\theta,\cos\sigma) &= \ln\frac{1-\cos(\theta+\sigma)}{2\sin((\theta+\sigma)/2)|\sin((\theta-\sigma)/2)|} \\ &= \ln\frac{\sin((\theta+\sigma)/2)}{|\sin((\theta-\sigma)/2)|} \ge 0, \qquad 0<\sigma, \qquad \theta\le\pi. \end{aligned}$$

Further, note that

$$\frac{1}{\pi}\int_{-1}^{1}|H(x,t)|\,dt = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\int_{-1}^{x}\left(\frac{1}{\sqrt{1-\tau^2}}\frac{d\tau}{t-\tau}\right)dt = \sqrt{1-x^2},$$

and hence one has the estimate

$$\frac{1}{\pi}\int_{-1}^{1}|H(x,t)|\,dt \le 1. \tag{10}$$

Next, we introduce a linear operator $\psi\mapsto K\psi$ by setting

$$K\psi \equiv K(\psi;x) = \gamma(x)\frac{1}{\pi}\int_{-1}^{1}H(x,t)\psi(t)\,dt. \tag{11}$$

Then the Cauchy problem (1), (2) reduces to the equations

$$\psi(x)+K(\psi;x) = f(x), \qquad \psi(x) = u'(x), \qquad u(x) = \int_{x_0}^{x}\psi(t)\,dt. \tag{12}$$

By definition (11), taking into account the Plemelj–Privalov theorem, we conclude that the operator K maps the space $C[-1,1]$ into itself provided that $\gamma(x)\in C[-1,1]$. In addition, it follows from the estimate (10) that

$$\|K\psi\|_C \le \rho\|\psi\|_C, \tag{13}$$

where

$$\rho = \max_{|x|\le 1}(\sqrt{1-x^2}\,|\gamma(x)|). \tag{14}$$

By virtue of the contraction mapping theorem, this implies the following assertion.

Theorem 1. *Let the function $\gamma(x)$ in Eq. (12) satisfy the condition*

$$\rho<1, \tag{15}$$

where the number ρ is defined by relation (14).

Then Eq. (12) in the space $C[-1,1]$ and hence the Cauchy problem (1), (2) in the space $C^1[-1,1]$ have a unique solution for each function $f(x)\in C[-1,1]$.

CONSTRUCTION OF A COMPUTATIONAL SCHEME FOR PROBLEM (1), (2)

Considering the constructive properties of the operator (11), let us calculate the integral

$$\begin{aligned}\frac{1}{\pi}\int_{-1}^{1} H(x,t)U_k(t)\,dt &= \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,U_k(t)\int_{-1}^{x}\left(\frac{1}{\sqrt{1-\tau^2}}\frac{d\tau}{t-\tau}\right)dt \\ &= \int_{-1}^{x}\frac{1}{\sqrt{1-\tau^2}}\left(\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,U_k(t)\frac{dt}{t-\tau}\right)d\tau \\ &= -\int_{-1}^{x}\frac{T_{k+1}(\tau)}{\sqrt{1-\tau^2}}\,d\tau = \sqrt{1-x^2}\,\frac{U_k(x)}{k+1}. \end{aligned} \tag{16}$$

It follows from the representation (16) that an approximate solution of problem (1), (2) can be sought in the form of a polynomial. Further, let $\psi_n(x)$ be the interpolation polynomial (6) of the function $\psi(x)$ on the Chebyshev nodes of the first kind,

$$\psi_n(x) = \sum_{k=0}^{n} c_k^n U_k(x), \tag{17}$$

where the c_k^n, $k = 0,\dots,n$, are yet unknown constants having the meaning of the constants indicated in (6).

We find an approximate solution $\psi_n(x)$ of problem (1), (2) as a solution of the auxiliary equation

$$\psi_n(x) + \gamma(x)\frac{1}{\pi}\int_{-1}^{1} H(x,t)\psi_n(t)\,dt = F_n(x) \tag{18}$$

constructed from Eq. (12). Here $F_n(x)$ is a function of the class $C[-1,1]$ with the property $F_n(x_j) = f(x_j)$, where $x_j = \cos((2j+1)\pi/(2n+2))$, $j = 0,\dots,n$.

It follows from the preceding formulas and (17) that

$$u_n(x) \triangleq \int_{x_0}^{x}\psi_n(t)\,dt = \sum_{k=0}^{n} c_k^n \int_{x_0}^{x} U_k(t)\,dt = \sum_{k=0}^{n}\frac{c_k^n}{k+1}(T_{k+1}(x) - T_{k+1}(x_0)). \tag{19}$$

Obviously, an analog of Theorem 1 holds for Eq. (18) with the operator (11), and Eq. (18) is solvable under condition (15) as well.

Let us simplify the expression for $K(\psi_n; x)$ using the calculations in (16). We have

$$\frac{1}{\pi}\int_{-1}^{1} H(x,t)\psi_n(t)\,dt = \int_{-1}^{x}\frac{1}{\sqrt{1-\tau^2}}\left(\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\frac{\psi_n(t)}{t-\tau}\,dt\right)d\tau = \sqrt{1-x^2}\sum_{k=0}^{n} c_k^n\frac{1}{k+1}U_k(x). \tag{20}$$

Therefore, based on the representation (20), the solvable integral equation (18) is equivalent to the equation

$$\sum_{k=0}^{n} c_k^n\left(\gamma(x)\frac{\sqrt{1-x^2}}{k+1} + 1\right)U_k(x) = F_n(x). \tag{21}$$

Setting $x_j = \cos((2j+1)\pi/(2n+2))$, $j = 0,\dots,n$, in (21), we arrive at the system of linear algebraic equations

$$\sum_{k=0}^{n} c_k^n\left(\gamma(x_j)\frac{\sqrt{1-x_j^2}}{k+1} + 1\right)U_k(x_j) = f(x_j), \qquad j = 0,\dots,n. \tag{22}$$

It follows from the preceding that system (22) has a unique solution. Having solved system (22), we find the constants c_k^n, $k = 0, \dots, n$. Therefore, based on (19), we have

$$u_n(x) = \sum_{k=0}^{n} \frac{c_k^n}{k+1}(T_{k+1}(x) - T_{k+1}(x_0)). \tag{23}$$

Let us estimate the approximation order by studying the structural properties of the function $u'(x)$. It is easily seen based on the Plemelj–Privalov theorem that if the functions $\gamma(x)$ and $f(x)$ belong to the Hölder class $H(\mu)$, $\mu \geq 1/2$, then $u'(x) \in H(1/2)$, $-1 \leq x \leq 1$ (e.g., see [8]). Further, let $F_n(x)$ be the interpolation polynomial (5) of the function $f(x)$. Then

$$\|\psi(x) - \psi_n(x)\|_C \leq \left\| \gamma(x) \frac{1}{\pi} \int_{-1}^{1} H(x,t)(\psi(t) - \psi_n(t))\, dt \right\|_C + \|f(x) - F_n(x)\|_C.$$

Hence it follows by Lemmas 1 and 2 that

$$\|\psi(x) - \psi_n(x)\|_C \left(1 - \left\| \gamma(x) \frac{1}{\pi} \int_{-1}^{1} H(x,t)\, dt \right\|_C \right) = O\left(\frac{\ln n}{\sqrt{n}}\right).$$

Based on inequality (10), we obtain

$$\|\psi(x) - \psi_n(x)\|_C = O\left(\frac{\ln n}{\sqrt{n}}\right).$$

Thus, the above argument proves the following assertion.

Theorem 2. *Assume that the functions $\gamma(x)$ and $f(x)$ occurring in Eq.* (1) *belong to the class $H(\mu)$, $\mu \geq 1/2$, and condition* (15) *is satisfied. Then system* (22) *is solvable for any positive integer n, and the approximate solutions $u_n(x)$ of problem* (1), (2) *constructed by formula* (23) *converge to its exact solution $u_n(x)$ at the rate*

$$\|u(x) - u_n(x)\|_C = O\left(\frac{\ln n}{\sqrt{n}}\right).$$

Example 1. Consider the integro-differential equation

$$u'(x) + \frac{\sqrt{2}(1-x^2)}{2(1+x^2)} \frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{\sqrt{1-t^2}} \frac{dt}{t-x} = -\frac{3(x^2-1)}{2(1+x^2)^2}, \qquad u(-1) = 0. \tag{24}$$

One can readily verify that in this case the function

$$u(x) = \frac{x}{1+x^2} + \frac{1}{2}$$

is the solution of problem (24).

Calculations performed in the MathCad 15 computer algebra system show that the approximate solution of Eq. (24) can be determined rather accurately even for relatively small n.

Solving system (22) for $n = 12$, 24, and 36, we find that the exact solution $u(x)$ differs from the approximate solution $u_n(x)$ calculated by formula (23) on the system of points $x = -0.99, -0.98, \dots, 0.99$ by at most $6.7 \cdot 10^{-6}$, $1.5 \cdot 10^{-10}$, and $4.0 \cdot 10^{-15}$, respectively. Moreover, the condition number *conde* of the system matrices does not exceed 31, 84, and 153, respectively.

CONSTRUCTION OF A COMPUTATIONAL SCHEME FOR PROBLEM (3), (4)

Let us reduce Eq. (3) to a Fredholm equation of the second kind with a logarithmic singularity. Let

$$u(x) \triangleq -\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma'(t)}{t-x}\,dt. \tag{25}$$

We apply the singular integral inversion formula in the function class H^* to (25) and obtain

$$\Gamma'(x) = \frac{1}{\sqrt{1-x^2}}\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}u(t)}{t-x}\,dt + \frac{c}{\sqrt{1-x^2}}.$$

Here c is an arbitrary constant. Hence, taking into account the boundary conditions (4), we conclude that

$$\Gamma(x) = \int_{-1}^{x}\Gamma'(\tau)\,d\tau = \int_{-1}^{x}\frac{1}{\sqrt{1-\tau^2}}\left(\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\frac{u(t)}{t-\tau}\,dt + \frac{c}{\sqrt{1-\tau^2}}\right)d\tau = \frac{1}{\pi}\int_{-1}^{1}H(x,t)u(t)\,dt + \mu(x),$$

where the function

$$H(x,t) = \sqrt{1-t^2}\int_{-1}^{x}\left(\frac{1}{\sqrt{1-\tau^2}}\frac{d\tau}{t-\tau}\right) = \ln\frac{1-xt+\sqrt{1-x^2}\sqrt{1-t^2}}{|t-x|}$$

is the same as in (8) and $\mu(x) = c(\arcsin x + \pi/2)$. Since $H(-1,t) = H(1,t)$, we find the constant $c = 0$.

Finally, we have

$$\Gamma(x) = \frac{1}{\pi}\int_{-1}^{1}H(x,t)u(t)\,dt.$$

We introduce a linear operator $u \mapsto Ku$ by setting

$$K(u;x) = \frac{1}{B(x)}\frac{1}{\pi}\int_{-1}^{1}H(x,t)u(t)\,dt. \tag{26}$$

Then the boundary value problem (3), (4) reduces to the equation

$$u(x) + K(u;x) = f(x). \tag{27}$$

Comparing definitions (11) and (26), we conclude by Theorem 1 that the following assertion holds.

Theorem 3. *Let the function $B(x)$ in Eq.* (3) *satisfy the condition*

$$\rho = \max_{|x|\le 1}(\sqrt{1-x^2}/|B(x)|) < 1. \tag{28}$$

Then the boundary value problem (3), (4) *has a unique solution in the class of functions $\Gamma'(x) \in H^*$ for each function $f(x) \in C[-1,1]$.*

Recall that the class H^* consists of functions satisfying the Hölder condition on the interval $[-1+\varepsilon_1, 1-\varepsilon_2]$ for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and admitting an integrable singularity in a neighborhood of the points -1 and 1.

Based on the argument in the proof of Theorem 2, let us briefly dwell on the approximate solution of Eq. (3).

Let $u_n(x)$ be the interpolation polynomial (6) of the function $u(x)$; i.e.,

$$u_n(x) = -\frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma_n'(t)}{t-x}\,dt = \sum_{k=0}^{n} c_k^n U_k(x). \tag{29}$$

We seek an approximate solution of Eq. (3) under condition (4) as a solution of the auxiliary equation

$$u_n(x) + \frac{1}{B(x)}\,\frac{1}{\pi}\int_{-1}^{1} H(x,t)u_n(t)\,dt = F_n(x), \qquad -1 < x < 1, \tag{30}$$

where $F_n(x)$ is a function of the class $C[-1,1]$ such that $F_n(x_j) = f(x_j)$, $x_j = \cos((2j+1)\pi/(2n+2))$, $j = 0,\dots,n$, which has a unique solution under condition (28).

Given (29), we simplify the expression for $\Gamma_n(x)$. We have

$$\Gamma_n(x) = \frac{1}{\pi}\int_{-1}^{1} H(x,t)u_n(t)\,dt = \sqrt{1-x^2}\sum_{k=0}^{n} c_k^n \frac{1}{k+1}U_k(x). \tag{31}$$

The representation (31) obviously implies that the conditions $\Gamma_n(-1) = \Gamma_n(+1) = 0$ are satisfied.

By analogy with Eq. (18), Eq. (30) can be written as

$$\sum_{k=0}^{n} c_k^n\left(\frac{\sqrt{1-x^2}}{B(x)}\,\frac{1}{k+1} + 1\right)U_k(x) = F_n(x). \tag{32}$$

Scheme 1. We take the nodes $x_j = \cos((2j+1)\pi/(2n+2))$, $j = 0,\dots,n$, for the external nodes in (32).

Then we arrive at the system of linear algebraic equations

$$\sum_{k=0}^{n} c_k\left(\frac{\sqrt{1-x_j^2}}{B(x_j)}\,\frac{1}{k+1} + 1\right)U_k(x_j) = f(x_j), \qquad j = 0,\dots,n. \tag{33}$$

The approximate solution $\Gamma_n(x)$ is calculated according to (31).

Theorem 4. *Let the functions $B(x)$ and $f(x)$ occurring in Eq.* (3) *belong to the class $H(\mu)$, $\mu \ge 1/2$, and let condition* (28) *be satisfied.*

Then system (33) *is solvable, and the approximate solutions $\Gamma_n(x)$ of problem* (3), (4) *constructed by formula* (31) *converge to the exact solution $\Gamma(x)$ at the rate*

$$\|\Gamma(x) - \Gamma_n(x)\|_C = O\left(\frac{\ln n}{\sqrt{n}}\right).$$

Example 2. Consider the integro-differential equation

$$\frac{\Gamma(x)}{B(x)} - \frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma'(t)}{t-x}\,dt = \frac{1}{B(x)}\frac{\sqrt{2}}{2}\,\ln\left(\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}+\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\frac{1}{1+x^2} + 1, \qquad -1 < x < 1. \tag{34}$$

Let $B(x) = \sqrt{1-x^2}(3+x^2)/(1+2x^2)$. One can readily verify that the function

$$\Gamma(x) = \frac{\sqrt{2}}{2} \ln\left(\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}+\sqrt{2}} \right)$$

is a solution of problem (34), (4) in this case.

Solving system (33) for $n = 10$ and 33, we find that the exact solution $\Gamma(x)$ differs from the approximate solution $\Gamma_n(x)$ computed by formula (31) on the system of points $x = -0.99, -0.98, \ldots, 0.99$ by at most $2.8 \cdot 10^{-6}$ and $4.0 \cdot 10^{-15}$, respectively. The condition number *conde* of the system matrices does not exceed 26 and 140, respectively.

Scheme 2. Considering the case of the function $B(x) = b\sqrt{1-x^2}$, $b =$ const, we take the interpolation polynomial (6) for $F_n(x)$ in Eq. (32). Then Eq. (32) becomes

$$\sum_{k=0}^{n} c_k^n \left(\frac{1}{b} \frac{1}{k+1} + 1 \right) U_k(x) = \sum_{k=0}^{n} f_k^n U_k(x), \tag{35}$$

where the f_k^n are calculated in accordance with (6). It follows that the numbers

$$c_k^n = f_k^n \left(\frac{1}{b} \frac{1}{k+1} + 1 \right)^{-1}, \qquad k = 0, \ldots, n,$$

are a solution of system (35). The approximate solution $\Gamma_n(x)$ of problem (3), (4) is calculated according to (31).

Comment. Note that in our paper [8] we present a computational scheme for the approximate solution of the boundary value problem (3), (4) based on the approximation of the integral in Eq. (27) by a quadrature sum.

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