ISSN 0012-2661, Differential Equations, 2019, Vol. 55, No. 6, pp. 841–848. © Pleiades Publishing, Ltd., 2019.
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CONTROL THEORY

# Group Pursuit Problem in a Differential Game with Fractional Derivatives, State Constraints, and Simple Matrix

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Received September 18, 2017; revised November 29, 2018; accepted February 12, 2019

**Abstract**—In a finite-dimensional Euclidean space, we consider the pursuit problem with one evader and a group of pursuers described by a system of the form $D^{(\alpha)}z_i = az_i + u_i - v$, where $D^{(\alpha)}f$ is the Caputo derivative of order $\alpha \in (1, 2)$ of a function $f$. The set of admissible solutions $u_i$ and $v$ is a convex compact set, the objective set is the origin, and $a$ is a real number. In addition, it is assumed that the evader does not leave a convex polyhedral cone with nonempty interior. We obtain sufficient conditions for the solvability of the pursuit problem in terms of the initial positions and the game parameters.

**DOI**: 10.1134/S0012266119060119

## INTRODUCTION

An important trend in the evolution of modern theory of differential games is related to the development of methods for solving pursuit–evasion game problems with several players [1–5]. Here not only classical solution methods are refined, but also new problems to which these methods apply are widely sought. For example, a pursuit problem with two evaders described by a fractional differential equation was considered in [6–8], where sufficient conditions for capture were obtained. A group pursuit problem and a problem about an evader escaping a group of pursuers in a differential game with state constraints and fractional derivatives of order $\alpha \in (0, 1)$ were studied in [9] and [10], respectively. A problem about multiple capture of the evader with no state constraints in a differential game with fractional derivatives of order $\alpha \in (1, 2)$ was studied in [11].

The present paper deals with a group pursuit problem with one evader in which the motion of all players is described by equations with Caputo fractional derivatives and the evader never leaves a convex polyhedral set. Sufficient conditions for capture are obtained. The paper continues the research initiated in [12–14].

## 1. STATEMENT OF THE PROBLEM

**Definition 1** [15]. Let $f : [0, \infty) \to \mathbb{R}^k$ be a function whose derivative $f'$ is absolutely continuous on $[0, +\infty)$. The function $D^{(\alpha)}f$, $\alpha \in (1, 2)$, defined by the rule

$$(D^{(\alpha)}f)(t) = \frac{1}{\Gamma(2-\alpha)} \int\limits_0^t \frac{f''(s)}{(t-s)^{\alpha-1}}\, ds, \quad \text{where} \quad \Gamma(\beta) = \int\limits_0^\infty e^{-s} s^{\beta-1}\, ds,$$

is called the *Caputo derivative of order* $\alpha$ of the function $f$.

In the space $\mathbb{R}^k$, $k \geq 2$, consider a differential game with $n + 1$ players, namely, $n$ pursuers $P_1, \ldots, P_n$ and an evader $E$. The law of motion for each of the pursuers $P_i$ has the form

$$D^{(\alpha)}x_i = ax_i + u_i, \qquad x_i(0) = x_i^0, \qquad \dot{x}_i(0) = x_i^1, \qquad u_i \in V, \tag{1}$$

and the law of motion for the evader $E$ is

$$D^{(\alpha)}y = ay + v, \qquad y(0) = y^0, \qquad \dot{y}(0) = y^1, \qquad v \in V. \tag{2}$$

Here $\alpha \in (1,2)$; $x_i, y, u_i, v \in \mathbb{R}^k$; $V$ is a strictly convex compact set in $\mathbb{R}^k$; and $a$ is a real number. In addition, it is assumed that the evader $E$ never leaves the convex polyhedral cone $\Omega$

$$\Omega = \{y \in \mathbb{R}^k : (p_j, y) \le 0, \quad j = 1, \dots, r\}$$

with nonempty interior, where $p_1, \dots, p_r$ are unit vectors in $\mathbb{R}^k$. If $\Omega = \mathbb{R}^k$, then we assume that $r = 0$.

Let $z_i = x_i - y$ and, along with systems (1) and (2), consider the system

$$D^{(\alpha)}z_i = az_i + u_i - v, \qquad z_i(0) = z_i^0 = x_i^0 - y^0, \qquad \dot{z}_i(0) = z_i^1 = x_i^1 - y^1, \qquad u_i, v \in V. \tag{3}$$

From now on, $i \in I_n = \{1, \dots, n\}$. We denote the vector of initial positions by $z^0 = \{z_i^0; z_i^1\}$ and assume that $z_i^0 \ne 0$ for all $i \in I_n$.

**Definition 2.** A *quasistrategy* $U_i$ of the pursuer $P_i$ is a mapping $U_i$ that takes any initial positions $z^0$, time $t$, and control prehistory $v_t(\cdot)$ of the evader $E$ to a measurable function $u_i(t)$ ranging in $V$.

**Definition 3.** We say that *capture* occurs in the game if there exists a time $T(z^0)$ and quasistrategies $U_1, \dots, U_n$ of the pursuers $P_1, \dots, P_n$ such that for each measurable function $v(\cdot)$ satisfying the conditions $v(t) \in V$ and $y(t) \in \Omega$, $t \in [0, T(z^0)]$, there exists a time $\tau \in [0, T(z^0)]$ and a number $q \in I_n$ such that $z_q(\tau) = 0$.

Let us introduce the following notation: $\operatorname{Int} A$ and $\operatorname{co} A$ are the interior and the convex hull, respectively, of a set $A$; $E_\rho(z, \mu) = \sum_{k=0}^{\infty} z^k / \Gamma(k\rho^{-1} + \mu)$ is the generalized Mittag-Leffler function;

$$f_i(t) = \begin{cases} t^{\alpha-1} E_{1/\alpha}(at^\alpha, 1) z_i^0 + t^\alpha E_{1/\alpha}(at^\alpha, 2) z_i^1 & \text{if } a \ne 0, \\ z_i^0/t + z_i^1 & \text{if } a = 0, \end{cases}$$

$$\lambda(z, v) = \sup\{\lambda \ge 0 : -\lambda z \in V - v\}, \quad \gamma = -a\Gamma(2-\alpha), \quad d = \max\{\|v\| : v \in V\},$$

$$\overline{E}(t, s, \alpha) = (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^\alpha, \alpha), \quad E(t) = \int_0^t |\overline{E}(t, s, \alpha)|\, ds,$$

$$r(t, s) = \begin{cases} 1 & \text{if } E_{1/\alpha}(a(t-s)^\alpha, \alpha) \ge 0, \\ -1 & \text{if } E_{1/\alpha}(a(t-s)^\alpha, \alpha) < 0, \end{cases}$$

$$\delta_0^+ = \min_{v \in V} \max\Big\{\max_{i \in I_n} \lambda(z_i^1/\gamma, v), \max_{l \in I_r}(p_l, v)\Big\}, \quad \delta_0^- = \min_{v \in V} \max\Big\{\max_{i \in I_n} \lambda(-z_i^1/\gamma, v), \max_{l \in I_r}(-p_l, v)\Big\},$$

$$\delta_t^+ = \min_{v \in V} \max\Big\{\max_{i \in I_n} \lambda(f_i(t), v), \max_{l \in I_r}(p_l, v)\Big\}, \quad \delta_t^- = \min_{v \in V} \max\Big\{\max_{i \in I_n} \lambda(-f_i(t), v), \max_{l \in I_r}(-p_l, v)\Big\},$$

$$\delta_0 = \min\{\delta_0^+, \delta_0^-\}, \qquad \delta_t = \min\{\delta_t^+, \delta_t^-\},$$

$$y_0(t) = E_{1/\alpha}(at^\alpha, 1) y^0 + t E_{1/\alpha}(at^\alpha, 2) y^1, \qquad D_\varepsilon(b) = \{z : \|z - b\| \le \varepsilon\}.$$

**Lemma 1** [11]. *Let $a < 0$ and $\delta_0 > 0$. Then there exists a time $T > 0$ such that $\delta_t > 0.5\delta_0$ for all $t > T$.*

**Lemma 2.** *Let $p \in \mathbb{R}^k$ and $a < 0$. Then the function $\mu(t) = t^{\alpha-1}(p, y_0(t))$ is bounded on $[0, \infty)$.*

**Proof.** The following asymptotic estimates hold as $t \to +\infty$ [16, p. 12]:

$$E_{1/\alpha}(at^\alpha, 1) = -\frac{1}{at^\alpha \Gamma(1-\alpha)} + O\left(\frac{1}{t^{2\alpha}}\right), \qquad E_{1/\alpha}(at^\alpha, 2) = -\frac{1}{at^\alpha \Gamma(2-\alpha)} + O\left(\frac{1}{t^{2\alpha}}\right). \tag{4}$$

The function $\mu(t)$ can be represented as

$$\mu(t) = t^{\alpha-1}E_{1/\alpha}(at^\alpha, 1)(p, y^0) + t^\alpha E_{1/\alpha}(at^\alpha, 2)(p, y^1).$$

It follows from the estimates (4) that

$$\mu(t) = -\frac{(p, y^0)}{at\Gamma(1-\alpha)} - \frac{(p, y^1)}{a\Gamma(2-\alpha)} + O\left(\frac{1}{t^\alpha}\right) \tag{5}$$

as $t \to +\infty$. Now the boundedness of the function $\mu(t)$ on $[0, \infty)$ follows from its continuity, the representation (5), and the condition $\alpha > 1$. The proof of the lemma is complete.

**Lemma 3.** *Let $r = 1$, $a < 0$, and $\delta_0 > 0$. Then there exists a time $T_0$ such that for each admissible function $v(\cdot)$ ($v(t) \in V$ and $y(t) \in \Omega$, $t \in [0, T_0]$) there exists a number $q \in I_n$ such that the inequality*

$$T_0^{\alpha-1} \int_0^{T_0} |\overline{E}(T_0, s, \alpha)| \lambda(f_q(T_0) r(T_0, s), v(s))\, ds \ge 1$$

*is satisfied.*

**Proof.** Lemma 1 implies that there exists a time $T_1 > 0$ such that $\delta_t > 0.5\delta_0$ for all $t > T_1$. Let $T > T_1$, and let $v(\cdot)$ be an arbitrary admissible function. Consider the functions

$$h_i(t, T, v(\cdot)) = t^{\alpha-1} \int_0^t |\overline{E}(t, s, \alpha)| \lambda(f_i(T) r(T, s), v(s))\, ds, \qquad t \in [0, T]. \tag{6}$$

Since the control $v(\cdot)$ of the evader $E$ is admissible, we have $(p_1, y(t)) \le 0$ for all $t \ge 0$. According to [17], the solution of problem (2) has the form

$$y(t) = y_0(t) + \int_0^t \overline{E}(t, s, \alpha) v(s)\, ds.$$

It follows that

$$\int_0^t \overline{E}(t, s, \alpha)(p_1, v(s))\, ds \le \mu_0(t) \equiv (-p_1, y_0(t)). \tag{7}$$

Let us define the sets

$$T^+(t) = \{s : s \in [0, t],\ \ \overline{E}(t, s, \alpha) \ge 0\}, \qquad T^-(t) = \{s : s \in [0, t],\ \overline{E}(t, s, \alpha) < 0\},$$
$$T_1^+(t) = \{s : s \in T^+(t),\ \ (p_1, v(s)) \ge \delta_1 = 0.5\delta_0\}, \qquad T_2^+(t) = \{s : s \in T^+(t),\ \ (p_1, v(s)) < \delta_1\},$$
$$T_1^-(t) = \{s : s \in T^-(t),\ \ (-p_1, v(s)) \ge \delta_1\}, \qquad T_2^-(t) = \{s : s \in T^-(t),\ \ (-p_1, v(s)) < \delta_1\}.$$

Then

$$\begin{aligned}\int_0^t \overline{E}(t, s, \alpha)(p_1, v(s))\, ds &= \int_{T^+(t)} \overline{E}(t, s, \alpha)(p_1, v(s))\, ds + \int_{T^-(t)} \overline{E}(t, s, \alpha)(p_1, v(s))\, ds \\ &= \int_{T_1^+(t)} \overline{E}(t, s, \alpha)(p_1, v(s))\, ds + \int_{T_2^+(t)} \overline{E}(t, s, \alpha)(p_1, v(s))\, ds \\ &\quad + \int_{T_1^-(t)} (-\overline{E}(t, s, \alpha))(-p_1, v(s))\, ds + \int_{T_2^-(t)} (-\overline{E}(t, s, \alpha))(-p_1, v(s))\, ds.\end{aligned}$$

In view of the obvious inequalities

$$\int\limits_{T_1^+(t)} \overline{E}(t,s,\alpha)(p_1, v(s))\,ds \ge \delta_1 \int\limits_{T_1^+(t)} \overline{E}(t,s,\alpha)\,ds,$$

$$\int\limits_{T_2^+(t)} \overline{E}(t,s,\alpha)(p_1, v(s))\,ds \ge -d \int\limits_{T_2^+(t)} \overline{E}(t,s,\alpha)\,ds,$$

$$\int\limits_{T_1^-(t)} (-\overline{E}(t,s,\alpha))(-p_1, v(s))\,ds \ge \delta_1 \int\limits_{T_1^-(t)} (-\overline{E}(t,s,\alpha))\,ds,$$

$$\int\limits_{T_2^-(t)} (-\overline{E}(t,s,\alpha))(-p_1, v(s))\,ds \ge -d \int\limits_{T_2^-(t)} (-\overline{E}(t,s,\alpha))\,ds,$$

we conclude that

$$\int\limits_0^t \overline{E}(t,s,\alpha)(p_1, v(s))\,ds \ge \delta_1 \int\limits_{T_1^+(t)\cup T_1^-(t)} |\overline{E}(t,s,\alpha)|\,ds - d \int\limits_{T_2^+(t)\cup T_2^-(t)} |\overline{E}(t,s,\alpha)|\,ds.$$

Hence

$$\delta_1 \int\limits_{T_1^+(t)\cup T_1^-(t)} |\overline{E}(t,s,\alpha)|\,ds - d \int\limits_{T_2^+(t)\cup T_2^-(t)} |\overline{E}(t,s,\alpha)|\,ds \le \mu_0(t)$$

by virtue of inequality (7). It follows from the definitions of the sets $T_i^{\pm}(t)$, $i = 1, 2$, and the function $E(t)$ that

$$\int\limits_{T_1^+(t)\cup T_1^-(t)} |\overline{E}(t,s,\alpha)|\,ds + \int\limits_{T_2^+(t)\cup T_2^-(t)} |\overline{E}(t,s,\alpha)|\,ds = E(t).$$

The last two relations imply that

$$\int\limits_{T_2^+(t)\cup T_2^-(t)} |\overline{E}(t,s,\alpha)|\,ds \ge \frac{\delta_1 E(t) - \mu_0(t)}{d + \delta_1}$$

for all $t \in [0, T]$, whence it follows that

$$\begin{aligned}
\max_{i\in I_n} h_i(t, T, v(\cdot)) &\ge \frac{1}{n}\sum_{i\in I_n} h_i(t, T, v(\cdot)) = \frac{t^{\alpha-1}}{n}\int\limits_0^t |\overline{E}(t,s,\alpha)| \sum_{i\in I_n} \lambda(f_i(T)r(T,s), v(s))\,ds \\
&\ge \frac{t^{\alpha-1}}{n}\int\limits_0^t |\overline{E}(t,s,\alpha)| \max_{i\in I_n} \lambda(f_i(T)r(T,s), v(s))\,ds \\
&\ge \frac{t^{\alpha-1}}{n}\int\limits_{T_2^+(t)\cup T_2^-(t)} |\overline{E}(t,s,\alpha)| \max_{i\in I_n} \lambda(f_i(T)r(T,s), v(s))\,ds \\
&\ge \frac{\delta_1}{n(d+\delta_1)}[\delta_1 t^{\alpha-1}E(t) - t^{\alpha-1}\mu_0(t)]
\end{aligned}$$

for all $t \in [0, T]$. Hence

$$\max_{i\in I_n} h_i(T, T, v(\cdot)) \ge \frac{\delta_1}{n(d+\delta_1)}[\delta_1 T^{\alpha-1}E(T) - T^{\alpha-1}\mu_0(T)]$$

for all $T > T_1$. Since [18, p. 120]

$$\int_0^t \overline{E}(t,s,\alpha)\,ds = t^\alpha E_{1/\alpha}(at^\alpha, \alpha+1), \tag{8}$$

we have

$$t^{\alpha-1}E(t) = t^{\alpha-1}\int_0^t |\overline{E}(t,s,\alpha)|\,ds \ge t^{\alpha-1}\int_0^t \overline{E}(t,s,\alpha)\,ds = t^{2\alpha-1}E_{1/\alpha}(at^\alpha,\alpha+1). \tag{9}$$

It follows from [16, p. 12] that

$$E_{1/\alpha}(at^\alpha,\alpha+1) = -\frac{1}{at^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right) \tag{10}$$

as $t \to +\infty$. By Lemma 2, there exists a $c > 0$ such that $|t^{\alpha-1}\mu_0(t)| \le c$ for all $t \ge 0$. Therefore,

$$\max_{i\in I_n} h_i(T,T,v(\cdot)) \ge \frac{\delta_1}{n(d+\delta_1)}\left[-\frac{\delta_1 T^{\alpha-1}}{a} - c + O\left(\frac{1}{T}\right)\right]$$

as $T \to +\infty$. Since $a < 0$ and $\alpha - 1 > 0$, it follows that there exists a time $T_0$ such that $\max_{i\in I_n} h_i(T_0,T_0,v(\cdot)) \ge 1$ for each admissible function $v(\cdot)$. The proof of the lemma is complete.

**Lemma 4.** *Let $\Omega = \mathbb{R}^k$, $a < 0$, and $\delta_0 > 0$. Then there exists a time $T_0$ such that for each admissible function $v(\cdot)$ $(v(t) \in V,\ t \in [0,T_0])$ there exists a number $q \in I_n$ such that*

$$T_0^{\alpha-1}\int_0^{T_0} |\overline{E}(T_0,s,\alpha)|\lambda(f_q(T_0)r(T_0,s),v(s))\,ds \ge 1.$$

**Proof.** It follows from Lemma 1 that there exists a time $T_1 > 0$ such that $\delta_t > 0.5\delta_0$ for all $t > T_1$. Let $T > T_1$, and let $v(\cdot)$ be an arbitrary admissible function. Consider the functions (6). Then

$$\max_{i\in I_n} h_i(t,T,v(\cdot)) \ge \frac{1}{n}\sum_{i\in I_n} h_i(t,T,v(\cdot)) = \frac{t^{\alpha-1}}{n}\int_0^t |\overline{E}(t,s,\alpha)| \sum_{i\in I_n}\lambda(f_i(T)r(T,s),v(s))\,ds$$

$$\ge \frac{t^{\alpha-1}}{n}\int_0^t |\overline{E}(t,s,\alpha)| \max_{i\in I_n}\lambda(f_i(T)r(T,s),v(s))\,ds \ge \frac{\delta_1 t^{\alpha-1}}{n}\int_0^t |\overline{E}(t,s,\alpha)|\,ds$$

for all $t \in [0,T]$. It follows from relations (8)–(10) that

$$\max_{i\in I_n} h_i(T,T,v(\cdot)) \ge \frac{\delta_1}{n}\left(-\frac{T^{\alpha-1}}{a} + O\left(\frac{1}{T^{2\alpha}}\right)\right)$$

for all $T > T_1$. Since $a < 0$ and $\alpha - 1 > 0$, we see that there exists a time $T_0$ such that $\max_{i\in I_n} h_i(T_0,T_0,v(\cdot)) \ge 1$ for each admissible function $v(\cdot)$. The proof of the lemma is complete.

By Lemmas 3 and 4,

$$\hat{T} = \min\left\{t : \inf_{v(\cdot)}\max_{i\in I_n} t^{\alpha-1}\int_0^t |\overline{E}(t,s,\alpha)|\lambda(f_i(t)r(t,s),v(s))\,ds \ge 1\right\} < \infty.$$

**Theorem 1.** *Let $r = 1$, $a < 0$, and $\delta_0 > 0$. Then the game ends in a capture.*

**Proof.** Let $v(s), s \in [0, \hat{T}]$, be an arbitrary control of the evader. Consider the function

$$H(t) = 1 - \max_{i \in I_n} \hat{T}^{\alpha-1} \int\limits_0^t |\overline{E}(\hat{T}, s, \alpha)| \lambda(f_i(\hat{T}) r(\hat{T}, s), v(s))\, ds$$

and denote its least positive root by $T_0$. Note that $T_0$ exists by Lemma 3 and the definition of $\hat{T}$. Moreover, $T_0 \le \hat{T}$. In addition, there exists a number $l \in I_n$ such that

$$1 - \hat{T}^{\alpha-1} \int\limits_0^{T_0} |\overline{E}(\hat{T}, s, \alpha)| \lambda(f_l(\hat{T}) r(\hat{T}, s), v(s))\, ds \le 0.$$

Therefore, there exists a time $\tau_l$ such that

$$1 - \hat{T}^{\alpha-1} \int\limits_0^{\tau_l} |\overline{E}(\hat{T}, s, \alpha)| \lambda(f_l(\hat{T}) r(\hat{T}, s), v(s))\, ds = 0. \tag{11}$$

For $i \ne l$, by $\tau_i$ we denote the times—if any—such that relation (11) is satisfied. Let us define the controls $u_i(s)$ of the pursuers $P_i$ by setting

$$u_i(s) = \begin{cases} v(s) - \lambda(f_i(\hat{T}) r(\hat{T}, s), v(s)) f_i(\hat{T}) r(\hat{T}, s), & s \in [0, \min\{\tau_i, \hat{T}\}], \\ v(s), & s \in [\min\{\tau_i, \hat{T}\}, \hat{T}]. \end{cases}$$

The solution of system (3) can be represented in the form [17]

$$z_i(t) = E_{1/\alpha}(at^\alpha, 1) z_i^0 + t E_{1/\alpha}(at^\alpha, 2) z_i^1 + \int\limits_0^t \overline{E}(t, s, \alpha)(u_i(s) - v(s))\, ds.$$

Hence we obtain

$$\begin{aligned} \hat{T}^{\alpha-1} z_l(\hat{T}) &= f_l(\hat{T}) + \hat{T}^{\alpha-1} \int\limits_0^{\hat{T}} \overline{E}(\hat{T}, s, \alpha)(u_l(s) - v(s))\, ds \\ &= f_l(\hat{T}) \left( 1 - \hat{T}^{\alpha-1} \int\limits_0^{\tau_l} |\overline{E}(\hat{T}, s, \alpha)| \lambda(f_l(\hat{T}) r(\hat{T}, s), v(s)\, ds \right) = 0. \end{aligned}$$

The proof of the theorem is complete.

**Theorem 2.** *Let $\Omega = \mathbb{R}^k$, $a < 0$, and $\delta_0 > 0$. Then the game ends in a capture.*

This theorem is a corollary of Theorem 1 in [11] under the condition of a single capture.

**Lemma 5.** *Let the following conditions be satisfied for vectors $b_1, \ldots, b_n \in \mathbb{R}^k$ :*
1. $\min_{v \in V} \max\{\max_{i \in I_n} \lambda(b_i, v), \max_{j \in I_r} (p_j, v)\} > 0$.
2. $\min_{v \in \overline{\operatorname{co} V_1}} \max_{j \in I_r} (p_j, v) > 0$, *where* $V_1 = \{v : \lambda(b_i, v) = 0$ *for all* $i \in I_n\}$.
*Then there exists a vector $p \in \mathbb{R}^k$ such that*
(a) $\Omega \subset \Omega_1 = \{y : (p, y) \le 0\}$;
(b) $\delta_0^+(p) > 0$, *where* $\delta_0^+(p) = \min_{v \in V} \max\{\max_{i \in I_n} \lambda(b_i, v), (p, v)\}$.

**Proof.** By the Bohnenblust–Karlin–Shapley theorem [19, p. 33 of the Russian translation], there exist nonnegative numbers $\gamma_1, \ldots, \gamma_r$, $\gamma_1 + \cdots + \gamma_r = 1$, such that

$$\inf_{v \in \overline{\operatorname{co} V_1}} (\gamma_1(p_1, v) + \cdots + \gamma_r(p_r, v)) > 0.$$

Set $p = \gamma_1 p_1 + \cdots + \gamma_r p_r$. Then $\Omega \subset \Omega_1 = \{y : (p, y) \leq 0\}$ and $(p, v) > 0$ for all $v \in \overline{\operatorname{co} V_1}$. Hence $\delta_0^+(p) > 0$. The proof of the lemma is complete.

**Lemma 6.** *Let $V = D_1(0)$. Then $\delta_0^+ > 0$ if and only if $\delta_0^- > 0$.*

**Proof.** The inequality $\delta_0^+ > 0$ is equivalent to the inclusion [5, p. 36]

$$0 \in \operatorname{Int} \operatorname{co} \{z_1^1, \ldots, z_n^1, p_1, \ldots, p_r\},$$

which is obviously equivalent to the inclusion

$$0 \in \operatorname{Int} \operatorname{co} \{-z_1^1, \ldots, -z_n^1, -p_1, \ldots, -p_r\},$$

which, in turn, is equivalent to the inequality $\delta_0^- > 0$. The proof of the lemma is complete.

**Theorem 3.** *Let $a < 0$, $V = D_1(0)$, $\delta_0 > 0$, and $\min_{v \in \overline{\operatorname{co} V_1}} \max_{j \in I_r} (p_j, v) > 0$, where*

$$V_1 = \{v : \lambda(z_i^1, v) = 0 \text{ for all } i \in I_n\}.$$

*Then the game ends in a capture.*

**Proof.** It follows from the assumptions of the theorem that there exists a vector $p \in \mathbb{R}^k$ such that properties (a) and (b) in Lemma 5 are satisfied. Therefore, $\delta_0^+(p) > 0$. It follows from Lemma 6 that $\delta_0^-(p) > 0$, where

$$\delta_0^+(p) = \min_{v \in V} \max\{\max_{i \in I_n}(z_i^1, v), (p, v)\}, \qquad \delta_0^-(p) = \min_{v \in V} \max\{\max_{i \in I_n}(-z_i^1, v), (-p, v)\}.$$

Consequently, $\delta_0(p) > 0$. It follows that capture occurs in the game with the state constraints $\Omega_1$. Therefore, the original game ends in a capture as well. The proof of the theorem is complete.

**Corollary.** *Let $a < 0$, $V = D_1(0)$, $n \geq k$, and*

$$0 \in \operatorname{Int} \operatorname{co} \{z_1^1, \ldots, z_n^1, p_1, \ldots, p_r\}. \tag{12}$$

*Then the game ends in a capture.*

**Proof.** It follows from the inclusion (12) that there exists an $\varepsilon > 0$ such that

$$0 \in \operatorname{Int} \operatorname{co} \{w_1, \ldots, w_n, p_1, \ldots, p_r\}$$

for all $w_i$, $i \in I_n$, satisfying the inequality $\|z_i^1 - w_i\| < \varepsilon$. Therefore, we can assume that the vectors $z_1^1, \ldots, z_k^1$ are linearly independent, because otherwise the pursuers $P_1, \ldots, P_k$ can choose their controls on a sufficiently small time interval $[0, \tau]$ to ensure that, at time $\tau$, the vectors $z_1^1(\tau), \ldots, z_k^1(\tau)$ are linearly independent and

$$0 \in \operatorname{Int} \operatorname{co} \{z_1^1(\tau), \ldots, z_n^1(\tau), p_1, \ldots, p_r\}.$$

In addition, it follows from the inclusion (12) that the vectors $z_1^1, \ldots, z_n^1$, $p_1, \ldots, p_r$ form a positive basis of the space $\mathbb{R}^k$ [20]. Therefore, there exist positive numbers $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_r$ such that

$$0 = \alpha_1 z_1^1 + \cdots + \alpha_n z_n^1 + \beta_1 p_1 + \cdots + \beta_r p_r. \tag{13}$$

Consider the vector $p_0 = \beta_1 p_1 + \cdots + \beta_r p_r$, and let us show that $0 \in \operatorname{Int} \operatorname{co} \{z_1^1, \ldots, z_n^1, p_0\}$. Let $x \in \mathbb{R}^k$. Since the vectors $z_1^1, \ldots, z_k^1$ form a basis of the space $\mathbb{R}^k$, we have $x = \gamma_1 z_1^1 + \cdots + \gamma_k z_k^1$. By (13),

$$x = \gamma_1 z_1^1 + \cdots + \gamma_k z_k^1 + \mu(\alpha_1 z_1^1 + \cdots + \alpha_n z_n^1 + \beta_1 p_1 + \cdots + \beta_r p_r).$$

Taking a sufficiently large $\mu$, we obtain $x = \gamma_1^0 z_1^1 + \dots + \gamma_n^0 z_n^1 + \mu p_0$, where all the coefficients are nonnegative. According to [20], the vectors $z_1^1, \dots, z_n^1$, $p_0$ form a positive basis, and therefore, $0 \in \operatorname{Int} \operatorname{co} \{z_1^1, \dots, z_n^1, p_0\}$. It follows that $\delta_0^+(p_0) > 0$. Hence $\delta_0^-(p_0) > 0$, and consequently, $\delta_0(p_0) > 0$.

Consider the set $\Omega_1 = \{x : x \in \mathbb{R}^k,\ (p_0, x) \le 0\}$. Then $\Omega \subset \Omega_1$. By Theorem 1, the game with the state constraints $\Omega_1$ ends in a capture. Therefore, the game with the state constraints $\Omega$ ends in a capture as well. The proof of the corollary is complete.

**Remark.** The case of $a = 0$ can be considered in a similar way.

## FUNDING

This work was supported by the Ministry of Education and Science of the Russian Federation in the framework of the basic part of the state order in the field of science, project no. 1.5211.2017/8.9, and by the Russian Foundation for Basic Research, project no. 16-01-00346.

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