ISSN 0012-2661, Differential Equations, 2019, Vol. 55, No. 4, pp. 510–514. © Pleiades Publishing, Ltd., 2019.
Russian Text © The Author(s), 2019, published in Differentsial'nye Uravneniya, 2019, Vol. 55, No. 4, pp. 524–528.

ORDINARY DIFFERENTIAL EQUATIONS

# Recovering Differential Operators with a Retarded Argument

**V. Yurko**

*Saratov State University, Saratov, 410012 Russia*
*e-mail: yurkova@info.sgu.ru*

Received December 5, 2018; revised December 5, 2018; accepted February 12, 2019

**Abstract**—We consider second-order differential operators with a constant delay. The properties of their spectral characteristics are established, and the inverse problem of recovering the operators from their spectra is studied. We develop constructive algorithms for inverse problems and prove the uniqueness of the solution.

**DOI**: 10.1134/S0012266119040086

## INTRODUCTION

We study inverse spectral problems for Sturm–Liouville differential operators with a constant delay. Such problems often arise in natural sciences and engineering (e.g., see the monographs [1, 2]). Inverse spectral problems are problems of recovering operators from given spectral characteristics. Inverse problems for classical differential operators have been studied fairly comprehensively; the main results can be found in the monographs [3–6].

However, delay differential operators are substantially more difficult to study, because the main methods of inverse problem theory (the transformation operator method and the spectral mapping method [5]) do not work for delay operators. Note that some special results on inverse problems for delay operators were obtained in [7–11].

Consider the following boundary value problems $L_j$, $j = 0, 1$:

$$-y''(x) + q(x)y(x-a) = \lambda y(x), \qquad 0 < x < \pi, \tag{1}$$
$$y'(0) - hy(0) = y^{(j)}(\pi) = 0.$$

Here $\lambda$ is the spectral parameter, $a \in (0, \pi)$, $h$ is a complex number, $q(x)$ is a complex-valued function, $q(x) \in L(a, \pi)$, and $q(x) = 0$ a.e. on $(0, a)$.

Let $S(x, \lambda)$ and $C(x, \lambda)$ be the solutions of Eq. (1) with the initial conditions

$$C(0, \lambda) = S'(0, \lambda) = 1, \qquad S(0, \lambda) = C'(0, \lambda) = 0.$$

Set $\varphi(x, \lambda) = C(x, \lambda) + hS(x, \lambda)$. For each $x$, the functions $C^{(j)}(x, \lambda)$, $S^{(j)}(x, \lambda)$, and $\varphi^{(j)}(x, \lambda)$, $j = 0, 1$, are entire of order $1/2$ in $\lambda$. Set

$$\Delta_j(\lambda) := \varphi^{(j)}(\pi, \lambda), \qquad j = 0, 1.$$

The eigenvalues $\{\lambda_{nj}\}_{n\ge 0}$ of the boundary value problem $L_j$ coincide with the zeros of the entire function $\Delta_j(\lambda)$. The function $\Delta_j(\lambda)$ is called the *characteristic function* of problem $L_j$.

The properties of spectral characteristics of $L_j$ are established in Section 1. Section 2 studies the inverse problem of recovering the potential $q(x)$ and the coefficient $h$ from given spectra $\{\lambda_{nj}\}_{n\ge 0}$, $j = 0, 1$. Similar results for the case of the Robin boundary conditions

$$y'(0) - hy(0) = y'(\pi) + H_j y(\pi) = 0$$

are obtained in Section 3. We consider the case of $a \in [\pi/2, \pi)$. The case of $a < \pi/2$ requires a separate study. Theorems 1 and 2 and Algorithms 1 and 2, which give constructive solution procedures for the inverse problems and prove the uniqueness of the solutions, are the main results of the paper.

## 1. PROPERTIES OF SPECTRAL CHARACTERISTICS

Let $\lambda = \rho^2$. The functions $C(x,\lambda)$ and $S(x,\lambda)$ are the unique solutions of the integral equations

$$C(x,\lambda) = \cos\rho x + \int_a^x G(x,t,\lambda)C(t-a,\lambda)\,dt, \qquad S(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_a^x G(x,t,\lambda)S(t-a,\lambda)\,dt,$$

where $G(x,t,\lambda) = \dfrac{q(t)\sin\rho(x-t)}{\rho}$. For $x \ge a$, this gives

$$\begin{aligned} C(x,\lambda) &= \cos\rho x + \frac{1}{\rho}\int_a^x q(t)\sin\rho(x-t)\cos\rho(t-a)\,dt, \\ S(x,\lambda) &= \frac{\sin\rho x}{\rho} + \frac{1}{\rho^2}\int_a^x q(t)\sin\rho(x-t)\sin\rho(t-a)\,dt. \end{aligned} \tag{2}$$

It follows from the representations (2) that for $x \ge a$ one has

$$S(x,\lambda) = \frac{\sin\rho x}{\rho} - \frac{\cos\rho(x-a)}{2\rho^2}\int_a^x q(t)\,dt + \frac{1}{2\rho^2 a}\int_a^x q(t)\cos\rho(x-2t+a)\,dt,$$

$$S'(x,\lambda) = \cos\rho x + \frac{\sin\rho(x-a)}{2\rho}\int_a^x q(t)\,dt - \frac{1}{2\rho}\int_a^x q(t)\sin\rho(x-2t+a)\,dt,$$

$$C(x,\lambda) = \cos\rho x + \frac{\sin\rho(x-a)}{2\rho}\int_a^x q(t)\,dt + \frac{1}{2\rho}\int_a^x q(t)\sin\rho(x-2t+a)\,dt,$$

$$C'(x,\lambda) = -\rho\sin\rho x + \frac{\cos\rho(x-a)}{2}\int_a^x q(t)\,dt + \frac{1}{2}\int_a^x q(t)\cos\rho(x-2t+a)\,dt,$$

and consequently,

$$\Delta_0(\lambda) = \cos\rho\pi + \frac{h\sin\rho\pi}{\rho} + \frac{A\sin\rho(\pi-a)}{\rho} - \frac{hA\cos\rho(\pi-a)}{\rho^2} + \frac{d_0(\rho)}{2\rho^2}, \tag{3}$$

$$\Delta_1(\lambda) = -\rho\sin\rho\pi + h\cos\rho\pi + A\cos\rho(\pi-a) + \frac{hA\sin\rho(\pi-a)}{\rho} + \frac{d_1(\rho)}{2}, \tag{4}$$

where $A = \dfrac{1}{2}\int_a^\pi q(t)\,dt$ and

$$\begin{aligned} d_0(\rho) &= \rho\int_a^\pi q(t)\sin\rho(\pi-2t+a)\,dt + h\int_a^\pi q(t)\cos\rho(\pi-2t+a)\,dt, \\ d_1(\rho) &= \int_a^\pi q(t)\cos\rho(\pi-2t+a)\,dt - \frac{h}{\rho}\int_a^\pi q(t)\sin\rho(\pi-2t+a)\,dt. \end{aligned} \tag{5}$$

We use (3) and (4) in conjunction with a well-known method (e.g., see [5, Ch. 1]) to obtain the asymptotic representations

$$\sqrt{\lambda_{n0}} = (n+1/2) + (h + A\cos(n+1/2)a)/(\pi n) + o(1/n),$$
$$\sqrt{\lambda_{n1}} = n + (h + A\cos na)/(\pi n) + o(1/n) \tag{6}$$

as $n \to \infty$. Further, the spectra $\{\lambda_{nj}\}_{n\ge 0}$, $j = 0, 1$, uniquely determine the respective characteristic functions by the formulas

$$\Delta_0(\lambda) = \prod_{n=0}^{\infty} \frac{\lambda_{n0} - \lambda}{(n+1/2)^2}, \qquad \Delta_1(\lambda) = \pi(\lambda_{01} - \lambda)\prod_{n=1}^{\infty} \frac{\lambda_{n1} - \lambda}{n^2}. \tag{7}$$

## 2. SOLUTION OF THE INVERSE PROBLEM

Let the spectra $\{\lambda_{nj}\}_{n\ge 0}$, $j = 0, 1$, be given. Our aim is to find the potential $q(x)$ and the coefficient $h$. First, we construct the characteristic functions $\Delta_j(\lambda)$, $j = 0, 1$, by formulas (7). Then we can use Eq. (3) or Eq. (4) to find the coefficients $h$ and $A$. Indeed, it follows from (3) that $A\sin an = (-1)^{n+1}(\Delta_0(n^2) - (-1)^n)n + o(1)$ as $n \to \infty$, and consequently,

$$A = \lim_{n_k\to\infty} (-1)^{n_k+1}(\sin an_k)^{-1}(\Delta_0(n_k^2) - (-1)^{n_k})n_k, \tag{8}$$

where the $n_k$ satisfy $|\sin an_k| > \delta > 0$. We again use (3) and find that

$$h = \lim_{n\to\infty}\Big((2n+1/2)\Delta_0((2n+1/2)^2) - A\sin(2n+1/2)(\pi - a)\Big). \tag{9}$$

Note that we can also calculate the coefficients $A$ and $h$ using the representations (6). Now that $A$ and $h$ are known, we can construct the functions $d_j(\rho)$, $j = 0, 1$, with the use of Eqs. (3) and (4).

Set

$$\delta_0(\rho) = \int_a^{\pi} q(t)\cos\rho(2t - \pi - a)\,dt = \frac{1}{2}\int_{-(\pi-a)}^{(\pi-a)} Q(x)\cos\rho x\,dx, \tag{10}$$

$$\delta_1(\rho) = \int_a^{\pi} q(t)\sin\rho(2t - \pi - a)\,dt = \frac{1}{2}\int_{-(\pi-a)}^{(\pi-a)} Q(x)\sin\rho x\,dx, \tag{11}$$

where $Q(x) = q\Big(\dfrac{x+\pi+a}{2}\Big)$. We can use the representations (5) to construct the functions $\delta_0(\rho)$ and $\delta_1(\rho)$ by the formulas

$$\delta_0(\rho) = -\frac{1}{\omega(\rho)}\Big(\frac{h}{\rho}d_0(\rho) + \rho d_1(\rho)\Big), \qquad \delta_1(\rho) = \frac{1}{\omega(\rho)}\Big(d_0(\rho) - hd_1(\rho)\Big), \tag{12}$$

where $\omega(\rho) = -(\rho + h^2/\rho)$. Now we can find the function $Q(x)$ and hence the potential $q(x)$ from Eqs. (10) and (11). Thus, we have proved the following theorem.

**Theorem 1.** *The spectra $\{\lambda_{nj}\}_{n\ge 0}$, $j = 0, 1$, uniquely determine the potential $q(x)$ and the coefficient $h$. The solution of the inverse problem can be found by the following algorithm.*

**Algorithm 1.** *Given the spectra $\{\lambda_{nj}\}_{n\ge 0}$, $j = 0, 1$,*

1. *Construct the characteristic functions $\Delta_j(\lambda)$, $j = 0, 1$ by formulas* (7).
2. *Calculate $A$ and $h$ using Eqs.* (3) *or* (4), *say, by formulas* (8) *and* (9).

3. *Find the functions* $d_j(\rho)$, $j = 0, 1$, *with the use of Eqs.* (3) *and* (4).
4. *Construct the functions* $\delta_0(\rho)$ *and* $\delta_1(\rho)$ *according to the representations* (12).
5. *Find the function* $Q(x)$ *using Eqs.* (10) *and* (11).
6. *Calculate the potential* $q(x) = Q(2x - \pi - a)$.

## 3. ROBIN BOUNDARY CONDITIONS

Let $\{\mu_{nj}\}_{n\geq 0}$ be the eigenvalues of the following boundary value problems $\mathcal{L}_j$, $j = 1, 2$:

$$-y''(x) + q(x)y(x-a) = \lambda y(x), \qquad 0 < x < \pi,$$
$$y'(0) - hy(0) = y'(\pi) + H_j y(\pi) = 0,$$

Here $a \in (0, \pi)$, $h$ and $H_j$ are complex numbers ($H_1 \neq H_2$), $q(x)$ is a complex-valued function, $q(x) \in L(a, \pi)$, and $q(x) = 0$ a.e. on $(0, a)$. In this section, we study the inverse spectral problem of recovering potential $q(x)$ and the coefficients $h$, $H_1$, and $H_2$ from given spectra $\{\mu_{nj}\}_{n\geq 0}$, $j = 1, 2$. The eigenvalues $\{\mu_{nj}\}_{n\geq 0}$ of the boundary value problem $\mathcal{L}_j$ coincide with the zeros of the entire function

$$\mathcal{P}_j(\lambda) := \varphi'(\pi, \lambda) + H_j \varphi(\pi, \lambda).$$

Since $\varphi(x, \lambda) = C(x, \lambda) + hS(x, \lambda)$, it follows that

$$\mathcal{P}_j(\lambda) = -\rho \sin \rho\pi + (h + H_j) \cos \rho\pi + A \cos \rho(\pi - a) + o(\exp(|Im\,\rho|\pi)), \qquad |\rho| \to \infty. \tag{13}$$

Using the asymptotic relation (13), we obtain

$$\sqrt{\mu_{nj}} = n + (h + H_j + A \cos na)/(\pi n) + o(1/n), \qquad n \to \infty. \tag{14}$$

Further, the spectrum $\{\mu_{nj}\}_{n\geq 0}$ uniquely determines the function $\mathcal{P}_j(\lambda)$ by the formula

$$\mathcal{P}_j(\lambda) = \pi(\mu_{0j} - \lambda) \prod_{n=1}^{\infty} \frac{\mu_{nj} - \lambda}{n^2}, \qquad j = 1, 2. \tag{15}$$

Since $\mathcal{P}_j(\lambda) = \Delta_1(\lambda) + H_j \Delta_0(\lambda)$, $j = 1, 2$, it follows that

$$\Delta_0(\lambda) = \frac{\mathcal{P}_1(\lambda) - \mathcal{P}_2(\lambda)}{H_1 - H_2}, \qquad \Delta_1(\lambda) = \frac{\mathcal{P}_1(\lambda)H_2 - \mathcal{P}_2(\lambda)H_1}{H_2 - H_1}. \tag{16}$$

Given the spectra $\{\mu_{nj}\}_{n\geq 0}$, $j = 1, 2$, our aim is to find the potential $q(x)$ and the coefficients $h$, $H_1$, and $H_2$. First, we use Eq. (15) to construct the functions $\mathcal{P}_j(\lambda)$, $j = 1, 2$. Then, using relation (14), we calculate the difference

$$H_1 - H_2 = \pi \lim_{n\to\infty} (\sqrt{\mu_{n1}} - \sqrt{\mu_{n2}})n. \tag{17}$$

Now we can construct the function $\Delta_0(\lambda)$ with the use of the first relation in (16). Then we find the coefficients $h$ and $A$ from formulas (8) and (9), and consequently, we can calculate $H_1$ and $H_2$ based on relation (14). After that, we construct the function $\Delta_1(\lambda)$ by formula (16). Once we know the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$, we can reproduce the argument in Section 2 and find the potential $q(x)$. Thus, we have proved the following theorem.

**Theorem 2.** *The spectra* $\{\mu_{nj}\}_{n\geq 0}$, $j = 1, 2$, *uniquely determine the potential* $q(x)$ *and the coefficients* $h$, $H_1$, *and* $H_2$. *The solution of the inverse problem can be found by the following algorithm.*

**Algorithm 2.** *Given the spectra* $\{\mu_{nj}\}_{n\geq 0}$, $j = 1, 2$,
1. *Construct the functions* $\mathcal{P}_j(\lambda)$, $j = 1, 2$, *by formulas* (15).
2. *Find* $H_1 - H_2$ *by formula* (17).

3. *Calculate the function* $\Delta_0(\lambda)$ *using the first relation in* (16).
4. *Calculate the coefficients* $A$ *and* $h$ *with the use of Eqs.* (8) *and* (9).
5. *Find the coefficients* $H_1$ *and* $H_2$ *using formula* (14).
6. *Construct the function* $\Delta_1(\lambda)$ *by the second relation in* (16).
7. *Find the functions* $d_j(\rho)$, $j = 0, 1$, *with the use of Eqs.* (3) *and* (4).
8. *Construct the functions* $\delta_0(\rho)$ *and* $\delta_1(\rho)$ *according to the representations* (12).
9. *Find the function* $Q(x)$ *using Eqs.* (10) *and* (11).
10. *Calculate the potential* $q(x) = Q(2x - \pi - a)$.

## ACKNOWLEDGMENTS

This work was supported by the Russian Ministry of Education and Science under grant no. 1.1660.2017/4.6 and by the Russian Foundation for Basic Research under grant no. 19-01-00102.

## REFERENCES

1. Hale, J., *Theory of Functional-Differential Equations*, New York: Springer-Verlag, 1977.
2. Myshkis, A.D., *Linear Differential Equations with a Delay Argument*, Moscow: Nauka, 1972.
3. Marchenko, V.A., *Sturm–Liouville Operators and Their Applications*, Kiev: Naukova Dumka, 1977; English transl.: Birkhäuser, 1986.
4. Levitan, B.M., *Inverse Sturm–Liouville Problems*, Moscow: Nauka, 1984; Engl. transl.: Utrecht: VNU Sci. Press, 1987.
5. Freiling, G. and Yurko, V.A., *Inverse Sturm–Liouville Problems and Their Applications*, New York: NOVA Science Publ., 2001.
6. Yurko, V.A., *Method of Spectral Mappings in the Inverse Problem Theory*, Inverse and Ill-Posed Problems Series, Utrecht: VSP, 2002.
7. Freiling, G. and Yurko, V.A., Inverse problems for Sturm–Liouville differential operators with a constant delay, *Appl. Math. Lett.*, 2012, vol. 25, pp. 1999–2004.
8. Yang, C.-F., Trace and inverse problem of a discontinuous Sturm–Liouville operator with retarded argument, *J. Math. Anal. Appl.*, 2012, vol. 395, no. 1, pp. 30–41.
9. Vladičić, V. and Pikula, M., An inverse problem for Sturm–Liouville-type differential equation with a constant delay, *Sarajevo J. Math.*, 2016, vol. 12 (24), no. 1, pp. 83–88.
10. Buterin, S.A. and Yurko, V.A., An inverse spectral problem for Sturm–Liouville operators with a large constant delay, *Anal. Math. Phys.*, 2017, doi: 10.1007/s13324-017-0176-6.
11. Bondarenko, N.P. and Yurko, V.A., An inverse problem for Sturm–Liouville differential operators with deviating argument, *Appl. Math. Lett.*, 2018, vol. 83, pp. 140–144.