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ORDINARY DIFFERENTIAL EQUATIONS

Definition and Some Properties of Perron Stability

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Abstract—The natural notions of Perron stability, Perron asymptotic stability, and Perron complete instability of the zero solution of a differential system are introduced. Peculiar features of these notions are noted in the one-dimensional, autonomous, and linear cases. Their connections with Perron exponents and with their counterparts in the sense of Lyapunov are described. The complete coincidence of the possibilities for studying the Perron and Lyapunov stability and asymptotic stability in the first approximation is revealed.

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INTRODUCTION

The present paper deals with the notion of *Perron stability*, which has emerged as a result of an attempt to precisely determine those properties of a differential system that its *Perron exponents* are responsible for (see the study [1], as well as the monograph [2, Sec. 2] and the extensive bibliography in that monograph).

In the case of a linear system, the negativity of the Perron exponents of all of its nonzero solutions is usually associated (not very precisely) with the *Poisson stability* of its zero solution [3, Ch. IV, Sec. 4], which is formally defined as an arbitrarily late *return* of the phase trajectory into any given neighborhood of the initial point.

In a similar way, the positivity of the Perron exponents of solutions of a linear system is (again, very remotely) reminiscent of the *complete instability* of a dynamical system [3, Ch. IV, Sec. 9], which, by definition, has the property that each of its points is *wandering* [3, Ch. IV, Sec. 5]; i.e., some neighborhood of the point, moving with time, no longer meets its initial position starting from some instance of time.

The relationship between the *Perron stability properties* to be introduced below and the signs of the Perron exponents is the same as the relationship between their Lyapunov counterparts and the signs of the *Lyapunov exponents* (see. [4, Ch. I] and [5, Ch. III, Sec. 5]). It is this relationship that accounts for the use of Oscar Perron's name in the definition of stability of the type being considered.

In this paper, we, first of all, list *all feasible combinations* of Perron and Lyapunov stability properties (Theorems 2 and 3) and provide examples demonstrating that the Perron instability of a system does not imply some at first sight obvious properties of its solutions (Theorems 1 and 5). Next, some peculiar features of Perron stability properties are revealed in the *one-dimensional* and *autonomous* cases (Theorems 4, 6, and 7), as well as in the *linear* case (Theorems 8 and 9) and, in particular, for an autonomous system (Theorem 10) or a regular system (Theorem 11). In addition, we describe how these properties are related to the Perron and Lyapunov *exponents* in the linear case (Theorems 12 and 14) and the nonlinear case (Theorem 13). Finally, we prove that the classes of linear systems suitable for studying Perron or Lyapunov stability or asymptotic stability *in the first approximation* coincide with each other (Theorem 15).

A substantial part of these results have been announced in the reports [6–9].

1. DEFINITION OF PERRON STABILITY

For a given neighborhood G of zero in the Euclidean space $\mathbb{R}^n$, consider the system

$$\dot{x} = f(t, x), \qquad f(t, 0) = 0, \qquad t \in \mathbb{R}^+ \equiv [0, \infty), \qquad x \in G, \tag{1}$$

with right-hand side $f \in C^1(\mathbb{R}^+ \times G)$ (admitting the zero solution). By $\mathcal{S}_*(f)$ we denote the set of all *nonextendable nonzero* solutions x of system (1), and $\mathcal{S}_\delta(f)$ and $\mathcal{S}^\delta(f)$ will stand for its subsets defined by the *initial conditions* with $|x(0)| < \delta$ and $|x(0)| = \delta$, respectively.

Definition 1. We say that system (1) (more precisely, its *zero* solution, which is implicitly meant in what follows) has the following *Perron stability property*:

1. *Perron stability* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every solution $x \in \mathcal{S}_\delta(f)$ is defined for all $t \in \mathbb{R}^+$ and satisfies the condition

$$\varliminf_{t\to\infty} |x(t)| < \varepsilon. \tag{2}$$

2. *Perron asymptotic stability* if there exists a $\delta > 0$ such that every solution $x \in \mathcal{S}_\delta(f)$ is defined for all $t \in \mathbb{R}^+$ and satisfies the condition

$$\varliminf_{t\to\infty} |x(t)| = 0. \tag{3}$$

3. *Perron instability* if there is no Perron stability, i.e., if there exists an $\varepsilon > 0$ such that for each $\delta > 0$ there exists a solution $x \in \mathcal{S}_\delta(f)$ that does not satisfy condition (2) (for example, is not defined on the entire half-line $\mathbb{R}^+$).

4. *Complete Perron instability* if there exist $\varepsilon, \delta > 0$ such that none of the solutions $x \in \mathcal{S}_\delta(f)$ satisfies condition (2).

At first glance, it may seem that, for a completely Perron unstable system, the property of going away from the zero solution is *automatically* transferred from solutions starting near zero to all nonzero solutions in general. However, this is false even in the two-dimensional case, as shown by the following theorem.

Theorem 1. *For* $n = 2$, *there exists a completely Perron unstable system* (1) *such that at least one solution* $x \in \mathcal{S}_*(f)$ *satisfies condition* (3) *and even the condition*

$$\lim_{t\to\infty} |x(t)| = 0. \tag{4}$$

Remark 1. In the case of complete Perron instability, it is in principle impossible (owing to the continuous dependence of solutions on the initial data) to find $\varepsilon, \delta > 0$ and $T \in \mathbb{R}$ such that *all solutions* $x \in \mathcal{S}_\delta(f)$ *simultaneously* satisfy the condition

$$\inf_{t\ge T} |x(t)| \ge \varepsilon. \tag{5}$$

Remark 2. In Definition 1, each of the four Perron stability properties:

(a) Can be in a standard manner (namely, by a plain coordinate shift) extended from the *zero* solution to any other solution rather than only to an equilibrium point of the system under study. (That is, the shift itself may be time dependent.)

(b) Is of *local* nature, i.e., only depends on the behavior of solutions that start near zero.

(c) Characterizes the behavior of solutions starting near zero from the viewpoint of the possibility of their arbitrarily *late approach* to the zero solution or, on the contrary, *ultimate departure* from it.

2. PERRON AND LYAPUNOV STABILITY

Recall the following definition (see [5, Ch. II, Sec. 1]).

Definition 2. We assign a *Lyapunov counterpart* to each of the four Perron stability properties introduced above as follows:

(a) *Lyapunov stability*, *Lyapunov instability*, and *complete Lyapunov instability* (the last term is not generally accepted) are obtained by replacing condition (2) in the first, third, and fourth parts of Definition 1 with the condition

$$\sup_{t\in\mathbb{R}^+} |x(t)| < \varepsilon. \tag{6}$$

(b) *Lyapunov asymptotic stability* is obtained by replacing condition (3) in the second part of Definition 1 with condition (4) assuming also that the Lyapunov stability condition is satisfied.

The simplest intrinsic *logical links* between Perron and Lyapunov stability properties are established by the following theorem.

Theorem 2. *The following assertions hold for any system* (1):

1. *It is either Perron* (*Lyapunov*) *stable or Perron* (*Lyapunov*) *unstable.*

2. *If it is Perron* (*Lyapunov*) *asymptotically stable*, *then it is also Perron* (*Lyapunov*) *stable.*

3. *If it is completely Perron* (*Lyapunov*) *unstable*, *then it is Perron* (*Lyapunov*) *unstable.*

4. *If it is Lyapunov stable* (*Lyapunov asymptotically stable*), *then it is also Perron stable* (*Perron asymptotically stable*).

5. *If it is Perron unstable* (*completely Perron unstable*), *then it is also Lyapunov unstable* (*completely Lyapunov unstable*).

To give a full description of all possible combinations of various Perron and Lyapunov stability properties, we give yet another, technical definition.

Definition 3. The following varieties of Perron (Lyapunov) stability properties are said to be *strict*:

(a) Asymptotic Perron (Lyapunov) stability.

(b) *Nonasymptotic* Perron (Lyapunov) stability, that is, stability but not asymptotic stability.

(c) Complete Perron (Lyapunov) instability.

(d) *Incomplete* Perron (Lyapunov) instability, that is, instability but not complete instability.

All combinations of stability properties in Definition 3 logically admitted by the preceding theorem turn out to be possible. Namely, the following assertion holds.

Theorem 3. *If a combination of a strict Perron stability property and a strict Lyapunov stability property does not contradict Theorem* 2, *then it is realized for some system* (1).

The following two most *natural* situations are especially important from the practical point of view:

1. Asymptotic Perron stability combined with Lyapunov stability.
2. Complete Perron (and hence Lyapunov) instability.

3. IMPORTANT SPECIAL CASES

For system (1) with *one-dimensional* ($n = 1$) phase space, the verification of Perron stability properties is somewhat simplified, because one can arrange the solutions in ascending order of their initial values on the *numerical* phase line. Namely, the following theorem holds.

Theorem 4. *The following assertions hold for a one-dimensional system* (1):

1. *Perron stability is equivalent to the property that for each* $\varepsilon > 0$ *there exist two solutions* $x \in \mathcal{S}_*(f)$ *of opposite sign each of which satisfies condition* (2).

2. *Asymptotic Perron stability is equivalent to the existence of two solutions* $x \in \mathcal{S}_*(f)$ *of opposite sign each of which satisfies condition* (3).

3. *Complete Perron instability is equivalent to the existence of an* $\varepsilon > 0$ *such that for each* $\delta > 0$ *there exist two solutions* $x \in \mathcal{S}_\delta(f)$ *of opposite sign any of which does not satisfy condition* (2).

The following theorem, close to Theorem 1, shows that there is no natural counterpart of the first two statements in Theorem 4 for systems *of dimension greater than* 1 even in the *autonomous* case (where the right-hand side of the system is independent of time).

Theorem 5. *There exists a Perron unstable two-dimensional autonomous system* (1) *and a* $\delta > 0$ *such that all solutions* $x \in \mathcal{S}^\delta(f)$ *of this system satisfy condition* (4).

The situation described in Theorem 1 is *impossible* not only in the one-dimensional case but even in the autonomous case, as shown by the following theorem.

Theorem 6. *If a one-dimensional or autonomous system* (1) *is completely Perron unstable, then there exists an* $\varepsilon > 0$ *such that none of the solutions* $x \in \mathcal{S}_*(f)$ *of this system satisfies condition* (2).

Notwithstanding Remark 1, in the one-dimensional and autonomous cases complete Perron instability is still *uniform* in the sense described in the following theorem.

Theorem 7. *If a one-dimensional or autonomous system* (1) *is completely Perron unstable, then for each* $\delta > 0$ *there exists an* $\varepsilon > 0$ *such that all solutions* $x \in \mathcal{S}_*(f) \backslash \mathcal{S}_\delta(f)$ *of this system satisfy condition* (5) *starting from* $T = 0$.

4. LINEAR SYSTEMS

Consider linear systems of the form

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}^+, \tag{7}$$

each of which is specified by a continuous operator function $A : \mathbb{R}^+ \to \operatorname{End} \mathbb{R}^n$. (If this function is bounded, then the system itself is said to be *bounded*.) By $\mathcal{S}_A^\delta$ we denote the set of solutions x of system (7) with initial condition satisfying $|x(0)| = \delta$.

Theorem 3 is to some extent refined by the following theorem.

Theorem 8. *Every pair of strict properties described in Theorem* 3 *is realized on some bounded linear system* (7) *for any* $n > 1$ *and even for* $n = 1$ *if the pair contains neither Perron nor Lyapunov incomplete instability.*

Each of the Perron stability properties is totally determined for a linear system by the properties of solutions issuing from some *sphere*, as claimed in the following theorem.

Theorem 9. *The Perron stability of the linear system* (7) *is equivalent to the condition*

$$\sup_{x \in \mathcal{S}_A^1} \varliminf_{t \to \infty} |x(t)| < \infty, \tag{8}$$

while the asymptotic Perron stability and the complete Perron instability of this system hold if and only if every solution $x \in \mathcal{S}_A^1$ *satisfies condition* (3) *or the condition*

$$\lim_{t \to \infty} |x(t)| = \infty, \tag{9}$$

respectively.

In the simplest case of a *linear autonomous* system, Perron stability analysis and Lyapunov stability analysis give the *same* result (uniquely recognizable from the real parts of the eigenvalues of the constant operator defining the system and the orders of the Jordan blocks corresponding to pure imaginary eigenvalues [5, Ch. II, Sec. 8]); namely, the following theorem holds.

Theorem 10. *A linear autonomous system* (7) *is Perron stable* (*asymptotically stable*, *unstable*, *or completely unstable*) *if and only if it is Lyapunov stable* (*respectively*, *asymptotically stable*, *unstable*, *or completely unstable*).

Theorem 10 *cannot be extended* from autonomous linear systems to the slightly wider class of Lyapunov *regular* linear systems [5, Ch. III, Sec. 11] (i.e., systems such that the sum of Lyapunov exponents of some fundamental solution system is equal to the lower mean value of the trace of the operator defining the system on the time half-line), as shown by the following theorem.

Theorem 11. *For every* $n \in \mathbb{N}$ *there exists a regular bounded linear system* (7) *that is asymptotically Perron stable but completely Lyapunov unstable.*

5. PERRON AND LYAPUNOV EXPONENTS

A special role in the stability analysis of linear (and not only linear) systems is intended for the *characteristic exponents* of their solutions, the upper exponent [4, Ch. I] and the lower exponent [2, Sec. 2], the latter having a much more complicated structure than the former.

Definition 4. The *Lyapunov* and *Perron* exponents of a solution $x \in \mathcal{S}_*(f)$ of system (1) are defined as

$$\lambda(x) \equiv \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |x(t)| \quad \text{and} \quad \pi(x) \equiv \underline{\lim}_{t\to\infty} \frac{1}{t} \ln |x(t)|.$$

Remark 3. The following assertions hold for the exponents introduced for a solution $x \in \mathcal{S}_*(f)$ in Definition 4:

(a) The inequality $\lambda(x) \geq \pi(x)$ holds.

(b) The inequalities $\pi(x) > 0$ and $\pi(x) < 0$ imply conditions (9) and (3), respectively.

(c) The inequalities $\lambda(x) < 0$ and $\lambda(x) > 0$ imply condition (4) and the condition

$$\overline{\lim_{t\to\infty}} |x(t)| = \infty, \tag{10}$$

respectively.

As is seen from the following two theorems, in the case of a *linear* system, the requirement that conditions (3) or (9) are satisfied not for all nonzero solutions but only for solutions in a *fundamental* solution system is not sufficient for Perron stability, let alone asymptotic stability (unlike asymptotic Lyapunov stability, for which it is quite sufficient that condition (4) be satisfied for all solutions in some fundamental solution system), or for complete Perron instability, respectively.

Theorem 12. *For each $n > 1$ there exists an n-dimensional bounded linear system* (7) *with Perron instability* (*incomplete Perron instability*) *for which the Perron exponents of all solutions in some fundamental solution system are negative* (*respectively*, *positive*).

Nevertheless, in some cases (even nonlinear ones) the *set of exponents of all* solutions close to the zero solution provides complete information about the Perron and Lyapunov stability properties, as shown by the following theorem.

Theorem 13. *If the Perron* (*respectively*, *Lyapunov*) *exponents of all solutions $x \in \mathcal{S}_\delta(f)$ of system* (1) *are negative for some $\delta > 0$, then the system is asymptotically Perron stable* (*respectively*, *asymptotically Lyapunov stable under the additional condition of Lyapunov stability*), *and if these exponents are positive*, *then the system is completely Perron* (*respectively*, *Lyapunov*) *unstable.*

Two particular statements in Theorem 3 are strengthened in the following theorem.

Theorem 14. *For each $n \in \mathbb{N}$ there exists a completely Lyapunov unstable but asymptotically* (*nonasymptotically*) *Perron stable n-dimensional linear system* (7) *for which all Lyapunov exponents are positive and all Perron exponents are negative* (*respectively, zero*).

6. STABILITY BY THE FIRST APPROXIMATION

Assume that the *linear part* of the right-hand side of system (1) has been isolated; i.e., the system is represented in the form

$$\dot{x} = A(t)x + h(t,x) \equiv f(t,x), \quad (t,x) \in \mathbb{R}^+ \times G, \quad \sup_{t\in\mathbb{R}^+} |h(t,x)| = o(x) \quad \text{as} \quad x \to 0, \tag{11}$$

where

$$A(t) \equiv f'_x(t,0), \qquad t \in \mathbb{R}^+. \tag{12}$$

Then the corresponding system (7) is called the *first approximation system* for system (1).

Definition 5. We say that the first approximation system (7) *ensures* a given Perron or Lyapunov stability property if any system (11) with this first approximation has this property.

An overwhelming number of papers (see [2, Sec. 11]) deal with asymptotic stability by the first approximation, which is the essence of Lyapunov's first method. However, the following theorem shows that stability and asymptotic stability analysis by the first approximation, regardless of whether Perron or Lyapunov, is only possible for *one and the same* systems.

Theorem 15. *If the linear approximation* (7) *ensures at least one of the four properties* (*Perron stability*, *Lyapunov stability*, *Perron asymptotic stability*, *or Lyapunov asymptotic stability*), *then it also ensures the remaining three properties.*

7. PROOFS OF STATEMENTS OF GENERAL NATURE

First, we prove Theorems 2; 4, 6, and 7; 9; 10 and 11; 13; and 15 in the order specified.

Proof of Theorem 2. The validity of the first and third statements in this theorem and also of the second one in the part related to Lyapunov stability follows directly from Definitions 1 and 2. The validity of all other statements follows from the fact that for each $\varepsilon > 0$ each of conditions (3) and (6) implies condition (2), while condition (4) implies condition (3).

The proof of Theorem 2 is complete.

Proof of Theorems 4, 6, and 7. Consider two cases, A and B.

A. First, let system (1) be *one-dimensional*. Then for this system:

(a) The necessity of the conditions in Theorem 4 for Perron stability, Perron asymptotic stability, and complete Perron instability is contained in the first, second, and fourth items of Definition 1.

(b) The sufficiency of the conditions in Theorem 4 for Perron stability and Perron asymptotic stability follows from the fact that if condition (2) or (3) is satisfied for two solutions $x = x_1 < 0$ and $x = x_2 > 0$, then the same condition is satisfied for any solution x such that $x_1(0) < x(0) < x_2(0)$ (because distinct integral curves cannot meet).

(c) The sufficiency of the conditions in Theorem 4 for complete Perron instability, as well as the validity of the assertion of Theorem 6 (or 7), follows from the fact that if condition (2) is not satisfied (or condition (5) is satisfied) for two solutions $x = x_1 < 0$ and $x = x_2 > 0$, then the same condition is not satisfied (respectively, is satisfied) for any solution x such that $x(0) < x_1(0)$ or $x(0) > x_2(0)$.

B. Now let system (1) be *autonomous* and *completely Perron unstable*. Then for some $\varepsilon', \delta' > 0$ none of the solutions $x \in \mathcal{S}_{\delta'}(f)$ satisfies condition (2) with ε replaced by ε'.

Fixing an arbitrary positive $\delta < \min\{\delta', \varepsilon'\}$, for each solution $x \in \mathcal{S}^\delta(f)$ with initial value $x(0) \equiv e$ belonging to the sphere $S \subset \mathbb{R}^n$ of radius δ we find positive numbers $T(e)$ and $\varepsilon(e) > 0$ such that

$$|x(T(e))| > \delta = |x(0)| \geq \inf_{0 \leq t \leq T(e)} |x(t)| > \varepsilon(e). \tag{13}$$

By virtue of the continuous dependence of solutions on the initial values (in the sense of the uniform topology on $[0, T(e)]$) for some neighborhood $U(e)\sqrt{S}$ of the point e both conditions in (13) will also be satisfied for all solutions $x \in \mathcal{S}^\delta(f)$ with initial conditions $x(0) \in U(e)$.

Selecting a finite subcover by neighborhoods $U(e_1), \ldots, U(e_N)$ from the resulting cover of the compact sphere S, we set

$$\varepsilon \equiv \min\{\varepsilon(e_1), \ldots, \varepsilon(e_N)\} \in (0, \delta).$$

Now let us prove that for each $\gamma \geq \delta$ each solution $x \in \mathcal{S}^\gamma(f)$ at $T = 0$ satisfies condition (5) (which excludes satisfying condition (2), which, by the way, is also not satisfied for the remaining solutions $x \in \mathcal{S}_\delta(f)$, because $\varepsilon < \delta < \varepsilon'$ and $\delta < \delta'$).

Indeed, assume that, on the contrary, there exists a solution x_0 of system (1) satisfying the equation $|x_0(0)| = \gamma$ at the initial time and the inequalities $|x_0(t_1)| < \varepsilon < \delta \leq \gamma$ at some other time $t_1 > 0$. Then, choosing the *greatest* $t_0 < t_1$ (which exists in view of the continuity of the function x_0) and some number $j \in \{1, \ldots, N\}$ that satisfy the conditions $|x_0(t_0)| = \delta$ and $x_0(t_0) \in U(e_j)$, we arrive at the following assertions.

(d) Being a solution of the same (autonomous) system (1), the function $x(t) \equiv x_0(t+t_0)$ $(t \in \mathbb{R}^+)$ satisfies conditions (13) for $e = e_j$.

(e) $|x(t)| < \delta$ for all $0 < t \le t_1 - t_0$, and hence $t_1 - t_0 < T(e_j)$. (The opposite inequality leads to a contradiction with the first inequality in Eq. (13) for $e = e_j$.)

(f) $|x(t_1 - t_0)| < \varepsilon \le \varepsilon(e_j)$, which contradicts the last inequality in (13) for $e = e_j$, thereby proving the desired assertion.

The proof of Theorems 4, 6, and 7 is complete.

Proof of Theorem 9. Any solution $x \in \mathcal{S}_*(f)$ satisfies the relation $\underline{Cx} = C\underline{x}$ for any $C > 0$, where $\underline{x}$ stands for the left-hand side of inequality (3). Therefore, given a $\delta > 0$, the following assertions hold *simultaneously for all* $x \in \mathcal{S}_A^\gamma$ and *all* $\gamma \in (0, \delta)$:

(a) Condition (3) is equivalent to the same condition for all $x \in \mathcal{S}_A^1$ simultaneously.

(b) If condition (2) is violated for some $\varepsilon > 0$, then condition (9) is violated for all $x \in \mathcal{S}_A^1$ simultaneously (and hence also for all solutions $x \neq 0$ in general), because the finiteness of $\underline{x}$ for at least one such solution $x = x_0$ would imply inequality (2) also for the solution $x = \gamma x_0 \in \mathcal{S}_A^\gamma$ for a sufficiently small $\gamma < \delta$.

(c) If condition (2) is satisfied for some $\varepsilon > 0$, then the least upper bound of $\underline{x}$ over all $x \in \mathcal{S}_A^\gamma$ is finite for at least one $\gamma > 0$ and hence also for $\gamma = 1$ (i.e., condition (8) is satisfied, which, in turn, implies that condition (2) is satisfied for all solutions $x \in \mathcal{S}_A^\gamma$ simultaneously for all sufficiently small $\gamma > 0$).

The proof of Theorem 9 is complete.

Proof of Theorem 10. It follows from the form of an arbitrary solution $x \neq 0$ of the autonomous linear system that the fact that the lower limit of the function $|x(t)|$ as $t \to \infty$ is zero or infinity is equivalent to the fact that its upper limit is zero or infinity, respectively, whereas the finiteness of its lower limit is equivalent to the finiteness of its upper limit (i.e., in simple terms, its boundedness). Therefore (see Theorem 9), for such a system we have the following assertions:

(a) Perron, as well as Lyapunov, asymptotic stability is equivalent to all such limits being zero.

(b) Complete Perron, as well as Lyapunov, instability is equivalent to all such limits being infinite.

(c) Lyapunov stability implies Perron stability, which, in turn, implies the boundedness of all solutions and hence Lyapunov stability.

(d) Perron instability is equivalent to Lyapunov instability (being the negations of Perron and Lyapunov stability, respectively).

The proof of Theorem 10 is complete.

Proof of Theorem 13 can be obtained from Definitions 1 and 2 with regard to the fact that the negativity (positivity) of the exponent $\pi(x)$ or $\lambda(x)$ implies the upper or lower limit $|x(t)|$ as $t \to \infty$ being zero (respectively, infinity).

Proof of Theorem 15. According to Theorem 2, Lyapunov asymptotic stability is logically the strongest stability property of the four appearing in the statement of Theorem 15, with Perron stability being the weakest one. Therefore, to prove Theorem 15 it suffices to verify that if the linear approximation (7) ensures Perron stability then it also ensures Lyapunov asymptotic stability.

Suppose that, on the contrary, some system (11) is not asymptotically Lyapunov stable. Then we select numbers $\varepsilon > \gamma > \gamma' > 0$ for which $B_\varepsilon \equiv \{x \in \mathbb{R}^n : |x| \le \varepsilon\} \subset G$ and also choose $\delta_1 \in (0, \gamma')$ such that for each $\delta \in (0, \delta_1]$ there exists a solution $x \in \mathcal{S}_\delta(f)$ that does not satisfy at least one of conditions (6) or (4).

A. Taking $t_0 \equiv 0$, we will construct a perturbed system with the same linear approximation (7) but with a right-hand side g that satisfies the condition $g(t, x) = x$ for all $t \in \mathbb{R}^+$ and $x \in G \backslash B_\varepsilon$.

1. Take a solution x_1 corresponding to the number δ_1 and hence satisfying the estimate $|x_1(t)| < \gamma < \varepsilon$ for $0 \le t \le t_0$.

2. If there exists at least one number $s \ge t_0$ such that $|x_1(s)| = \gamma$, then:

(a) Selecting the *least* such number s_1 and setting $t_1 \equiv s_1 + 1$ and $\gamma_1 = \gamma'$, for the function $y_1 \equiv x_1$ we obtain the relations

$$|y_1(s_1)|^{\cdot} \ge 0, \quad \gamma = |y_1(s_1)| > |y_1(t)|, \quad 0 \le t < s_1, \quad 0 < \beta_1 \equiv \min_{0 \le t \le s_1} |y_1(t)| \le \delta_1 < \gamma'. \qquad (14)$$

(b) For sufficiently small $\sigma_1 \in [0,1)$, we *adjust* the function y_1 for $s_1 - \sigma \le t \le s_1$ to obtain the new function

$$z_1(t) \equiv \chi_{\sigma_1}(t-s_1)y_1(t), \qquad 0 \le t \le s_1, \qquad \chi_\sigma(\tau) \equiv \begin{cases} 1+\sigma\tau e^{((\tau/\sigma)^2-1)^{-1}} \le 1, & -\sigma \le \tau \le 0, \\ 1, & \tau \le -\sigma \end{cases}$$

(here $\chi_\sigma \in \mathrm{C}^2(\mathbb{R})$, $\dot\chi_\sigma(0) > 0$ and $\chi_0 \equiv 1$) satisfying the conditions

$$|z_1(s_1)|^{\cdot} > 0, \quad \gamma = |z_1(s_1)| > |z_1(t)|, \quad 0 \le t < s_1, \quad |z_1(t)| > \gamma', \quad s_1 - \sigma_1 \le t \le s_1. \tag{15}$$

(c) Now we *adjust* the system itself so that the function z_1 serves as its solution; to this end, we modify its right-hand side (preserving its smoothness) only in the small domain defined by the conditions $|t-s_1| \le \sigma_1$ and $\gamma' < |x| \le \gamma$.

3. If there exists no number $s \ge t_0$ with these properties, then the solution x_1 satisfies condition (6) and hence fails to satisfy condition (4). Then

(d) For some unbounded sequence $0 < \tau_1 < \tau_2 < \ldots$ and some $\gamma_1 \in (0, \gamma']$, we have

$$\gamma \ge \sup_{t \in \mathbb{R}^+} |x_1(t)| \ge |x_1(\tau_i)| \ge \gamma_1 > 0, \qquad i \in \mathbb{N}.$$

(e) Select the least root s_1 of the equation $|y_1(s)| = \gamma$ in which the function

$$y_1(t) \equiv \begin{cases} x_1(t)e^{\delta_1\varphi(t-t_0)}, & t \ge t_0, \\ x_1(t), & 0 \le t \le t_0, \end{cases} \qquad \varphi(\tau) \equiv \begin{cases} \tau - 2/3, & \tau \ge 1, \\ \tau^3/3 \ge \tau - 2/3, & 0 \le \tau \le 1, \\ 0, & \tau \le 0 \end{cases}$$

(here $\varphi \in \mathrm{C}^2(\mathbb{R})$) satisfies the condition $|y_1(\tau_i)| \ge \gamma_1 e^{\delta_1(\tau_i - t_0 - 2/3)} \to \infty$ as $i \to \infty$, which guarantees the existence of the number s_1, and set $t_1 \equiv s_1 + 1$ by defining β_1 by formula (14).

(f) *Perturb* the original system so that it admits a solution $y_1(t)$ for $0 \le t \le s_1$; namely, supplement its right-hand side with a term that vanishes (together with its first derivatives) for all $t = t_0, t_1$ and $|x| \le \beta_1/2$ and has the form

$$\Delta(t,x) \equiv (1-\theta_{s_1}^{t_1}(t))\theta_{\beta_1/2}^{\beta_1}(|x|)\psi(t,x), \qquad t \in [t_0,t_1], \qquad |x| \le \gamma, \qquad \theta_a^b(\tau) \equiv \begin{cases} 0, & \tau \le a, \\ 1, & \tau \ge b, \end{cases} \tag{16}$$

where the function $\theta_a^b \in \mathrm{C}^1(\mathbb{R})$ is nonstrictly increasing for each pair $a < b$ and

$$\psi(t,x) \equiv \delta_1\dot\varphi(t-t_0)x - h(t,x) + e^{\delta_1\varphi(t-t_0)}h(t, xe^{-\delta_1\varphi(t-t_0)}). \tag{17}$$

(g) Adjust the solution y_1 thus obtained and the system itself in accordance with 2 (b) and 2 (c) above to obtain a solution z_1 and a parameter value σ_1 that satisfy all conditions (15).

4. Using the continuity of solutions with respect to initial data, find a $\delta_2 \in (0, \beta_1/2)$ such that every solution $x \in \mathcal{S}_{\delta_2}(f)$ satisfies the condition $|x(t)| < \beta_1/2 < \gamma$ for all $t \in [0, t_1]$.

5. Extend the definition of the function z_1 previously defined on the segment $[0, s_1]$ and satisfying conditions (15) to the interval $[s_1, t_1]$ so that the following relations be satisfied:

$$\gamma = |z_1(s_1)| < |z_1(t)| < |z_1(t_1)| = \varepsilon, \quad s_1 < t < t_1, \quad \dot z_1(t_1) = z_1(t_1), \quad z_1 \in \mathrm{C}^2([0,t_1]). \tag{18}$$

6. Repeat the reasoning in 1–5 above, increasing the indices on all parameters (except for τ_i) first by one, then again by one, and so on. As a result, we obtain sequences $x_i, y_i, z_i, \sigma_i, \gamma_i$ ($i \in \mathbb{N}$); a sequence $\delta_1 \ge \beta_1 > \delta_2 \ge \beta_2 > \ldots$ decreasing to zero; an unbounded sequence $0 \equiv t_0 < s_1 < t_1 < s_2 < \ldots$; and a new system for $|x| \le \gamma$.

Let us complete the definition of the right-hand side of the perturbed system for $\gamma \le |x| \le \varepsilon$ arbitrarily (while retaining its smoothness) so that on each section $[s_i, t_i]$ ($i \in \mathbb{N}$) it admits the corresponding solution z_i satisfying conditions (15) and (18).

B. The system thus constructed:

1. Is *Perron unstable*, because it has a sequence of solutions z_i that do not satisfy condition (2) with the initial values $z_i(0) \to 0$ as $i \to \infty$ for a given $\varepsilon > 0$.

2. Has the *original linear approximation*, because for every $\alpha > 0$, given some $N(\alpha) \in \mathbb{N}$ and $\delta(\alpha) \in (0, \gamma')$, the following estimates hold (see the representation (11)):

$$\delta_i < \alpha/3, \qquad i > N(\alpha), \qquad \eta(\delta) \equiv \sup_{\substack{t\in\mathbb{R}^+ \\ 0<|x|\le\delta}} \frac{|h(t,x)|}{|x|} < \alpha/3, \qquad 0 < \delta \le \delta(\alpha),$$

whence for $|x| \le \min\{\beta_{N(\alpha)}/2, \delta(\alpha)\}$ we find that if $t \le t_{N(\alpha)}$, then $\Delta(t,x) = 0$, and if $t > t_{N(\alpha)}$, then for some $i > N(\alpha)$ we obtain the condition $t \in [t_{i-1}, t_i]$ and (see relations (16) and (17)) the estimates

$$|\Delta(t,x)| \le \delta_i|x| + \eta(\delta)|x| + e^{\delta_i\varphi(t-t_{i-1})}\eta(\delta)|x|e^{-\delta_i\varphi(t-t_{i-1})} = (\delta_i + 2\eta(\delta))|x| \le \alpha|x|.$$

The proof of Theorem 15 is complete.

8. PROOF OF THE ASSERTIONS ON THE EXISTENCE OF EXAMPLES

Now let us prove the remaining theorems in the following order: 1; 5; 3, 8 and 14; 11; and 12.

Proof of Theorem 1. Consider two autonomous two-dimensional systems

$$\dot z = Az, \qquad A \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad z \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \dot y = (4^2 - y_1^2)(1^2 - y_2^2)Ay, \qquad y \equiv \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The first system is linear and has the only singular point at the origin, which is an unstable degenerate node with a vertical (under the canonical representation of coordinates) eigenline and nontrivial phase curves of the form $z_2 = z_1(C + \ln|z_1|)$ ($C \in \mathbb{R}$).

In addition to the node at the origin, the second system has singular points that fill the boundary of the rectangle $P \equiv \{y \in \mathbb{R}^2 : |y_1| \le 4,\ |y_2| \le 1\}$ and from each of the previous phase curves (geometrically coinciding with the new ones) they isolate parts that have the origin as a common α-limit point and are located inside the rectangle P. Every such part that passes under the curve $y_2 = y_1(-1 + \ln|y_1|)$ (which has a punctured point $(0,0)^\mathsf{T}$ and touches the horizontal rectangle sides at the points $y_+ \equiv (1,-1)^\mathsf{T}$ and $y_- \equiv (-1,1)^\mathsf{T}$) or along its right-hand branch has a concrete ω-limit point at the lower boundary, whereas each of the remaining parts, at the upper boundary (by the estimate $4(-1 + \ln 4) > 1$).

If we pass to the new variables $x_1 \equiv e^{-t}y_1$ and $x_2 \equiv y_2$ in the second system, then, say, at $\delta = 1$ any solution $x \in \mathcal{S}_\delta(f)$ of the resulting *nonautonomous* system (1) will possess the property $x(t) \to (0,1)^\mathsf{T}$ or $x(t) \to (0,-1)^\mathsf{T}$ as $t \to \infty$ and hence will not satisfy condition (2) even at $\varepsilon = 1$. System (1) is thus completely Perron unstable.

Nevertheless, the solution $e^{-t}x_0$ of the same system in Eq. (1), issuing from the (previously fixed) point $x_0 \equiv (4,0)^\mathsf{T}$, satisfies condition (4).

The proof of Theorem 1 is complete.

Remark 4. Unfortunately, system (1) constructed in the proof of Theorem 1 has an *unbounded* first approximation system (7) with the matrix function (12).

Proof of Theorem 5. In the plane with the coordinates $y_1 = \rho\cos\varphi$ and $y_2 = \rho\sin\varphi$, consider two autonomous two-dimensional systems

$$\begin{pmatrix} \dot\varphi \\ \dot\rho \end{pmatrix} = (1-\rho)^2 \begin{pmatrix} 1 - \cos\varphi \\ 1 - \rho \end{pmatrix}, \qquad \rho^2 \equiv y_1^2 + y_2^2 \ge 1, \qquad \dot y = (1-\rho)^2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad 0 \le \rho^2 \le 1,$$

and treat the first system, which is 2π-periodic with respect to $\varphi \in \mathbb{R}$, as written in polar coordinates. The singular points of these two systems fill the unit circle ($\rho^2 = 1$), and in each of them, both right-hand sides have zero derivatives with respect to the phase variables. Therefore, in the merged system, the right-hand side (specified piecewise) is continuously differentiable on the entire plane.

In the first system, the phase curves lying outside the unit circle are given by the formula $\rho = 1 + Ce^{\operatorname{ctg}(\varphi/2)}$ ($C > 0$) and have a common (and unique) ω-limit point $y_0 \equiv (1,0)^{\mathsf{T}}$, whereas for the second system the phase curves lying inside the circle are horizontal chords with right-to-left motion along them, and for one of these chords, namely, the *diameter*, the points y_0 and $y_1 \equiv (-1,0)^{\mathsf{T}}$ are α- and ω-limit points, respectively.

Now if we make the change of variables $x = y - y_0$ for the constructed system by shifting the point y_0 to the origin, then the resulting system (1) (also autonomous) will have the zero fixed point.

Further, system (1) will be Perron unstable, because for each $\delta > 0$ there exists a solution $x \in \mathcal{S}_\delta(f)$ issuing from the above-mentioned (shifted) diameter and having the property $x(t) \to (-2,0)^{\mathsf{T}}$ as $t \to \infty$, which contradicts condition (2) even at $\varepsilon = 1$.

Finally, if we take $\delta = 3$, then all solutions $x \in \mathcal{S}^\delta(f)$ issue from outside the unit circle after the performed shift as well, and hence all of them, having the limit $x(t) \to (0,0)^{\mathsf{T}}$ as $t \to \infty$, satisfy condition (4).

The proof of Theorem 5 is complete.

Proof of Theorems 3, 8, and 14. Consider the two-parameter family of *scalar* linear systems (7) with a bounded matrix function of the form $A(t) \equiv a(t)E$, where

$$a(t) \equiv \dot{\varphi}_{\alpha,\beta}(t), \qquad \varphi_{\alpha,\beta}(t) \equiv (\alpha + (\beta - \alpha)\sin^2 \ln(t+1))t, \qquad t \in \mathbb{R}^+. \tag{19}$$

A. If $\alpha \le \beta$, then all nonzero solutions of such a scalar system will have the properties

$$x(t) \equiv x_0 e^{\varphi_{\alpha,\beta}(t)} \qquad (0 \ne x_0 \in \mathbb{R}^n), \qquad \lambda(x) = \beta, \qquad \pi(x) = \alpha,$$

$$\lim_{t\to\infty} |x(t)| = \begin{cases} \infty, & \alpha > 0, \\ 0, & \beta < 0, \end{cases} \qquad \varliminf_{t\to\infty} |x(t)| = \begin{cases} |x_0|, & \alpha = 0, \\ 0, & \alpha < 0, \end{cases} \qquad \sup_{t\in\mathbb{R}^+} |x(t)| = \begin{cases} \infty, & \beta > 0, \\ |x_0|, & \beta \le 0, \end{cases}$$

and the following situations are possible for each $n \in \mathbb{N}$.

1. Asymptotic, both Lyapunov and Perron, stability for $\alpha, \beta = -1$.

2. Nonasymptotic Lyapunov stability for $\beta = 0$, and, in terms of Perron properties, either nonasymptotic stability at $\alpha = 0$ or asymptotic stability at $\alpha = -1$.

3. Complete Lyapunov instability for $\beta = 1$, and, in terms of Perron properties, complete instability at $\alpha = 1$, nonasymptotic stability at $\alpha = 0$, or asymptotic stability at $\alpha = -1$ (the last two cases provide examples for Theorem 14).

B. If, in addition, $n > 1$, then by fixing some decomposition into a direct sum of two mutually orthogonal subspaces

$$\mathbb{R}^n = \mathbb{R}^{n'} \dotplus \mathbb{R}^{n''}, \qquad n' = 1, \qquad n'' = n - 1, \tag{20}$$

we define on each of them its own scalar linear system with coefficients of the form in Eq. (19) and with parameters $\alpha' \le \beta'$ and $\alpha'' \le \beta''$, respectively. For the resulting linear system, the following cases may further be possible.

4. Complete Lyapunov instability for $\beta' = \beta'' = 1$ and complete Perron instability at $\alpha' = -1$ and $\alpha'' = 1$.

5. Complete Lyapunov instability for $\beta' = -1$ and $\beta'' = 1$, and, in terms of Perron properties, complete instability at $\alpha' = -1$ and $\alpha'' = 1$, nonasymptotic stability at $\alpha' = -1$ and $\alpha'' = 0$, or asymptotic stability at $\alpha' = \alpha'' = -1$.

The proof of Theorems 3, 8, and 14 is complete.

Proof of Theorem 11. If we replace the function $\varphi_{\alpha,\beta}$ in relations (19) with

$$\varphi(t) \equiv (\alpha + (\beta - \alpha)\sin^2 \ln(t+1)) \ln(t+1), \qquad \alpha = -1, \qquad \beta = 1,$$

then all nonzero solutions of the corresponding scalar linear system, which is also bounded, will possess the properties

$$x(t) \equiv x_0 e^{\varphi(t)} = x_0 (t+1)^{\alpha + (\beta-\alpha)\sin^2 \ln(t+1)} \quad (0 \neq x_0 \in \mathbb{R}^n), \qquad \lambda(x) = 0 = \varliminf_{t\to\infty} \frac{1}{t} \int_0^t a(\tau)\,d\tau$$

(the last property ensures the regularity of this system) as well as properties (3) and (10).

The proof of Theorem 11 is complete.

Proof of Theorem 12. For a given $n > 1$, fix some decomposition (20) and on each of the subspaces $\mathbb{R}^{n'}$ and $\mathbb{R}^{n''}$ define its own scalar linear system with coefficients a' and a'', respectively.

1. First, let $a' \equiv -1$ and $a'' \equiv 1$ in this system. Then any of its solutions $x' \neq 0$ with initial value $x'(0) \in \mathbb{R}^{n'}$ will have the exponents $\pi(x') = \lambda(x') = -1$, while any of its remaining solutions $x \neq 0$ will have the exponents $\pi(x) = \lambda(x) = 1$. In this case, it is possible to select a fundamental system from the latter solutions. Hence, incomplete Perron instability takes place here.

2. Now let $a \equiv \dot\varphi_{-1,3}$ and $b \equiv \dot\varphi_{3,-1}$ in this system, where the functions $\varphi_{\alpha,\beta}$ are defined by formulas (19). (The idea of such a combination of functions was borrowed form the research [1].) Then the Perron exponent for any of its solutions $x', x'' \neq 0$ with the initial data $x'(0) \in \mathbb{R}^{n'}$ and $x''(0) \in \mathbb{R}^{n''}$ is equal to -1, and it is possible to choose a fundamental solution system from them. However, the following chain of relations holds for any solution $x \equiv x' + x''$ with $t \in \mathbb{R}^+$:

$$|x(t)|^2 = |x'(t)|^2 + |x''(t)|^2 \geq 2|x'(0)|e^{\varphi_{-1,3}(t)}|x''(0)|e^{\varphi_{3,-1}(t)} = 2|x'(0)||x''(0)|e^{(3-1)t},$$

which implies that $\pi(x) \geq 1$. Therefore, no Perron stability can takes place here.

The proof of Theorem 12 is complete.

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