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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Application of the Leray–Schauder Principle to the Analysis of a Nonlinear Integral Equation

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Abstract— We study a nonlinear integral equation arising from the parametric closure for the third spatial moment in the Dieckmann–Law model of stationary biological communities. The existence of a fixed point of the integral operator defined by this equation is analyzed. The noncompactness of the resulting operator is proved. Conditions are stated under which the equation in question has a nontrivial solution.

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INTRODUCTION

This paper continues the study of integral equations arising in the Dieckmann–Law model of stationary biological communities initiated in [1–3]. One most important result in the cited papers is the derivation of the system of integro-differential equations

$$\dot{N} = (b-d)N - d' \int\limits_{\mathbb{R}} C(\xi)w(\xi)\,d\xi,$$

$$\dot{C}(\xi) = bm(\xi)N + \int\limits_{\mathbb{R}} bm(\xi')C(\xi+\xi')\,d\xi' - (d+d'\omega(\xi))C(\xi) - \int\limits_{\mathbb{R}} d'\omega(\xi')T(\xi,\xi')\,d\xi'$$

describing the spatial moment dynamics. Here the constants b, d, and d', as well as the functions m and ω, are assumed to be known, while the functions N, C, and T (the first, second, and third moments, respectively) are to be determined.

Considering this system to be at equilibrium (the moments do not change in time) and expressing the third spatial moment via the first two moments [4], we arrive at various linear and nonlinear integral equilibrium equations. The linear integral equilibrium equation

$$(b+d'w(x))C(x) = \int\limits_{\mathbb{R}} bm(y)C(x+y)\,dy + \frac{b}{b-d}m(x)\int\limits_{\mathbb{R}} d'w(y)C(y)\,dy$$

has been studied by many authors (e.g., see [5–8]), and it has been shown by now that it fails to describe the biological system under study well enough. Therefore, the analysis of this equation is not important when studying the operation of stationary biological communities.

The main object of study in the present paper is the nonlinear integral equilibrium equation

$$\left(\overline{\omega} + b - \frac{\alpha}{2}\left(b-d-\frac{d'(b-d)}{Y}\right)\right)C = \frac{Y\overline{m}}{b-d} - \overline{\omega} + [\overline{m} * C] - \alpha\frac{b-d}{2Y}((C+2)[\overline{\omega} * C] + [\overline{\omega}C * C]), \qquad (1)$$

where $\alpha \in [0,1]$, $\overline{m} = bm$, $\overline{\omega} = d'\omega$, $Y = \int_{\mathbb{R}}(C(x)+1)\overline{\omega}(x)\,dx$, and $*$ stands for the operation of convolution of functions.

The derivation of Eq. (1) is described in [9, 10]. It was indicated in these papers that the analytical study of this equation is rather difficult, and so it was investigated numerically.

Let us write Eq. (1) in the operator form $\mathcal{A}f = f$, where

$$\mathcal{A}f = \left(\frac{Y\overline{m}}{b-d} - \overline{\omega} + [\overline{m}*f] - \alpha\frac{b-d}{2Y}((f+2)[\overline{\omega}*f] + [\overline{\omega}f*f])\right)\left(\overline{\omega} + b - \frac{\alpha}{2}\left(b-d-\frac{d'(b-d)}{Y}\right)\right)^{-1}. \quad (2)$$

The difficulties associated with the analytical study of this nonlinear integral operator are primarily due to the fact that it is neither compact nor contracting; this substantially complicates studying the existence of a fixed point of this operator.

1. AUXILIARY ASSERTIONS

Let us present some assertions to be used in the paper.

Lemma 1. *Let f be a measurable function, let $g \in L_1(\mathbb{R})$, and let the inequality $|f(x)| \le g(x)$ hold for almost all $x \in \mathbb{R}$. Then $f \in L_1(\mathbb{R})$.*

Corollary 1. *If $f \in L_1(\mathbb{R})$ and g is a bounded measurable function, then $fg \in L_1(\mathbb{R})$.*

Lemma 2. *Let $f \in C(\mathbb{R})$ and $\lim_{|x|\to+\infty} f(x) = 0$. Then the function f is bounded.*

Lemma 3. *Let $f \in C(\mathbb{R})$ and $\lim_{|x|\to+\infty} f(x) = 0$. Then the function f is uniformly continuous on $\mathbb{R}$.*

Lemma 4. *If $f \in L_1(\mathbb{R})$, then for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the estimate*

$$\int\limits_{\mathbb{R}} |f(x+h) - f(x)|\,dx \le \varepsilon$$

holds for each $h \in \mathbb{R}$ satisfying the inequality $|h| \le \delta$.

Lemma 5 (Riesz criterion). *A set $K \subset L_p(\mathbb{R})$ is precompact if and only if the following two conditions are satisfied:*

1. *There exists a positive constant M such that $\|f\|_p \le M$ for each $f \in K$.*
2. *For each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the estimate*

$$\left(\int\limits_{\mathbb{R}} |f(x+h) - f(x)|^p\,dx\right)^{1/p} \le \varepsilon$$

holds for any $h \in \mathbb{R}$ with $|h| \le \delta$ and any $f \in K$.

Theorem (Fubini). *Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a measurable function such that the integral $\int_{\mathbb{R}} |f(x,y)|\,dx = F(y)$ exists for almost all $y \in \mathbb{R}$ and $\int_{\mathbb{R}} F(y)\,dy < +\infty$. Then the integrals*

$$\int\limits_{\mathbb{R}\times\mathbb{R}} f(x,y)\,dx\,dy \quad \text{and} \quad \int\limits_{\mathbb{R}}\left(\int\limits_{\mathbb{R}} f(x,y)\,dy\right)dx$$

exist, and one has

$$\int\limits_{\mathbb{R}}\left(\int\limits_{\mathbb{R}} f(x,y)\,dx\right)dy = \int\limits_{\mathbb{R}}\left(\int\limits_{\mathbb{R}} f(x,y)\,dy\right)dx = \int\limits_{\mathbb{R}\times\mathbb{R}} f(x,y)\,dx\,dy.$$

Remark 1. We will use the following obvious assertion:

$$\int\limits_{\mathbb{R}} |f(x+y)|\,dx = \|f\|_1$$

for any $f \in L_1(\mathbb{R})$ and $y \in \mathbb{R}$. We will also take it for granted that a function is integrable if and only if so is its absolute value [11, Sec. 9, item 111].

Remark 2. From now on, we assume that the functions ω and m belong to the space $C(\mathbb{R}) \cap L_1(\mathbb{R})$ and are nonnegative and even and also that $\|m\|_1 = \|\omega\|_1 = 1$ and $\lim_{|x|\to\infty} m(x) = \lim_{|x|\to\infty} \omega(x) = 0$. Then ω and m are bounded and uniformly continuous on $\mathbb{R}$ by Lemmas 2 and 3.

Remark 3. In what follows, for $R > 0$, by $B(R)$ we denote the closed ball of radius R centered at zero in the space $L_1(\mathbb{R})$; i.e., $B(R) = \{f \in L_1(\mathbb{R}) : \|f\|_1 \le R\}$.

2. MAIN PROOF

Lemma 6. *The operator* $\mathcal{B}_\omega f = [\omega * f]$ *is compact from* $L_1(\mathbb{R})$ *to* $L_1(\mathbb{R})$.

Proof. First, note that the properties of convolution readily imply that the operator indeed maps $L_1(\mathbb{R})$ into itself.

Let $M > 0$. For each function $f \in B(M)$,

$$\begin{aligned}\|\mathcal{B}_\omega f\|_1 &= \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}} \omega(x-y) f(y)\,dy \right| dx \le \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |\omega(x-y) f(y)|\,dy\,dx \\ &= \int\limits_{\mathbb{R}} |f(y)| \int\limits_{\mathbb{R}} |\omega(x-y)|\,dx\,dy = \|\omega\|_1 \|f\|_1 \le M.\end{aligned}$$

Changing the order of integration is justified by the Fubini theorem, which can be used here because the convolution is an integrable function, and hence the inner integral exists for each x and is integrable.

Further, let us estimate the expression $\int_{\mathbb{R}} |[\mathcal{B}_\omega f](x+h) - [\mathcal{B}_\omega f](x)|\,dx$ from above. Using the Fubini theorem and Lemma 4, we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}} \Big| [\mathcal{B}_\omega f](x+h) - [\mathcal{B}_\omega f](x) \Big|\,dx &= \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}} [\omega(x+h-y) - \omega(x-y)] f(y)\,dy \right| dx \\ &\le \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |\omega(x+h-y) - \omega(x-y)||f(y)|\,dy\,dx \\ &= \int\limits_{\mathbb{R}} |f(y)| \int\limits_{\mathbb{R}} |\omega(x+h-y) - \omega(x-y)|\,dx\,dy \le \varepsilon \|f\|_1 \le \varepsilon M.\end{aligned}$$

Therefore, the expression to be estimated is small for sufficiently small h; i.e., the image of $B(M)$ under the operator $\mathcal{B}_\omega$ is uniformly bounded and equicontinuous in the sense of $L_1(\mathbb{R})$. By the Riesz criterion, this set is precompact. Thus, the operator $\mathcal{B}_\omega$ takes bounded sets to precompact ones and hence is compact. The proof of the lemma is complete.

Remark 4. It is obvious that the operator $\mathcal{B}_m f = [m * f]$ is compact as well.

Lemma 7. *The operator*

$$\mathcal{C} f = \varphi(x) \int\limits_{\mathbb{R}} \omega(y) f(y)\,dx + \psi(x),$$

where φ *and* ψ *are continuous integrable functions, is compact from* $L_1(\mathbb{R})$ *to* $L_1(\mathbb{R})$.

Proof. By Lemmas 2 and 4, the functions φ and ψ are bounded and continuous in the sense of L_1 on $\mathbb{R}$. Let us use the Riesz criterion. Let $M>0$ and $f\in B(M)$. Then

$$\|\mathcal{C}f\| = \int\limits_{\mathbb{R}} |\varphi(x)|\,dx \left|\int\limits_{\mathbb{R}} \omega(y)f(y)\,dy\right| + \int\limits_{\mathbb{R}} |\psi(x)|\,dx \le M\|\varphi\|_1\|\omega\|_C + \|\psi\|_1,$$

$$\int\limits_{\mathbb{R}} \Big|[\mathcal{C}f](x+h) - [\mathcal{C}f](x)\Big|\,dx \le \int\limits_{\mathbb{R}} |\varphi(x+h)-\varphi(x)|\,dx \int\limits_{\mathbb{R}} |\omega(y)f(y)|\,dy + \int\limits_{\mathbb{R}} |\psi(x+h)-\psi(x)|\,dx \le \varepsilon M\|\omega\|_C + \varepsilon.$$

Hence the operator $\mathcal{C}$ is compact. The proof of the lemma is complete.

Now consider the operator $\mathcal{A}$ defined in (2). We represent this operator as the sum $\mathcal{A}=\mathcal{K}+\mathcal{S}$, where

$$\mathcal{K}f = \left(\frac{Y\overline{m}}{b-d} - \overline{\omega} + [\overline{m}*f] - \alpha\frac{b-d}{Y}[\overline{\omega}*f]\right)\left(\overline{\omega} + b - \frac{\alpha}{2}\left(b-d-\frac{d'(b-d)}{Y}\right)\right)^{-1},$$

$$\mathcal{S}f = -\alpha\frac{b-d}{2Y}(f[\overline{\omega}*f] + [\overline{\omega}f*f])\left(\overline{\omega} + b - \frac{\alpha}{2}\left(b-d-\frac{d'(b-d)}{Y}\right)\right)^{-1}.$$

Theorem 1. *Let $b>d\ge 0$, $d'\ge 0$, and $\alpha\in[0,1]$. If $R<1/(\|\omega\|_C)$, then the operator $\mathcal{K}$ is well defined as an operator from $B(R)$ to $L_1(\mathbb{R})$ and is compact.*

Proof. Consider the function

$$g_s(x) = \left(\overline{\omega}(x) + b - \frac{\alpha}{2}\left(b-d-\frac{d'(b-d)}{s}\right)\right)^{-1}.$$

Fix an $x=x_0$ for which the denominator of this function is zero and express the parameter s from the equation

$$\overline{\omega}(x_0) + b - \frac{\alpha}{2}\left(b-d-\frac{d'(b-d)}{s}\right) = 0.$$

Then

$$s = \frac{\alpha d'(b-d)}{(\alpha-2)b - \alpha d - 2\overline{\omega}(x_0)}.$$

Since $b>d$ and $\alpha\in[0,1]$, we have $s\le 0$; in this case, the estimate

$$s \le \frac{\alpha d'(b-d)}{(\alpha-2)b-\alpha d} = q$$

holds. It follows that the function $g_s(x)$ is continuous for $s>q$.

On the other hand, $Y\ge d' - \|\overline{\omega}\|_C\|f\|_1$; thus, if $\|f\|_1 < (d'-q)/\|\overline{\omega}\|_C$, then $Y>q$. If we set $s=Y$, then it is obvious that the function $g_Y(x)$ is bounded and continuous; in particular, it is measurable on $\mathbb{R}$. It follows that the product of this function by any integrable function lies in the class $L_1(\mathbb{R})$ (Corollary 1).

The operator $\mathcal{K}$ can be represented in the form

$$\mathcal{K}f = g_Y\left(\mathcal{C} + \mathcal{B}_{\overline{m}} - \alpha\frac{b-d}{Y}\mathcal{B}_{\overline{\omega}}\right)f,$$

where

$$\mathcal{C}f = \frac{Y\overline{m}}{b-d} - \overline{\omega}, \qquad \mathcal{B}_{\overline{m}}f = [\overline{m}*f], \qquad \mathcal{B}_{\overline{\omega}}f = -\alpha\frac{b-d}{Y}[\overline{\omega}*f].$$

If

$$\|f\|_1 \le R < \frac{d'}{\|\overline{\omega}\|_C} = \frac{1}{\|\omega\|_C},$$

then the expression

$$-\alpha\frac{b-d}{2Y} = -\alpha(b-d)\left(2\left(\int\limits_{\mathbb{R}} f(x)\overline{\omega}(x)\,dx + d'\right)\right)^{-1}$$

is uniformly bounded with respect to f. This condition is sufficient for the function $g_Y(x)$ to be bounded and measurable. In this case, the operators $\mathcal{C}$, $\mathcal{B}_{\overline{m}}$, and $\mathcal{B}_{\overline{\omega}}$ act on $L_1(\mathbb{R})$ and are compact by Lemmas 6 and 7; therefore, the operator $\mathcal{K}$ is compact as well. The proof of the theorem is complete.

Theorem (Leray–Schauder). *If an operator $\mathcal{A}$ defined on a closed ball B in a Banach space is compact and $\mathcal{A}[\partial B] \subset B$, then there exists an $f \in B$ such that $f = \mathcal{A}f$.*

The proof of this theorem can be found, e.g., in the monograph [12, pp. 129–130]. We only note that this proof is based on the notion of rotation of a mapping. In particular, if the rotation of a compact operator on the boundary of a ball is nonzero, then such an operator has fixed points.

Theorem 2. *Under the assumptions of Theorem* 1, *if $\rho = 1 - R\|\omega\|_C > 0$ and $\alpha > 0$, then there exists a $d' \in (0, 3\rho/4)$ such that the operator $\mathcal{K}$ has a fixed point in $B(R)$. If $\alpha = 0$, then for a fixed point to exist it suffices that $d' = 0$.*

Proof. Let us estimate the norm of the image of a function $f \in B(R)$ under the operator $\mathcal{K}$. First, we establish an upper bound for Y. Since $0 < R\|\omega\|_C < 1$, we have $1 + R\|\omega\|_C < 2$, and then

$$0 < d'\rho = d'(1 - R\|\omega\|_C) \le Y \le d'(1 + R\|\omega\|_C) < 2d'.$$

Thus,

$$\frac{1}{d'\rho} \ge \frac{1}{Y} > \frac{1}{2d'}.$$

Hence the following estimate holds for the function $g_Y(x)$ defined in the proof of Theorem 1:

$$|g_Y(x)| < \left(\omega(x) + \frac{\alpha}{2}\frac{d'(b-d)}{2d'} + b - \frac{\alpha}{2}(b-d)\right)^{-1} < \left(\frac{3}{4}\alpha(b-d) + b\right)^{-1}.$$

Let us estimate the norm of the convolutions occurring in the definition of the operator $\mathcal{K}$. We have

$$\begin{aligned}\Big\|[\overline{m} * f]\Big\|_1 &= \int\limits_{\mathbb{R}} \left|\int\limits_{\mathbb{R}} \overline{m}(x-y)f(y)\,dy\right| dx \le \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |\overline{m}(x-y)||f(y)|\,dy\,dx \\ &= \int\limits_{\mathbb{R}} |f(y)| \int\limits_{\mathbb{R}} |\overline{m}(x-y)|\,dx\,dy = \|\overline{m}\|_1\|f\|_1 \le bR.\end{aligned}$$

The order of integration can be changed by the Fubini theorem, because the convolution of integrable functions is integrable. In a similar way, we obtain

$$\|[\overline{\omega} * f]\|_1 \le d'R.$$

As a result, we arrive at the estimates

$$\|\mathcal{K}f\|_1 \le \int\limits_{\mathbb{R}} \left|\overline{\omega}(x) + b - \frac{\alpha}{2}\left(1 - \frac{d'}{Y}\right)(b-d)\right|^{-1}$$
$$\times \left(\left|\frac{Y\overline{m}(x)}{b-d}\right| + |\overline{\omega}(x)| + |[\overline{m} * f](x)| + \left|\alpha\frac{b-d}{Y}[\overline{\omega} * f](x)\right|\right) dx$$
$$\le \int\limits_{\mathbb{R}} \left(b + \frac{3}{4}\alpha(b-d)\right)^{-1}\left(\frac{2bd'}{b-d}|\overline{m}(x)| + |\overline{\omega}(x)| + \Big|[\overline{m} * f](x)\Big| + \alpha\frac{b-d}{\rho}\Big|[\overline{\omega} * f](x)\Big|\right) dx$$
$$\le \left(b + \frac{3}{4}\alpha(b-d)\right)^{-1}\left(\frac{2bd'}{b-d} + d' + bR + \alpha d'\frac{b-d}{\rho}R\right)$$
$$= \left(b + \frac{3}{4}\alpha(b-d)\right)^{-1}\left(d'\left(\frac{2bd'}{b-d} + 1\right) + \left(b + \alpha d'\frac{b-d}{\rho}\right)R\right).$$

Set

$$\xi = d'\left(b + \frac{3}{4}\alpha(b-d)\right)^{-1}\left(\frac{2bd'}{b-d} + 1\right), \qquad \eta = \left(b + \frac{3}{4}\alpha(b-d)\right)^{-1}\left(b + \frac{d'}{\rho}\alpha(b-d)\right).$$

Then $\|\mathcal{K}f\|_1 \le \xi + \eta R$. It is obvious that if $d' < 3\rho/4$ and $\alpha > 0$, then $\eta < 1$. Moreover, $\xi \to 0$ as $d' \to 0$. These two facts imply the existence of a $d' \in (0, 3\rho/4)$ such that the inequality $\|\mathcal{K}f\|_1 < R$ holds for each function $f \in B(R)$; i.e., $\mathcal{K}f \in \operatorname{int} B(R)$.

Thus, there exists a small number $d' \in (0, 3\rho/4)$ such that the operator $\mathcal{K}$ maps the ball $B(R)$ into the interior of $B(R)$. Since this operator is compact in $B(R)$, we see that it has a fixed point in $B(R)$ by the Leray–Schauder theorem.

If $\alpha = 0$, then $\eta = 1$ and $\mathcal{K}f \le R$ provided that $d' = 0$. In a similar way, the operator $\mathcal{K}$ has a fixed point in $B(R)$ by the Leray–Schauder theorem.

Note that it is important in the subsequent theorems that for $\alpha > 0$ the ball $B(R)$ is taken by the operator $\mathcal{K}$ into some closed subball $B' \subset B(R)$ such that $d(\partial B', \partial B(R)) > 0$. Here $d(A, B)$ stands for the distance between sets A and B in the metric generated by the norm of the space $L_1(\mathbb{R})$. The proof of the theorem is complete.

Remark 5. Note that the operator $\mathcal{S}$ is also well defined as an operator from $B(R)$ to $L_1(\mathbb{R})$, where the condition imposed on R is the same as in Theorem 1, but is not compact.

Let us prove the assertions contained in this remark. Let $f \in B(R)$. It follows from the properties of the convolution that $[\overline{\omega}f * f] \in L_1(\mathbb{R})$; moreover,

$$|[\overline{\omega} * f](x)| = \left|\int\limits_{\mathbb{R}} \overline{\omega}(x-y)f(y)\,dy\right| \le R\|\overline{\omega}\|_C;$$

i.e., according to the corollary of Lemma 1, we have the inclusion $f[\overline{\omega} * f] \in L_1(\mathbb{R})$. It follows from the proof of Theorem 2 that, under the condition $f \in B(R)$, the function

$$h_Y(x) = -\alpha\frac{b-d}{2Y}g_Y(x) = -\alpha\frac{b-d}{2Y}\left(\omega(x) + b - \frac{\alpha}{2}\left(b - d - \frac{d'(b-d)}{Y}\right)\right)^{-1}$$

is separated from infinity uniformly with respect to f. Thus, the operator $\mathcal{S}$ acts from $B(R)$ to $L_1(\mathbb{R})$.

To prove that this operator is noncompact, it suffices to find a sequence of functions $f_n \in B(R)$, $n \in \mathbb{N}$, that does not contain a Cauchy subsequence.

Since $\overline{\omega}(x)$ is a nonnegative continuous function with unit norm, it follows that there exists an $x_0 \in \mathbb{R}$ and a $\delta > 0$ such that the inequality $\overline{\omega}(x_0) \ge \mu > 0$ holds for each $x \in (x_0 - \delta; x_0 + \delta)$.

Without loss of generality, we take $x_0 > 0$, because the function ω is even. Let $I_n = [nx_0, (n+1)x_0]$, $n \in \mathbb{N}$. Define the function f_n by the relation

$$f_n(x) = \begin{cases} R/(2x_0), & x \in I_n, \\ 0, & x \notin I_n. \end{cases}$$

Obviously, the f_n, $n \in \mathbb{N}$, lie in the ball $B(R)$. Note also that since the function $h_Y(x)$ is separated from zero and infinity uniformly with respect to f, it suffices to prove that the operator $\mathcal{P}f = f[\overline{\omega} * f] + [\overline{\omega} f * f]$ is noncompact. In the subsequent computations, we take into account the fact that

$$f_n(x-y) = f_m(x-y) = 0$$

for $x \in (nx_0, (n+1)x_0)$ and $y \in (mx_0, (m+1)x_0)$, $m \neq n$.

Let $n, p \in \mathbb{N}$. Then

$$\begin{aligned}
&\|\mathcal{P}f_{n+p} - \mathcal{P}f_n\|_1 \\
&\quad= \int_{\mathbb{R}} |f_{n+p}(x)[\overline{\omega} * f_{n+p}](x) + [\overline{\omega} f_{n+p} * f_{n+p}](x) - f_n(x)[\overline{\omega} * f_n](x) - [\overline{\omega} f_n * f_n](x)|\, dx \\
&\quad\geq \int_{I_n} |f_{n+p}(x)[\overline{\omega} * f_{n+p}](x) + [\overline{\omega} f_{n+p} * f_{n+p}](x) - f_n(x)[\overline{\omega} * f_n](x) - [\overline{\omega} f_n * f_n](x)|\, dx \\
&\quad= \int_{I_n} \left| \int_{\mathbb{R}} \overline{\omega}(y) f_{n+p}(y) f_{n+p}(x-y)\, dy - f_n(x)[\overline{\omega} * f_n](x) - [\overline{\omega} f_n * f_n](x) \right| dx \\
&\quad= \int_{I_n} \left| \frac{R}{2x_0} \int_{I_{n+p}} \overline{\omega}(y) f_{n+p}(x-y)\, dy - f_n(x)[\overline{\omega} * f_n](x) - [\overline{\omega} f_n * f_n](x) \right| dx \\
&\quad= \int_{I_n} |f_n(x)[\overline{\omega} * f_n](x) + [\overline{\omega} f_n * f_n](x)|\, dx.
\end{aligned}$$

Both terms in the last expression are nonnegative, because so are the functions f_n and $\overline{\omega}$; therefore,

$$\begin{aligned}
\int_{I_n} |f_n(x)[\overline{\omega} * f_n](x) + [\overline{\omega} f_n * f_n](x)|\, dx &= \int_{I_n} (f_n(x)[\overline{\omega} * f_n](x) + [\overline{\omega} f_n * f_n](x))\, dx \\
&\geq \int_{I_n} f_n(x)[\overline{\omega} * f_n](x)\, dx = \frac{R}{2x_0} \int_{I_n} [\overline{\omega} * f_n](x)\, dx \\
&= \frac{R^2}{4x_0^2} \int_{I_n}\int_{I_n} \overline{\omega}(x-y)\, dy\, dx \geq \frac{R^2 \delta^2}{4x_0^2} \mu > 0.
\end{aligned}$$

Thus, there exists an $\varepsilon = R^2\delta^2\mu/(4x_0^2) > 0$ such that the estimate

$$\|\mathcal{P}f_{n+p} - \mathcal{P}f_n\|_1 \geq \varepsilon$$

holds for any positive integer n and p; this implies the noncompactness of the operator $\mathcal{P}$ and hence of the operator $\mathcal{S}$. The proof of the assertions in Remark 5 is complete.

Theorem (on fixed points of a perturbed compact operator). *Suppose that a compact operator with nonzero rotation on the boundary is defined on a domain G in a Banach space and that this operator takes the domain G into some subdomain $H \subset G$ such that*

$$d(\partial G, \partial H) = \delta > 0.$$

If this operator is perturbed by a Lipschitz operator whose norm does not exceed δ, then the perturbed operator has fixed points in G.

The proof of this theorem can be found in the monograph [12, pp. 162–163].

Theorem 3. *Let the assumptions of Theorems* 1 *and* 2 *be satisfied. If* $\alpha > 0$, *then the operator* $\mathcal{A}$ *has a fixed point in* $B(R)$ *for sufficiently small* d'.

Proof. Let us establish the Lipschitz property of the operator $\mathcal{S}$. Let $f \in B(R)$. By virtue of the condition imposed on R, the expression

$$-\alpha\frac{b-d}{2Y}\left(\overline{\omega}+b-\frac{\alpha}{2}\left(b-d-\frac{d'(b-d)}{Y}\right)\right)^{-1}$$

is separated from zero and infinity uniformly with respect to f; therefore, it suffices to prove the Lipschitz property of the operator $\mathcal{P}f = f[\overline{\omega}*f]+[\overline{\omega}f*f]$.

The following estimates hold:

$$\begin{aligned}
&\|f[\overline{\omega}*f]-g[\overline{\omega}*g]\|_1 = \|f[\overline{\omega}*f]-f[\overline{\omega}*g]+f[\overline{\omega}*g]-g[\overline{\omega}*g]\|_1\\
&\quad = \|f[\overline{\omega}*(f-g)]+(f-g)[\overline{\omega}*g]\|_1 \le \|f[\overline{\omega}*(f-g)]\|_1+\|(f-g)[\overline{\omega}*g]\|_1\\
&\quad = \int\limits_{\mathbb{R}}\left|f(x)\int\limits_{\mathbb{R}}\overline{\omega}(x-y)[f(y)-g(y)]\,dy\right|dx+\int\limits_{\mathbb{R}}\left|[f(x)-g(x)]\int\limits_{\mathbb{R}}\overline{\omega}(x-y)g(y)\,dy\right|dx\\
&\quad \le \|\overline{\omega}\|_C\int\limits_{\mathbb{R}}|f(x)|\int\limits_{\mathbb{R}}|f(y)-g(y)|\,dy\,dx+\|\overline{\omega}\|_C\int\limits_{\mathbb{R}}|f(x)-g(x)|\int\limits_{\mathbb{R}}|g(y)|\,dy\,dx\\
&\quad = \|\overline{\omega}\|_C\|f-g\|_1(\|f\|_1+\|g\|_1)\le 2R\|\overline{\omega}\|_C\|f-g\|_1 < 2d'\|f-g\|_1,\\
&\|[\overline{\omega}f*f]-[\overline{\omega}g*g]\|_1 = \|[\overline{\omega}f*f]-[\overline{\omega}f*g]+[\overline{\omega}f*g]-[\overline{\omega}g*g]\|_1\\
&\quad = \|[\overline{\omega}f*(f-g)]+[\overline{\omega}(f-g)*g]\|_1 \le \|[\overline{\omega}f*(f-g)]\|_1+\|[\overline{\omega}(f-g)*g]\|_1\\
&\quad = \int\limits_{\mathbb{R}}\left|\int\limits_{\mathbb{R}}\overline{\omega}(y)f(y)[f(x-y)-g(x-y)]\,dy\right|dx+\int\limits_{\mathbb{R}}\left|\int\limits_{\mathbb{R}}\overline{\omega}(y)[f(y)-g(y)]g(x-y)\,dy\right|dx\\
&\quad \le \|\overline{\omega}\|_C\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}|f(y)||f(x-y)-g(x-y)|\,dy\,dx+\|\overline{\omega}\|_C\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}|f(y)-g(y)||g(x-y)|\,dy\,dx\\
&\quad = \|\overline{\omega}\|_C\int\limits_{\mathbb{R}}|f(y)|\int\limits_{\mathbb{R}}|f(x-y)-g(x-y)|\,dx\,dy+\|\overline{\omega}\|_C\int\limits_{\mathbb{R}}|f(y)-g(y)|\int\limits_{\mathbb{R}}|g(x-y)|\,dx\,dy\\
&\quad = \|\overline{\omega}\|_C\|f-g\|_1(\|g\|_1+\|f\|_1)\le 2R\|\overline{\omega}\|_C\|f-g\|_1 < 2d'\|f-g\|_1.
\end{aligned}$$

The Fubini theorem permits changing the order of integration. Thus, if $f, g \in B(R)$, then

$$\|\mathcal{P}f-\mathcal{P}g\|_1 \le 4d'\|f-g\|_1.$$

Let us estimate the norm of the operator $\mathcal{P}$. To this end, we separately consider the expressions $f[\overline{\omega}*f]$ and $[f\overline{\omega}*f]$,

$$\begin{aligned}
\|f[\overline{\omega}*f]\|_1 &= \int\limits_{\mathbb{R}}\left|f(x)\int\limits_{\mathbb{R}}\overline{\omega}(x-y)f(y)\,dy\right|dx \le \int\limits_{\mathbb{R}}|f(x)|\int\limits_{\mathbb{R}}|\overline{\omega}(x-y)||f(y)|\,dy\,dx\\
&\le \|\overline{\omega}\|_C\int\limits_{\mathbb{R}}|f(x)|\int\limits_{\mathbb{R}}|f(y)|\,dy\,dx = \|\overline{\omega}\|_C\|f\|_1^2 \le \|\overline{\omega}\|_C R\|f\|_1 < d'\|f\|_1,
\end{aligned}$$

$$\|[f\overline{\omega} * f]\|_1 = \int\limits_{\mathbb{R}} \left| \int\limits_{\mathbb{R}} f(x-y)\overline{\omega}(x-y)f(y)\,dy \right| dx$$
$$\le \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |f(x-y)||\overline{\omega}(x-y)||f(y)|\,dy\,dx = \int\limits_{\mathbb{R}} |f(y)| \int\limits_{\mathbb{R}} |f(x-y)||\overline{\omega}(x-y)|\,dx\,dy$$
$$\le \|\overline{\omega}\|_C \int\limits_{\mathbb{R}} |f(y)| \int\limits_{\mathbb{R}} |f(x-y)|\,dx\,dy = \|\overline{\omega}\|_C \|f\|_1^2 \le d'\|f\|_1.$$

Thus, the norm of the operator $\mathcal{P}$ does not exceed $2d'$, and hence it tends to zero as d' tends to zero. This means that the norm of the operator $\mathcal{S}$ tends to zero as d' tends to zero. It follows from the proof of Theorem 2 that for $\alpha > 0$ the quantity $d(\partial B(R), \partial\mathcal{K}[B(R)])$ is positive and independent of d'. Thus, by the theorem on fixed points of a perturbed compact operator, the operator $\mathcal{K}+\mathcal{S} = \mathcal{A}$ has a fixed point in $B(R)$ for sufficiently small d'. The proof of the theorem is complete.

Corollary 2. *The equilibrium equation of the form $C = \mathcal{A}C$ has a solution under the assumptions of Theorem* 3.

CONCLUSIONS

Despite some progress in the analytical study of the well-posedness of the problem stated by biologists, there remain many unanswered important questions related to Eq. (1). One of these questions is that on the uniqueness of its solution. It is shown in the papers [5, 6] that for $\alpha = 0$ and $d = 0$, in addition to the trivial solution $C \equiv 0$, this equation, subject to some conditions imposed on the functions m and ω, also has a solution C that is not identically zero. Of interest is also the stability of the solution to the problem posed.

Note that all the results in this paper can readily be carried over to multidimensional cases. In particular, under conditions similar to those described in Theorem 3, the nonlinear equilibrium equation has a solution in the two- and three-dimensional cases, which are of most interest from the biological point of view.

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