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INTEGRAL EQUATIONS

Well-Posed Solvability and the Representation of Solutions of Integro-Differential Equations Arising in Viscoelasticity

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Abstract—For abstract integro-differential equations with unbounded operator coefficients in a Hilbert space, we study the well-posed solvability of initial problems and carry out spectral analysis of the operator functions that are symbols of these equations. This allows us to represent the strong solutions of these equations as series in exponentials corresponding to points of the spectrum of operator functions. The equations under study are the abstract form of linear integro-partial differential equations arising in viscoelasticity and several other important applications.

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INTRODUCTION

The present paper deals with integro-differential equations with unbounded operator coefficients in Hilbert space. The considered equations are abstract hyperbolic equations perturbed by a term containing Volterra integral operators. These equations can be realized as integro-partial differential equations that arise in the viscoelasticity (see [1, 2]) and as Gurtin–Pipkin integro-differential equations (see [3–5]) that describe the process of heat propagation in media with memory at a finite rate; moreover, these equations arise in problems of homogenization in multiphase media (the Darcy law) (see [6, 7]).

Let H be a separable Hilbert space, and let A be a self-adjoint positive operator ($A^* = A$) in H with compact inverse. For the second-order integro-differential equation

$$\frac{d^2u}{dt^2} + A^2u - \int_0^t K(t-s)A^2u(s)\,ds = f(t), \qquad t \in \mathbb{R}_+ \equiv (0,\infty), \tag{1}$$

defined on the positive half-line, consider the initial problem

$$u(+0) = \varphi_0, \tag{2}$$

$$u^{(1)}(+0) = \varphi_1. \tag{3}$$

In what follows, we assume that the scalar function $K(t)$ can be represented as

$$K(t) = \sum_{j=1}^{\infty} c_j R_j(t); \tag{4}$$

here $c_j > 0$, $j \in \mathbb{N}$, and $R_j(t)$ are fractionally exponential functions (see [2, Ch. I]) of the form

$$R_j(t) = t^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-\beta_j)^n t^{n\alpha}}{\Gamma((n+1)\alpha)}, \qquad 0 < \alpha \le 1, \tag{5}$$

where $\Gamma(\cdot)$ is the Euler gamma function and the sequence $\{\beta_j\}$ satisfies the condition that $\beta_j \to +\infty$ as $j \to +\infty$. Without loss of generality, we assume that $0 < \beta_j < \beta_{j+1,}$, $j \in \mathbb{N}$. Moreover, we assume that

$$\sum_{j=1}^{\infty} \frac{c_j}{\beta_j} < 1. \tag{6}$$

The Laplace transform of the function $R_j(t)$ has the form $\hat{R}_j(\lambda) = 1/(\lambda^\alpha + \beta_j)$ (see [2, Ch. I]). Here and below, λ^α $(0 < \alpha \le 1)$ is understood as the principal branch of the multivalued function $f(\lambda) = \lambda^\alpha$, $\lambda \in \mathbb{C}$, with the cut along the negative real semiaxis: $\lambda^\alpha = |\lambda^\alpha| e^{i\alpha \arg \lambda}$, $-\pi < \arg \lambda < \pi$. Considering the Laplace transform of Eq. (1) under homogeneous initial conditions, we obtain the operator function

$$L(\lambda) = \lambda^2 I + A^2 - \hat{K}(\lambda) A^2, \tag{7}$$

which is the symbol of this equation. Here $\hat{K}(\lambda)$ is the Laplace transform of the kernel $K(t)$; it can be represented as

$$\hat{K}(\lambda) = \sum_{j=1}^{\infty} \frac{c_j}{\lambda^\alpha + \beta_j}, \qquad 0 < \alpha \le 1. \tag{8}$$

In the present paper, we prove the well-posed solvability of the initial problem (2), (3) for Eq. (1) in weighted Sobolev spaces on the positive semiaxis, study the problem of localization of the spectrum of the operator function $L(\lambda)$ which is the symbol of this equation, and obtain the representation of strong solutions of Eq. (1).

In our previous papers [8–13], problem (1)–(3) was studied in detail for the case in which the kernel $K(t)$ can be represented by a series in decaying exponentials with positive coefficients, which is equivalent to the case of $\alpha = 1$ in the representation (4). Our approach to the study was based on the spectral analysis of the operator function (7), which also permits proving the well-posed solvability and representing the solution of this problem in the form of a series in exponentials corresponding to points of the spectrum of the operator function $L(\lambda)$. Note that the results of [9–12] were summarized in Chapter 3 of the monograph [13].

We also note that the method we use to prove the well-posed solvability of initial problems for abstract integro-differential equations significantly differs from the more traditional approach applied by Pandolfi in [14], where the solvability is studied in the function space on a finite time interval $(0, T)$. In the present paper, we study the solvability in the weighted Sobolev spaces $W^2_{2,\gamma}(\mathbb{R}_+, A_0)$ of vector functions on the positive semiaxis $\mathbb{R}_+$, where A_0 is a positive self-adjoint operator in a Hilbert space. The proof of our Theorem 1 on solvability is significantly based on the use of the Hilbert structure of the spaces $W^2_{2,\gamma}(\mathbb{R}_+, A_0)$ and $L_{2,\gamma}(\mathbb{R}_+, H)$ and on the Paley–Wiener theorem, while in [14] the study is carried out in a Banach function space of smooth functions on a finite time interval $(0, T)$.

1. STATEMENT OF THE RESULTS

We transform the domain $\operatorname{Dom}(A^\beta)$ of the operator A^β, $\beta > 0$, into a Hilbert space H_β by introducing the norm $\|\cdot\|_\beta = \|A^\beta \cdot\|$, equivalent to the graph norm of the operator A^β on $\operatorname{Dom}(A^\beta)$.

1.1. Well-Posed Solvability

By $W^n_{2,\gamma}(\mathbb{R}_+, A^n)$ we denote the Sobolev space of vector functions defined on the semiaxis $\mathbb{R}_+$ and ranging in the space H equipped with the norm

$$\|u\|_{W^n_{2,\gamma}(\mathbb{R}_+, A^n)} \equiv \left(\int_0^\infty e^{-2\gamma t} (\|u^{(n)}(t)\|^2_H + \|A^n u(t)\|^2_H)\, dt \right)^{1/2}, \qquad \gamma \ge 0.$$

For more details about the spaces $W^n_{2,\gamma}(\mathbb{R}_+, A^2)$, see the monograph [15, Ch. 1]. For $n = 0$, we set $W^0_{2,\gamma}(\mathbb{R}_+, A^0) = L_{2,\gamma}(\mathbb{R}_+, H)$, where $L_{2,\gamma}(\mathbb{R}_+, H)$ denotes the space of measurable functions ranging in the space H equipped with the norm

$$\|f\|_{L_{2,\gamma}}(\mathbb{R}_+, H) = \left(\int\limits_0^{+\infty} e^{-2\gamma t} \|f(t)\|_H^2 \, dt \right)^{1/2}.$$

Definition 1. A vector function $u(\cdot)$ is called a *strong solution* of problem (1)–(3) if it belongs to the space $W_{2,\gamma}^2(\mathbb{R}_+, A^2)$ for some $\gamma \geq 0$, satisfies Eq. (1) on the semiaxis $\mathbb{R}_+$ almost everywhere and the initial conditions (2), (3).

Definition 2. A vector function $u(\cdot)$ is called a *generalized solution* of problem (1)–(3) if it belongs to the space $W_{2,\gamma}^1(\mathbb{R}_+, A)$ for some $\gamma \geq 0$ and satisfies the initial condition (2) and the identity

$$\left\langle A\left[u(t) - \int\limits_0^t K(t-s)u(s)ds\right], Av(t)\right\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} - \langle u'(t), v'(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)}$$
$$+ 2\gamma\langle u'(t), v(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} = \langle f(t), v(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} + (\varphi_1, v(0)) \quad (9)$$

for all $v(t) \in W_{2,\gamma}^1(\mathbb{R}_+, A)$ for which the relation $\lim_{t\to+\infty} v(t)e^{-2\gamma t} = 0$ holds.

The following theorem gives a sufficient condition for the well-posed solvability of problem (1)–(3).

Theorem 1. *Assume that a vector function $Af(t)$ belongs to the space $L_{2,\gamma_0}(\mathbb{R}_+, H)$ for some $\gamma_0 > 0$, the kernel $K(t)$ can be represented as* (4), (5) *with a constant α $(0 < \alpha < 1)$, condition* (6) *is satisfied, and in addition, $\varphi_0 \in H_3$ and $\varphi_1 \in H_2$. Then there exists a $\gamma_1 > \gamma_0$ such that, for all $\gamma \geq \gamma_1$, problem* (1)–(3) *has a unique strong solution $u(\cdot)$ in the space $W_{2,\gamma}^2(\mathbb{R}_+, A^2)$ and the estimate*

$$\|u\|_{W_{2,\gamma}^2(\mathbb{R}_+,A^2)} \leq d(\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|A^3\varphi_0\|_H + \|A^2\varphi_1\|_H)$$

is satisfied with a constant d that is independent of the vector function f and the vectors φ_0 and φ_1.

Note that Theorem 1 sharpens the assertion of Theorem 1 in [16], because the condition $(1/2 < \alpha < 1)$ was imposed in the statement of Theorem 1 in [16].

Theorem 2. *Assume that the vector function $f(t)$ belongs to the space $L_{2,\gamma_0}(\mathbb{R}_+, H)$ for some $\gamma_0 > 0$, the kernel $K(t)$ can be represented as* (4), (5) *with a constant α $(0 < \alpha < 1)$, condition* (6) *is satisfied, and in addition, $\varphi_0 \in H_2$ and $\varphi_1 \in H$. Then there exists a $\gamma_1 > \gamma_0$ such that, for all $\gamma \geq \gamma_1$, problem* (1)–(3) *has a generalized solution in the space $W_{2,\gamma}^1(\mathbb{R}_+, A)$ which satisfies the inequality*

$$\|u\|_{W_{2,\gamma}^2(\mathbb{R}_+,A^2)} \leq d(\|f\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|A^2\varphi_0\|_H + \|A\varphi_1\|_H) \quad (10)$$

with a constant d independent of the vector function f and the vectors φ_0 and φ_1.

1.2. Spectral Analysis

Let a_j denote the eigenvalues of the operator A $(Ae_j = a_je_j)$ numbered in ascending order (with multiplicities taken into account), $0 < a_1 < a_2 < \dots < a_n < \dots$, $a_n \to +\infty$ as $n \to +\infty$. The corresponding eigenvectors $\{e_j\}_{j=1}^\infty$ form an orthonormal basis of the space H.

Consider the restriction of the operator function $L(\lambda)$ to the one-dimensional space spanned by the vector e_n:

$$l_n(\lambda) = (L(\lambda)e_n, e_n) = \lambda^2 + a_n^2\left(1 - \sum_{k=1}^\infty \frac{c_k}{\lambda^\alpha + \beta_k}\right).$$

The following two theorems describe some properties of the spectrum of the operator function $L(\lambda)$ for the case in which condition (6) is satisfied.

Theorem 3. *Assume that conditions* (4)–(6) *are satisfied. Then the spectrum of the operator function* $L(\lambda)$ *lies in the open left half-plane.*

Theorem 4. *Assume that conditions* (4)–(6) *are satisfied and* $c_j = 0$ *for all* j *starting from some* $N \in \mathbb{N}$. *Then for each sufficiently large* $n \in \mathbb{N}$ *there exist two nonreal complex conjugate zeros* ${\lambda_n}^+ = \bar{\lambda}_n^-$ *of the function* $l_n(\lambda)$ *with the asymptotics*

$$\lambda_n^{\pm} = -\sin\left(\frac{\pi\alpha}{2}\right) a_n^{1-\alpha}\frac{Q}{2} \pm i a_n\left(1 - \cos\left(\frac{\pi\alpha}{2}\right) a_n^{-\alpha}\frac{Q}{2}\right) + o(a_n^{1-\alpha}) \quad \text{as} \quad n \to +\infty, \tag{11}$$

where $Q = c_1 + \cdots + c_N$.

We mention the following important fact.

Remark 1. For $\alpha = 1$, the asymptotic formula (11) turns into the previously known asymptotic formula (2.15) in [8] (also see [13]).

The complete proofs of Theorem 3 and 4 are given in [16], and the statements of close results can be found in [17].

Note that the operator functions of the form (7) for the case in which the kernels of integral operators are series in decaying exponentials with positive coefficients were studied in [8, 9]. Theorems 3 and 4 are natural generalizations of the results obtained in [8, 9].

Here it is appropriate to reproduce Remark 3.1 in [16].

Remark 2 [16]. Under the condition $\sum_{j=1}^{\infty} c_j/\beta_j > 1$, in the right half-plane there are infinitely many nonreal eigenvalues of the operator function $L(\lambda)$.

The assertion of this remark follows from simple graphical considerations.

Consider the restrictions of the vector functions $l_n(\lambda)$ to the real axis. We write the equation $l_n(x) = 0$ in the form $\varphi_n(x) = \psi(x)$, where

$$\varphi_n(x) = \frac{x^2}{a_n^2} + 1, \qquad \psi(x) = \sum_{j=1}^{\infty} \frac{c_j}{x^{\alpha}} + \beta_j.$$

Note that the function $\psi(x)$ on the semiaxis $[0, +\infty)$ monotonically decreases, and hence, at the point $x = 0$, it takes the greatest value equal to $\sum_{j=1}^{\infty} c_j/\beta_j > 1$. Therefore, the graph of the function $\psi(x)$ intersects the graphs of the parabolas $\varphi_n(x)$ for positive x_n. In this case, as n increases, the zeros x_n tend to a point x^* that is a solution of the equation $\psi(x) = 1$.

1.3. Representation of the Solutions

Now let us state the results on the representation of the strong solution of problem (1)–(3).

Theorem 5. *Let the assumptions of Theorem* 1 *be satisfied, and let* $\alpha \in (0, 1/2)$ *and* $f(t) \equiv 0$. *Then the strong solution of problem* (1)–(3) *can be represented as*

$$u(t) = v(t) + w(t),$$

where the vector function $v(t)$ *is given by*

$$v(t) = \sum_{n=1}^{\infty} (u_n^+(t) + u_n^-(t)) e_n, \tag{12}$$

$$u_n^+(t) = 2\pi i \frac{(\lambda_n^+ \varphi_{0n} + \varphi_{1n}) e^{\lambda_n^+ t}}{2\lambda_n^+ - a_n^2 \hat{K}^{(1)}(\lambda_n^+)}, \qquad u_n^-(t) = 2\pi i \frac{(\lambda_n^- \varphi_{0n} + \varphi_{1n}) e^{\lambda_n^- t}}{2\lambda_n^- - a_n^2 \hat{K}(\lambda_n^-)},$$

$$\varphi_{0n} = (\varphi_0, e_n), \qquad \varphi_{1n} = (\varphi_1, e_n), \qquad n \in \mathbb{N},$$

the vector function $w(t)$ *has the form*

$$w(t) = \sum_{n=1}^{\infty} w_n(t) e_n, \tag{13}$$

$$w_n(t) = \int_0^{\infty} \frac{e^{-t\tau}(-\tau\varphi_{0n} + \varphi_{1n}) a_n^2 (\hat{K}_-(-\tau) - \hat{K}_+(-\tau))}{(\tau^2 + a_n^2(1 - \hat{K}_+(-\tau)))(\tau^2 + a_n^2(1 - \hat{K}_-(-\tau)))} \, d\tau, \qquad \hat{K}_\pm(-\tau) = \sum_{k=1}^{n} \frac{c_k}{\tau^\alpha e^{\pm i\pi\alpha} + \beta_k},$$

and the series (12), (13) *converge with respect to the norm of the space* H.

2. PROOF OF THE MAIN RESULTS

To prove Theorems 1 and 2, we need the following assertion.

Proposition 1. *For any* $\gamma \ge \gamma_1 > 0$ *and all* $n \in \mathbb{N}$, *there exist positive constants* d_1 *and* d_2 *such that the inequalities*

$$\sup_{\operatorname{Re}\lambda > \gamma} |a_n / l_n(\lambda)| \le d_1 < \infty, \qquad \sup_{\operatorname{Re}\lambda > \gamma} |\lambda / l_n(\lambda)| \le d_2 < \infty \tag{14}$$

hold in the half-plane $\{\lambda : \operatorname{Re}\lambda > \gamma\}$.

Proof. Let $\lambda = x + iy = |\lambda|(\cos\varphi + i\sin\varphi)$. Note that $\operatorname{sgn} y = \operatorname{sgn} \sin(\alpha\varphi)$ for $0 < \alpha \le 1$. The Laplace transform $\hat{K}(\lambda)$ of the kernel $K(t)$ (see (8)) admits the representation

$$\hat{K}(\lambda) = \sum_{j=1}^{\infty} \left[c_j \frac{(|\lambda|^\alpha \cos(\alpha\varphi) + \beta_j) - i|\lambda|^\alpha \sin(\alpha\varphi)}{(|\lambda|^\alpha \cos(\alpha\varphi) + \beta_j)^2 + (|\lambda|^\alpha \sin(\alpha\varphi))^2} \right].$$

Consider the scalar functions

$$\mathrm{M}_n(\lambda) = \frac{l_n(\lambda)}{a_n^2} = \frac{1}{a_n^2}(L(\lambda)e_n, e_n) = \frac{\lambda^2}{a_n^2} + 1 - \sum_{k=1}^{\infty} \frac{c_k}{\lambda^\alpha + \beta_k}, \qquad n \in \mathbb{N},$$

and separate their real and imaginary parts

$$\operatorname{Re}\mathrm{M}_n(\lambda) = \frac{x^2 - y^2}{a_n^2} + 1 - \operatorname{Re}\hat{K}(\lambda), \qquad \operatorname{Im}\mathrm{M}_n(\lambda) = \frac{2xy}{a_n^2} - \operatorname{Im}\hat{K}(\lambda).$$

1. We divide the right half-plane into two the domains

$$\Omega_1 = \{\lambda : |y| > \operatorname{Re}\lambda := x > \gamma_0, \quad y = \operatorname{Im}\lambda\}, \qquad \Omega_2 = \{\lambda : \operatorname{Re}\lambda = x > |y|, \quad y = \operatorname{Im}\lambda\}.$$

First, we estimate the expression $|a_n / l_n(\lambda)|$ in the domain Ω_1. We have

$$\begin{aligned}
\frac{|l_n(\lambda)|}{a_n^2} &\ge |\operatorname{Im}\mathrm{M}_n(\lambda)| = \left| \frac{2xy}{a_n^2} + \sum_{k=1}^{\infty} c_k \frac{|\lambda|^\alpha \sin(\alpha\varphi)}{|\lambda|^{2\alpha} + 2|\lambda|^\alpha \beta_k \cos(\alpha\varphi) + \beta_k^2} \right| \\
&\ge \frac{2x|y|}{a_n^2} + \sum_{k=1}^{\infty} c_k \frac{|y|^\alpha |\sin(\pi\alpha/4)|}{(|\lambda|^\alpha + \beta_k)^2} \ge \frac{2\gamma|y|}{a_n^2} + c_1 \frac{|y|^\alpha |\sin(\pi\alpha/4)|}{((\sqrt{2}\,|y|)^\alpha + \beta_1)^2} \\
&\ge \frac{2\gamma|y|((\sqrt{2}\,|y|)^\alpha + \beta_1)^2 + c_1|y|^\alpha a_n^2 \sin^2(\pi\alpha/4)}{a_n^2((\sqrt{2}\,|y|)^\alpha + \beta_1)^2} \\
&\ge \frac{\sqrt{2\gamma|y|}\,((\sqrt{2}\,|y|)^\alpha + \beta_1)\sqrt{c_1}\,|y|^{\alpha/2} a_n |\sin(\pi\alpha/4)|}{a_n^2((\sqrt{2}\,|y|)^\alpha + \beta_1)^2} \ge \frac{\sqrt{2\gamma c_1}\,|y|^{(\alpha+1)/2} |\sin(\pi\alpha/4)|}{a_n((\sqrt{2}\,|y|)^\alpha + \beta_1)} \\
&\ge \frac{\sqrt{2\gamma c_1}\,|\sin(\pi\alpha/4)|}{((\sqrt{2}\,)^\alpha + \beta_1/\gamma^\alpha)} \frac{|y|^{(1-\alpha)/2}}{a_n} \ge \frac{\sqrt{2\gamma c_1}\,|\sin(\pi\alpha/4)|}{((\sqrt{2}\,)^\alpha + \beta_1/\gamma^\alpha)} \ge \frac{\gamma^{(1-\alpha)/2}}{a_n} = \frac{k(\alpha,\gamma)}{a_n},
\end{aligned}$$

where $k(\alpha,\gamma)$ is a positive constant depending on the parameters α ($0<\alpha<1$) and $\gamma>0$. Therefore, the estimate $a_n/|l_n(\lambda)| \le 1/k(\alpha,\gamma)$ holds for all $\lambda\in\Omega_1$.

Now let us estimate the expression $|a_n/l_n(\lambda)|$ in the domain Ω_2. We use condition (6) to obtain

$$\frac{|l_n(\lambda)|}{a_n^2} \ge |\operatorname{Re} \mathrm{M}_n(\lambda)| = \left|\frac{x^2-y^2}{a_n^2} + \sum_{k=1}^{\infty} c_k \frac{|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k}{(|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k)^2+(|\lambda|^\alpha\sin(\alpha\varphi))^2}\right| \ge 1-\sum_{k=1}^{\infty}\frac{c_k}{\beta_k} > 0.$$

Indeed, note that

$$\sum_{k=1}^{\infty} c_k \frac{|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k}{(|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k)^2+(|\lambda|^\alpha\sin(\alpha\varphi))^2} \le \sum_{k=1}^{\infty}\frac{c_k}{|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k} \le \sum_{k=1}^{\infty}\frac{c_k}{\beta_k} < 1.$$

Therefore, for all $\lambda\in\Omega_2$ we have the estimate

$$\frac{a_n}{|l_n(\lambda)|} \le a_1^{-1}\left(1-\sum_{k=1}^{\infty}\frac{c_k}{\beta_k}\right)^{-1}.$$

Thus, we finally obtain the estimate

$$\sup_{\operatorname{Re}\lambda>\gamma}\frac{a_n}{|l_n(\lambda)|} \le \left(\min\left\{k(\alpha,\gamma), a_1\left(1-\sum_{k=1}^{\infty}\frac{c_k}{\beta_k}\right)\right\}\right)^{-1} =: d_1.$$

2. Let us estimate the expression $|\lambda/l_n(\lambda)|$ for all $\lambda = x+iy$, $x>\gamma$. We have

$$\begin{aligned}\left|\frac{l_n(\lambda)}{\lambda}\right| &= \left|\lambda + \frac{a_n^2}{\lambda}\left(1-\sum_{k=1}^{\infty}\frac{c_k}{\lambda^\alpha+\beta_k}\right)\right| \\ &= \left|x+iy+\frac{a_n^2(x-iy)}{x^2+y^2}\left(1-\sum_{k=1}^{\infty}c_k\frac{|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k-i|\lambda|^\alpha\sin(\alpha\varphi)}{(|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k)^2+(|\lambda|^\alpha\sin(\alpha\varphi))^2}\right)\right| \\ &\ge x+\frac{a_n^2 x}{x^2+y^2}\left(1-\sum_{k=1}^{\infty}c_k\frac{|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k}{(|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k)^2+(|\lambda|^\alpha\sin(\alpha\varphi))^2}\right) \\ &\quad+\frac{a_n^2 y}{x^2+y^2}\sum_{k=1}^{\infty}c_k\frac{|\lambda|^\alpha\sin(\alpha\varphi)}{(|\lambda|^\alpha\cos(\alpha\varphi)+\beta_k)^2+(|\lambda|^\alpha\sin(\alpha\varphi))^2} > x > \gamma.\end{aligned}$$

This implies the estimate $|\lambda/l_n(\lambda)| < 1/\gamma =: d_2$ for all $\lambda = x+iy$, $x>\gamma$. The proof of the proposition is complete.

In turn, inequalities (14), respectively, imply the estimates

$$\sup_{\operatorname{Re}\lambda>\gamma}\|AL^{-1}(\lambda)\| \le d_1 < \infty, \qquad \sup_{\operatorname{Re}\lambda>\gamma}\|\lambda L^{-1}(\lambda)\| \le d_2 < \infty. \tag{15}$$

Proof of Theorem 1. First, we prove Theorem 1 for homogeneous (zero) initial data $\varphi_0 = \varphi_1 = 0$. In this case, the proof follows the scheme of the proof of the well-posed solvability of the Cauchy problem for hyperbolic-type equations based on the use of the Laplace transform. In this connection, for the reader's convenience, we recall known facts which can be used later.

Definition 2. The Hardy space $H_2(\operatorname{Re}\lambda>\gamma, H)$ is the class of vector functions $\hat f(\lambda)$ ranging in the space H and holomorphic in the half-plane $\{\lambda\in\mathbb{C}: \operatorname{Re}\lambda>\gamma\ge 0\}$ for which the following inequality holds:

$$\sup_{x>\gamma}\int_{-\infty}^{+\infty}\|\hat f(x+iy)\|_H^2\,dy < \infty \qquad (\lambda = x+iy).$$

Let us state the Paley–Wiener theorem for the Hardy spaces $H_2(\operatorname{Re}\lambda > \gamma, H)$.

Theorem (Paley–Wiener). 1. *The space $H_2(\operatorname{Re}\lambda > \gamma, H)$ coincides with the set of vector functions (Laplace transforms) that admit the representation*

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda t} f(t)\, dt, \tag{16}$$

where $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$, $\lambda \in \mathbb{C}$, and $\operatorname{Re}\lambda > \gamma \geq 0$.

2. *For any vector function $\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, H)$, the representation* (16), *where the vector function $f(t)$ belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$, exists and is unique, and the inversion formula holds:*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\gamma + iy) e^{(\gamma+iy)t}\, dy, \qquad t \in \mathbb{R}_+, \qquad \gamma \geq 0.$$

3. *The vector functions $\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma, H)$ and $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$ related by* (16) *satisfy the relation*

$$\|\hat{f}\|^2_{H_2(\operatorname{Re}\lambda>\gamma,H)} \equiv \sup_{x>\gamma} \int_{-\infty}^{+\infty} \|\hat{f}(x+iy)\|^2_H\, dy = \int_0^{+\infty} e^{-2\gamma t} \|f(t)\|^2_H\, dt \equiv \|f\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)}.$$

This theorem is widely known for scalar functions. But it can readily be generalized to the case of vector functions ranging in a separable Hilbert space.

Let us return to the proof of Theorem 1 for the zero initial data $\varphi_0 = \varphi_1 = 0$. We apply the Laplace transform to Eq. (1) and obtain the following representation for the Laplace transform of the solution of problem (1)–(3): $\hat{u}(\lambda) = L^{-1}(\lambda)\hat{f}(\lambda)$. Let us prove the unique solvability of problem (1)–(3) in the space $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ for any $\gamma \geq \gamma_1 > \gamma_0$.

First, let us show that the vector function $A^2u(t)$ belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$. It is easily seen that

$$A^2\hat{u}(\lambda) = A^2L^{-1}(\lambda)\hat{f}(\lambda) = AL^{-1}(\lambda)A\hat{f}(\lambda). \tag{17}$$

By the Paley–Wiener theorem, $A\hat{f}(\lambda) \in H_2(\operatorname{Re}\lambda > \gamma_0, H)$, because $Af(t) \in L_{2,\gamma_0}(\mathbb{R}_+, H)$. Moreover, we have

$$\|Af\|_{L_{2,\gamma_0}(\mathbb{R}_+,H)} = \|A\hat{f}\|_{H_2(\operatorname{Re}\lambda>\gamma_0,H)}. \tag{18}$$

By the first estimate in (15) and formulas (17) and (18), we have

$$\|A^2u\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} = \|A^2\hat{u}\|^2_{H_2(\operatorname{Re}\lambda>\gamma,H)} = \|AL^{-1}(\lambda)A\hat{f}(\lambda)\|^2_{H_2(\operatorname{Re}\lambda>\gamma,H)} \leq d_1^2\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)}.$$

Thus, the vector function $A^2u(t)$ belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$ and the following estimate holds:

$$\|A^2u\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \leq d_1\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)}. \tag{19}$$

Now let us show that the vector function $\lambda^2\hat{u}(\lambda)$ also belongs to the space $H_2(\operatorname{Re}\lambda > \gamma, H)$. Note that $I = \lambda^2L^{-1}(\lambda) + (1 - \hat{K}(\lambda))A^2L^{-1}(\lambda)$ for $\operatorname{Re}\lambda > \gamma$. Therefore, for $\operatorname{Re}\lambda > \gamma$ we have

$$\hat{f}(\lambda) = \lambda^2\hat{u}(\lambda) + (1 - \hat{K}(\lambda))A^2L^{-1}(\lambda)\hat{f}(\lambda). \tag{20}$$

By the assumptions about the function $K(t)$, the function $1 - \hat{K}(\lambda)$ is bounded and analytic in the half-plane $\{\lambda : \operatorname{Re}\lambda > \gamma\}$. Indeed, the following inequality holds:

$$|1 - \hat{K}(\lambda)| \leq 1 + \sum_{k=1}^{\infty} \frac{c_k}{|\lambda^\alpha + \beta_k|} \leq 1 + \sum_{k=1}^{\infty} \frac{c_k}{\beta_k} < 2. \tag{21}$$

By (6), the regularity (analyticity) follows from the uniform convergence of the series.

Let us estimate the norm of the vector function $\lambda^2\hat{u}(\lambda)$ in the Hardy space $H_2(\operatorname{Re}\lambda > \gamma, H)$. It follows from the representation (20), the first estimate in (15), and the estimate (21) that

$$\begin{aligned}\|\lambda^2\hat{u}(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)} &\le \|\hat{f}(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)} + |1-\hat{K}(\lambda)|\|AL^{-1}(\lambda)Af(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)} \\ &\le \text{ const } \|Af(\lambda)\|_{H_2(\operatorname{Re}\lambda>\gamma,H)}.\end{aligned}$$

Thus, the Paley–Wiener theorem implies the inequality

$$\|d^2u/dt^2\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} \le d_1\|Af\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)}. \tag{22}$$

Finally, combining the estimates (19) and (22), we see that the vector function $u(t)$ belongs to the space $W^2_{2,\gamma}(\mathbb{R}_+, H)$ and the following estimate holds:

$$\|u\|_{W^2_{2,\gamma}(\mathbb{R}_+,A^2)} \le d_2\|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)}.$$

Now consider problem (1)–(3) with inhomogeneous initial data φ_0 and φ_1. Set

$$u(t) = \cos(At)\varphi_0 + A^{-1}\sin(At)\varphi_1 + w(t).$$

Then the vector function $w(t)$ is a solution of the problem

$$\frac{d^2w}{dt^2} + A^2w(t) - \int_0^t K(t-s)A^2w(s)ds = f_1(t),$$
$$w(+0) = w^{(1)}(+0) = 0,$$

where $f_1(t) = f(t) - h(t)$ and

$$h(t) = \int_0^t K(t-s)A^2(\cos(As)\varphi_0 + A^{-1}\sin(As)\varphi_1)\, ds. \tag{23}$$

To prove the theorem, it suffices to prove the inequality

$$\|Af_1\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \le \|Af\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|Ah\|_{L_{2,\gamma}(\mathbb{R}_+,H)} < \infty.$$

To this end, we first prove the following assertion.

Proposition 2. *Let the assumptions of Theorem* 1 *be satisfied. Then for any* $\gamma \ge \gamma_1 > \gamma_0$ *the function* h *defined by formula* (23) *satisfies the estimate*

$$\|h(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \le d_3(\|A^2\varphi_0\| + \|A\varphi_1\|) \tag{24}$$

with a constant d_3 *independent of the vectors* φ_0 *and* φ_1.

Proof. To estimate the norm of the vector function $h(t)$ in the space $L_{2,\gamma}(\mathbb{R}_+, H)$, it suffices, by the Paley–Wiener theorem, to estimate the norm of the vector function $\hat{h}(\lambda)$ in the Hardy space $H_2(\operatorname{Re}\lambda > \gamma, H)$. The vector function $\hat{h}(\lambda)$ admits the representation

$$\hat{h}(\lambda) = \hat{K}(\lambda)[\lambda(\lambda^2I + A^2)^{-1}A^2\varphi_0 + A(\lambda^2I + A^2)^{-1}A\varphi_1].$$

Therefore, its norm satisfies the relation

$$\|\hat{h}(\lambda)\|^2_{H_2(\operatorname{Re}\lambda>\gamma,H)} = \sup_{x>\gamma}\int_{-\infty}^{+\infty} \|\hat{K}(x+iy)[(x+iy)((x+iy)^2I + A^2)^{-1}A^2\varphi_0 + A((x+iy)^2I + A^2)^{-1}A\varphi_1]\|^2_H\, dy. \tag{25}$$

Let us estimate the resulting integral:

$$\begin{aligned}
&\|\hat{K}(x+iy)[(x+iy)((x+iy)^2I+A^2)^{-1}A^2\varphi_0 + A((x+iy)^2I+A^2)^{-1}A\varphi_1]\|_H^2 \\
&\le C\left|\sum_{k=1}^{\infty}\frac{c_k}{(x+iy)^{\alpha}+\beta_k}\right|^2 (\|(x+iy)((x+iy)^2I+A^2)^{-1}A^2\varphi_0\|_H^2 + \|A((x+iy)^2I+A^2)^{-1}A\varphi_1\|_H^2) \\
&\le C\left(\sum_{k=1}^{\infty}\frac{c_k}{|(x+iy)^{\alpha}+\beta_k|}\right)^2\left(\sum_{n=1}^{\infty}\frac{|x+iy|^2a_n^4|\varphi_{0n}|^2}{|(x+iy)^2+a_n^2|^2} + \sum_{n=1}^{\infty}\frac{a_n^4|\varphi_{1n}|^2}{|(x+iy)^2+a_n^2|^2}\right) \\
&= C\left(\sum_{k=1}^{\infty}\frac{c_k}{|(x+iy)^{\alpha}+\beta_k|}\right)^2\left(\sum_{n=1}^{\infty}\frac{(x^2+y^2)a_n^4|\varphi_{0n}|^2}{(x^2-y^2+a_n^2)^2+4x^2y^2} + \sum_{n=1}^{\infty}\frac{a_n^4|\varphi_{1n}|^2}{(x^2-y^2+a_n^2)^2+4x^2y^2}\right). \quad (26)
\end{aligned}$$

Note that

$$(x^2-y^2+a_n^2)^2+4x^2y^2 = (x^2+(y-a_n)^2)(x^2+(y+a_n)^2).$$

Moreover, for $x>\gamma$ we have the inequality

$$\frac{c_k}{|(x+iy)^{\alpha}+\beta_k|} = \frac{c_k}{\sqrt{((x^2+y^2)^{\alpha/2}\cos(\alpha\varphi)+\beta_k)^2 + (x^2+y^2)^{\alpha}\sin^2(\alpha\varphi)}} \le \frac{c_k}{\beta_k}.$$

Then, using the estimate (26), we obtain the following estimate of the integral in (25):

$$\begin{aligned}
\|\hat{h}(\lambda)\|^2_{H_2(\operatorname{Re}\lambda>\gamma,H)} &\le C\left(\sum_{k=1}^{\infty}\frac{c_k}{\beta_k}\right)^2 \sup_{x>\gamma}\sum_{n=1}^{\infty}\left(\int_{-\infty}^{+\infty}\frac{(x^2+y^2)a_n^4|\varphi_{0n}|^2}{(x^2+(y-a_n)^2)(x^2+(y+a_n)^2)}\,dy\right. \\
&\quad \left.+\int_{-\infty}^{+\infty}\frac{a_n^4|\varphi_{1n}|^2}{(x^2+(y-a_n)^2)(x^2+(y+a_n)^2)}\,dy\right) \\
&\le 2C\left(\sum_{k=1}^{\infty}\frac{c_k}{\beta_k}\right)^2 \sup_{x>\gamma}\sum_{n=1}^{\infty}\left(\int_{0}^{+\infty}\frac{a_n^4|\varphi_{0n}|^2}{(x^2+(y-a_n)^2)}\,dy + \int_{0}^{+\infty}\frac{a_n^4|\varphi_{1n}|^2}{a_n^2(x^2+(y-a_n)^2)}\,dy\right) \\
&\le C\frac{2\pi}{\gamma}\left(\sum_{k=1}^{\infty}\frac{c_k}{\beta_k}\right)^2\left(\sum_{n=1}^{\infty}a_n^4|\varphi_{0n}|^2 + \sum_{n=1}^{\infty}a_n^2|\varphi_{1n}|^2\right) = C\frac{2\pi}{\gamma}(\|A^2\varphi_0\|_H^2 + \|A\varphi_1\|_H^2).
\end{aligned}$$

The proof of the proposition is complete.

The estimate (24) implies the estimate

$$\|Ah(t)\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \le d_4(\|A^3\varphi_0\| + \|A^2\varphi_1\|)$$

with a constant d_4 independent of the vectors φ_0 and φ_1; we need this estimate to prove Theorem 1.

Now let us prove the uniqueness of the strong solution of problem (1)–(3). Assume that there exist two distinct strong solutions $u_1(t)$ and $u_2(t)$ of problem (1)–(3). Then the vector function $v(t) = u_1(t) - u_2(t)$ is a strong solution of problem (1)–(3) with zero right-hand side $f(t) \equiv 0$ and zero initial vectors $\varphi_0 = \varphi_1 = 0$, and its Laplace transform $\hat{v}(\lambda)$ satisfies the equation $L(\lambda)\hat{v}(\lambda) = 0$. Therefore, we have $\hat{v}(\lambda) = 0$ and, by the inversion formula for the Laplace transform, $v(t) \equiv 0$. The proof of Theorem 1 is complete.

Proof of Theorem 2. Assume that $u(t)$ is the strong solution of problem (1)–(3) with zero initial data $\varphi_0 = \varphi_1 = 0$. Applying the Laplace transform to Eq. (1), we obtain the following representation for the Laplace transform of the strong solution $u(t)$ of problem (1)–(3); i.e., $\hat{u}(\lambda) = L^{-1}(\lambda)\hat{f}(\lambda)$.

Consider the projection $u_n(t)$ of the vector function $u(t)$ onto the one-dimensional subspace spanned by the vector e_n; i.e., $u_n(t) = (u(t), e_n)_H$. Then we have $\hat{u}_n(\lambda) = (\hat{u}(\lambda), e_n)_H = l_n^{-1}(\lambda)\hat{f}_n(\lambda)$.

First, let us prove the generalized solvability in the space $W^1_{2,\gamma}(\mathbb{R}_+, A)$ of problem (1), (2) with zero initial data $\varphi_0 = \varphi_1 = 0$ for any $\gamma \geq \gamma_1 > \gamma_0$. To this end, we prove the following assertion.

Claim 1. *If a function $u_n(t)e_n$ is a strong solution of the problem*

$$\frac{d^2u}{dt^2} + A^2u - \int\limits_0^t K(t-s)A^2u(s)\,ds = f_n(t)e_n, \qquad t \in \mathbb{R}_+, \tag{27}$$

$$u(+0) = 0, \qquad u^{(1)}(+0) = 0, \tag{28}$$

then the function $u_n(t)e_n$ is a generalized solution of this problem.

Proof. Indeed, taking the inner product of both sides of Eq. (27) by the function $v(t) \in W^1_{2,\gamma}(\mathbb{R}_+, A)$ in the space $L_{2,\gamma}(\mathbb{R}_+, H)$, where $\gamma \geq \gamma_1 > \gamma_0$, we obtain

$$\int\limits_0^{+\infty} (u_n''(t)e_n, v(t))_H e^{-2\gamma t}\,dt + \int\limits_0^{+\infty} \left(A^2\left[u_n(t)e_n - \int\limits_0^t K(t-s)u_n(s)e_n ds\right], v(t)\right)_H e^{-2\gamma t}\,dt$$
$$= \int\limits_0^{+\infty} (f_n(t)e_n, v(t))_H e^{-2\gamma t}\,dt.$$

We integrate the first term by parts and obtain

$$\int\limits_0^{+\infty} (u_n''(t)e_n, v(t))_H e^{-2\gamma t}\,dt = -(\varphi_1, v(0))_H - \langle u_n'(t), v'(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} + 2\gamma\langle u_n'(t), v(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)}.$$

Transforming the second term, we obtain

$$\int\limits_0^{+\infty} \left(A^2\left[u_n(t)e_n - \int\limits_0^t K(t-s)u_n(s)e_n\,ds\right], v(t)\right)_H e^{-2\gamma t}\,dt$$
$$= \left\langle A\left[u_n(t)e_n - \int\limits_0^t K(t-s)u_n(s)e_n\,ds\right], Av(t)\right\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)}.$$

Thus, the function $u_n(t)e_n$ satisfies the identity

$$\left\langle A\left[u_n(t)e_n + \int\limits_0^t K(t-s)u_n(s)e_n ds\right], Av(t)\right\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} - \langle u'_n(t)e_n, v'(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)}$$
$$+ 2\gamma\langle u'_n(t)e_n, v(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} = \langle f_n(t)e_n, v(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)}$$

and hence is a generalized solution of problem (27), (28). The proof of the claim is complete.

Corollary. *If the vector function $S_N(t) = \sum_{n=1}^N u_n(t)e_n$ is a strong solution of the problem*

$$\frac{d^2u}{dt^2} + A^2u - \int\limits_0^t K(t-s)A^2u(s)ds = F_N(t), \qquad t \in \mathbb{R}_+, \tag{29}$$

$$u(+0) = 0, \qquad u^{(1)}(+0) = 0, \tag{30}$$

where $F_N(t) = \sum_{n=1}^{N} f_n(t)$, *then the vector function* $S_N(t)$ *is a generalized solution of problem* (29), (30); *i.e., the following identity holds*:

$$\left\langle A\left[S_N(t) + \int_0^t K(t-s)S_N(s)ds\right], Av(t)\right\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} - \langle S'_N(t), v'(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+)} + 2\gamma\langle S'_N(t), v(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)} = \langle F_N(t), v(t)\rangle_{L_{2,\gamma}(\mathbb{R}_+,H)}. \quad (31)$$

Now we return to the proof of Theorem 2. Let us show that if its conditions are satisfied, then the vector function $u(t) = \sum_{n=1}^{\infty} u_n(t)e_n$ (where, for each $n \in \mathbb{N}$, the function $u_n(t)e_n$ is a strong solution of the corresponding problem (27), (28)) is a generalized solution of problem (1)–(3).

To this end, using the estimate (14), we show that the sequence $S_N(t) = \sum_{n=1}^{N} u_n(t)e_n$ is a Cauchy sequence in the space $W_{2,\gamma}^1(\mathbb{R}_+, A)$. We have

$$\begin{aligned}
\|S_N(t) - S_M(t)\|^2_{W^1_{2,\gamma}(\mathbb{R}_+,A)} &= \left\|\sum_{n=M+1}^{N} u_n(t)e_n\right\|^2_{W^1_{2,\gamma}(\mathbb{R}_+,A)} = \left\|A\sum_{n=M+1}^{N} u_n(t)e_n\right\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} \\
&+ \left\|\sum_{n=M+1}^{N} u'_n(t)e_n\right\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} = \left\|A\sum_{n=M+1}^{N} \hat{u}_n(\lambda)e_n\right\|^2_{H_2(\operatorname{Re}\lambda>0,H)} + \left\|\sum_{n=M+1}^{N} \lambda\hat{u}_n(\lambda)e_n\right\|^2_{H_2(\operatorname{Re}\lambda>0,H)} \\
&= \sup_{x>\gamma}\int_{-\infty}^{+\infty}\left(\sum_{n=M+1}^{N}|a_n\hat{u}_n(x+iy)|^2 + \sum_{n=M+1}^{N}|(x+iy)\hat{u}_n(x+iy)|^2\right)dy \\
&= \sup_{x>\gamma}\int_{-\infty}^{+\infty}\left(\sum_{n=M+1}^{N}\left|\frac{a_n\hat{f}_n(x+iy)}{l_n(x+iy)}\right|^2 + \sum_{n=M+1}^{N}\left|\frac{(x+iy)\hat{f}_n(x+iy)}{l_n(x+iy)}\right|^2\right)dy \\
&\le \int_{-\infty}^{+\infty}\sum_{n=M+1}^{N}\sup_{x>\gamma}\left(\left|\frac{a_n}{l_n(x+iy)}\right|^2 + \left|\frac{x+iy}{l_n(x+iy)}\right|^2\right)|\hat{f}_n(x+iy)|^2\,dy \le d_5\sup_{x>\gamma}\int_{-\infty}^{+\infty}\sum_{n=M+1}^{N}|\hat{f}_n(x+iy)|^2\,dy \\
&= d_5\left\|\sum_{n=M+1}^{N}\hat{f}_n(\lambda)e_n\right\|^2_{H_2(\operatorname{Re}\lambda>0,H)} = d_5\left\|\sum_{n=M+1}^{N} f_n(t)e_n\right\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)}. \quad (32)
\end{aligned}$$

By the assumptions of Theorem 2, the vector function $f(t)$ belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$, and therefore, the sequence of vector functions $S_N(t)$ converges in the space $W_{2,\gamma}^1(\mathbb{R}_+, A)$ to the vector function $u(t)$ if the sequence $F_N(t)$ converges to the vector function $F(t)$ in the space $L_{2,\gamma}(\mathbb{R}_+, H)$. Passing to the limit as $N \to +\infty$ in identity (31), we obtain identity (9) for $\varphi_1 = 0$; i.e., the function $u(t) = \sum_{n=1}^{\infty} u_n(t)e_n$ is a generalized solution of problem (1)–(3).

Now let us estimate the norm of the generalized solution $u(t) = \sum_{n=1}^{\infty} u_n(t)e_n$ in the space $W_{2,\gamma}^1(\mathbb{R}_+, A)$. By setting $M = 0$ and $S_M(t) = 0$ in the chain of inequalities (32), we obtain the inequality

$$\|S_N(t)\|^2_{W^1_{2,\gamma}(\mathbb{R}_+,A)} \le d_5\left\|\sum_{n=1}^{N} f_n(t)e_n\right\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} \le d_5\|f(t)\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)},$$

from which, passing to the limit as $N \to +\infty$, we obtain the estimate

$$\|u(t)\|^2_{W^1_{2,\gamma}(\mathbb{R}_+,A)} \le d_5\|f(t)\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)}. \quad (33)$$

Now consider problem (1)–(3) with inhomogeneous initial data φ_0 and φ_1. Set

$$u(t) = \cos(At)\varphi_0 + A^{-1}\sin(At)\varphi_1 + w(t). \tag{34}$$

Claim 2. *The vector function $u(t)$ is a generalized solution of problem (1)–(3) whenever the vector function $w(t)$ is a generalized solution of the problem*

$$\frac{d^2w}{dt^2} + A^2w(t) - \int_0^t K(t-s)A^2w(s)ds = f_1(t), \tag{35}$$

$$w(+0) = w^{(1)}(+0) = 0, \tag{36}$$

where $f_1(t) = f(t) - h(t)$, $h(t) = \int_0^t K(t-s)A^2(\cos(As)\varphi_0 + A^{-1}\sin(As)\varphi_1)\,ds$.

The conditions of Theorem 2 and Proposition 2 imply the estimate

$$\|f_1\|_{L_{2,\gamma}(\mathbb{R}_+,H)} \le \|f\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|h\|_{L_{2,\gamma}(\mathbb{R}_+,H)} < \infty.$$

Thus, the assumptions of Theorem 2 are satisfied for problem (35), (36). A straightforward substitution of the vector function $u(t)$ given by (34) into identity (9) readily proves the assertion of Claim 2. Moreover, the estimate (33) and Proposition 2 imply the estimate (10). The proof of Theorem 2 is complete.

Proof of Theorem 5. Let us outline the proof of Theorem 5. The proof is based on the path integration of the inversion of the Laplace transform of the strong solution of problem (1)–(3).

Expanding the vector function $\hat{u}(\lambda)$ with respect to the orthonormal basis of eigenvectors $\{e_n\}_{n=1}^{\infty}$ of the operator A corresponding to the eigenvalues a_n, we obtain

$$\hat{u}_n(\lambda) = (\hat{u}_n(\lambda), e_n) = l_n^{-1}(\lambda)(\lambda\varphi_{0n} + \varphi_{1n}),$$

where $l_n(\lambda) = \lambda^2 + a_n^2(1 - \hat{K}(\lambda))$, $\varphi_{0n} = (\varphi_0, e_n)$, and $\varphi_{1n} = (\varphi_1, e_n)$. Considering the inversion of the Laplace transform of the solution $u(t)$, we obtain the following representation for the coordinate functions $u_n(t)$:

$$u_n(t) = (u(t), e_n) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} l_n^{-1}(\lambda)(\lambda\varphi_{0n} + \varphi_{1n})e^{\lambda t}\,d\lambda, \qquad n \in \mathbb{N}. \tag{37}$$

The integral (37) is calculated by the path integration as follows. By Theorem 2.3 in [16], the function $\hat{u}_n(\lambda)$ has simple poles at the points $\lambda_n^{\pm}$ and the cut along the negative real semiaxis $(-\infty, 0]$.

In the complex plane, consider the contour $\Gamma = \Gamma_0 \cup \Gamma_1 \cup C_R^+ \cup R^+ \cup R^- \cup C_R^- \cup \Gamma_2$, where

$$\Gamma_0 = \{\lambda : \operatorname{Re}\lambda = \gamma, -R \le \operatorname{Im}\lambda \le R\}, \qquad \Gamma_1 = \{\lambda : 0 \le \operatorname{Re}\lambda \le \gamma, \operatorname{Im}\lambda = R\},$$
$$C_R^+ = \{\lambda : \lambda = R\mathrm{e}^{i\varphi}, \pi/2 \le \varphi \le \pi\}, \qquad R^+ = \{\lambda : \operatorname{Im}\lambda = 0, -R \le \operatorname{Re}\lambda \le 0\},$$
$$R^- = \{\lambda : \operatorname{Im}\lambda = 0, -R \le \operatorname{Re}\lambda \le 0\}, \qquad C_R^- = \{\lambda : \lambda = R\mathrm{e}^{i\varphi}, -\pi \le \varphi \le -3\pi/2\},$$
$$\Gamma_2 = \{\lambda : 0 \le \operatorname{Re}\lambda \le \gamma, \operatorname{Im}\lambda = -R\}.$$

Note that the contour Γ is bypassed anticlockwise. If R tends to $+\infty$, then the integral over the path Γ_0 tends to the integral (37). In turn, by the Jordan lemma, the integrals over the arcs of the circles $C_R^{\pm}$ tend to zero as $R \to +\infty$. A straightforward verification readily shows that the integrals over the contours Γ_1 and Γ_2 tend to zero as $R \to +\infty$. Thus, passing to the limit as $R \to +\infty$, we obtain the following representation of the function $u_n(t)$:

$$u_n(t) = u_n^+(t) + u_n^-(t) + w_n(t),$$

where

$$u_n^+(t) = 2\pi i \operatorname*{res}_{\lambda=\lambda_n^+} \hat{u}_n(\lambda) = 2\pi i \frac{(\lambda_n^+\varphi_{0n} + \varphi_{1n})e^{\lambda_n^+ t}}{2\lambda_n^+ - a_n^2 \hat{K}^{(1)}(\lambda_n^+)}, \quad u_n^-(t) = 2\pi i \operatorname*{res}_{\lambda=\lambda_n^-} \hat{u}_n(\lambda) = 2\pi i \frac{(\lambda_n^-\varphi_{0n} + \varphi_{1n})}{2\lambda_n^- - a_n^2 \hat{K}^{(1)}(\lambda_n^-)},$$

$$w_n(t) = \int\limits_0^{+\infty} \frac{e^{-t\tau}(-\tau\varphi_{0n} + \varphi_{1n})a_n^2(\hat{K}_+(-\tau) - \hat{K}_-(-\tau))}{(\tau^2 + a_n^2(1 - \hat{K}_+(-\tau)))(\tau^2 + a_n^2(1 - \hat{K}_-(-\tau)))}\, d\tau,$$

and

$$\hat{K}_\pm(-\tau) = \sum_{k=1}^{n} \frac{c_k}{\tau^\alpha e^{\pm i\pi\alpha} + \beta_k}.$$

Note that the term $w_n(t)$ corresponds to the cut from $-\infty$ to 0. Thus, the solution $u(t)$ of problem (1)–(3) can be represented as the sum of the series

$$u(t) = \sum_{n=1}^{\infty} (u_n^+(t) + u_n^-(t) + w_n(t))e_n. \tag{38}$$

The convergence of the series (38) follows from the estimates that hold for sufficiently large n and $\delta > 0$:

$$|u_n^+(t) + u_n^-(t)| \le c e^{-k a_n^{1-\alpha} t}(a_n|\varphi_{0n}| + |\varphi_{1n}|), \tag{39}$$

$$|w_n(t)| \le \int\limits_0^{+\infty} \frac{e^{-t\tau}(\tau|\varphi_{0n}| + |\varphi_{1n}|)a_n^2(\hat{K}_+(-\tau) - \hat{K}_-(-\tau))}{(\tau^2 + \delta^2 a_n^2)^2}\, d\tau, \tag{40}$$

where

$$\hat{K}_+(-\tau) - \hat{K}_-(-\tau) = (-\sin\pi\alpha)\tau^\alpha \sum_{k=1}^{n} \frac{c_k}{(\tau^\alpha\cos\pi\alpha + \beta_k)^2 + \tau^{2\alpha}\sin^2\pi\alpha}.$$

The proof of the estimates (39) and (40) is significantly based on the use of the asymptotics (11) of the eigenvalues. The proof of Theorem 5 is complete.

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