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NUMERICAL METHODS

Monotone Finite-Difference Schemes of Second-Order Accuracy for Quasilinear Parabolic Equations with Mixed Derivatives

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Abstract—We consider the initial–boundary value problem for quasilinear parabolic equation with mixed derivatives and an unbounded nonlinearity. We construct unconditionally monotone and conservative finite-difference schemes of the second-order accuracy for arbitrary sign alternating coefficients of the equation. For the finite-difference solution, we obtain a two-sided estimate completely consistent with similar estimates for the solution of the differential problem, and also obtain an important a priori estimate in the uniform C-norm. These estimates are used to prove the convergence of finite-difference schemes in the grid L_2-norm. All theoretical results are obtained under the assumption that some conditions imposed only on the input data of the differential problem are satisfied.

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INTRODUCTION

The maximum principle is successfully used to prove the existence and uniqueness of the solutions of initial–boundary value problems for parabolic and elliptic equations. In contrast to the method of energy inequalities, it permits establishing a priori estimates of the solution in the strongest uniform norm for problems of arbitrary dimension with a nonself-adjoint elliptic operator [1, p. 500].

A similar mathematical apparatus is also used in the theory of finite-difference schemes [2, p. 226; 3, p. 140; 4, p. 296]. In particular, it is used to study the stability and convergence of the finite-difference solution in the uniform norm. Computational methods satisfying the grid maximum principle are usually said to be monotone [2, p. 228; 4, p. 296]. Monotone finite-difference schemes play an important role, because there do not arise nonphysical oscillations in the computer simulation of applied problems described by partial differential equations [5].

The lower estimates of solutions of differential-difference problems or, in general, their two-sided estimates are no less important. This is especially important when studying the theoretical properties of computational methods approximating the problems with unbounded nonlinearity when it is necessary to prove that the grid solution lies in a neighborhood of the values of the exact solution [6–8]. As an example, one can consider the gamma equation [9] obtained by transforming the nonlinear Black–Scholes equation for option pricing into the quasilinear parabolic equation for the second derivative of the option price in financial mathematics.

In the case of linear problems, two-sided estimates of the desired solution are used to determine its range in terms of input data of the problem (the coefficients of the equation, the right-hand side, and the initial and boundary conditions). In the nonlinear case, such estimates allow one to prove the nonnegativity of the exact solution, which is important in physical problems, and determine the conditions on the output data under which the problem is parabolic or elliptic. Of course, it is desirable in this case that the estimates be as sharp as possible. To this end, starting from the classical monograph [10, p. 22], there is a special technique related to the change of variables and minimization or maximization of some functions with respect to a parameter.

Of course, such estimates are also required for computational algorithms approximating the problem data. We also note that, to state the grid maximum principle, it is usually required that the input data of the problem be of constant sign. But in [7, 8], to write a general finite-difference scheme in canonical form, a two-sided estimate of the grid solution was proved in terms of input data of the problem without the assumption that they are of constant sign. We also note the paper [11], where the technique proposed in O.A. Ladyzhenskaya's pioneering paper [12] was developed and used to obtain two-sided estimates of the solution of grid schemes completely consistent with the differential problem. Based on the application of such estimates, the convergence of a linearized finite-difference scheme approximating the quasilinear heat equation with unboundedly increasing nonlinearities and generalized solution satisfying the balance equation in the nonstationary case was studied in [6].

When developing higher-order finite-difference schemes, it is important to preserve both the monotonicity property and the conservativeness, because the systems of linear algebraic equations obtained by applying such methods are well conditioned [13, p. 64]. The iteration methods converge much better in the case of diagonally dominating matrices. The problems of developing finite-difference schemes for equations with mixed derivatives were studied in [14–16]. Note that equations with mixed derivatives arise in the construction of computational methods already for the classical equations (Laplace, Poisson, and others) on arbitrary nonorthogonal grids. For elliptic and parabolic equations with mixed derivatives, monotone and conservative finite-difference schemes were proposed in [17–19], but these schemes can be used only in the case of sign-constant coefficients. If the coefficient of mixed derivatives changes sign, then the differential equations can be written in nondivergence form with the first derivatives, and monotone schemes are constructed for it by using the regularization principle [2–4]. But the conservativeness property is lost after such a transformation. For elliptic equations with mixed derivatives, new monotone and conservative finite-difference schemes were developed in [20] for both sign-constant and sign-alternating coefficients. The main idea of these schemes is based on the use of standard functionals with absolute values of the coefficients of mixed derivatives. For a nonlinear problem with mixed derivatives in nondivergence form, a finite-difference scheme and an iteration process implementing it were constructed and investigated in [21]. The convergence of this iteration process was rigorously studied, and this process was used to prove the existence and uniqueness of the solution of a nonlinear finite-difference scheme approximating the original differential problem. The estimates consistent with the smoothness of the desired solution were obtained for the rate of convergence of finite-difference schemes in the grid $W^2_{2,0}(\omega)$-norm.

In the present paper, we consider the initial–boundary value problem for a quasilinear parabolic equation with mixed derivatives and unbounded nonlinearity. Based on the combination of two known finite-difference schemes of the second order of approximation [18, 19], we construct and investigate unconditionally monotone and conservative finite-difference schemes of the second-order accuracy for arbitrary sign-alternating coefficients of the mixed derivatives. The results of [7, 8] are used to obtain a two-sided estimate for the finite-difference solution, which is completely consistent with a similar estimate for the solution of the differential problem, and an a priori estimate is proved in the uniform C-norm. These estimates are used to prove the convergence of finite-difference schemes in the grid L_2-norm. All theoretical results are proved under the only assumption that some conditions are satisfied by the input data of the differential problem.

1. AUXILIARY RESULTS

Assume that Ω_h is a finite set of nodes (grid) in some bounded domain of the n-dimensional Euclidean space, and $x \in \Omega_h$ is a point of the grid Ω_h. Consider the equation

$$A(x)y(x) = \sum_{\xi\in\mathcal{M}'(x)} B(x,\xi)y(\xi) + F(x), \qquad x \in \Omega_h, \tag{1}$$

which is called the canonical form of the finite-difference scheme [2, p. 226]. Here $\mathcal{M}'(x) = \mathcal{M}(x)\backslash x$, and $\mathcal{M}(x)$ is the grid stencil. Since any finite-difference scheme can be written as (1), monotonicity is understood as the following conditions saying that the coefficients of Eq. (1) are positive:

$$A(x) > 0, \quad B(x,\xi) > 0 \quad \text{for all} \quad \xi \in \mathcal{M}'(x), \tag{2}$$

$$D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x,\xi) > 0 \quad \text{for all} \quad \xi \in \mathcal{M}'(x). \tag{3}$$

To obtain a two-sided estimate of the solution of a finite-difference scheme, it is most convenient to use the following lemma.

Lemma [7, 8]. *Assume that conditions* (2), (3) *that the coefficients are positive are satisfied. Then the maximum and minimum values of the solution of the finite-difference scheme* (1) *belong to the range of the input data*,

$$\min_{x\in\Omega_h} \frac{F(x)}{D(x)} \le y(x) \le \max_{x\in\Omega_h} \frac{F(x)}{D(x)}, \qquad x \in \Omega_h. \tag{4}$$

Corollary [2, p. 231]. *Assume that conditions of the lemma are satisfied. Then in the grid analog of the C-norm, the solution of finite-difference problem* (1) *satisfies the estimate*

$$\|y\|_C = \max_{x\in\Omega_h} |y(x)| \le \|F/D\|_C. \tag{5}$$

2. STATEMENT OF THE PROBLEM AND TWO-SIDED ESTIMATE OF THE EXACT SOLUTION

Let $\bar{G} = \{x = (x_1, x_2) : 0 \le x_\alpha \le l_\alpha,\ \alpha = 1,2\}$ be a rectangle whose boundary is denoted by Γ. It is required to obtain a continuous function $u(x,t)$ in $\bar{Q}_T = \bar{G} \times [0,T]$ satisfying the initial–boundary value problem for the quasilinear parabolic equation with mixed derivatives

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad x \in G, \quad t \in (0,T], \quad u(x,0) = u_0(x), \quad u|_\Gamma = \mu(x,t), \tag{6}$$

$$Lu = \sum_{\alpha,\beta=1}^{2} L_{\alpha\beta} u, \qquad L_{\alpha\beta} u = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(u)\frac{\partial u}{\partial x_\beta}\right). \tag{7}$$

It is assumed that the following ellipticity conditions are satisfied:

$$c_1 \sum_{\alpha=1}^{2} \xi_\alpha^2 \le \sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}(u)\xi_\alpha\xi_\beta \le c_2 \sum_{\alpha=1}^{2} \xi_\alpha^2 \quad \text{for any} \quad u \in \bar{D}_u, \tag{8}$$

where $\bar{D}_u = \{u(x,t) : m_1 \le u(x,t) \le m_2,\ (x,t) \in \bar{Q}_T,\ m_1, m_2 \text{ are constants}\}$, $c_1 > 0$, $c_2 > 0$ are constants, and $\xi = (\xi_1, \xi_2)$ is an arbitrary vector. In particular, it follows from inequalities (8) that,

$$0 < c_1 \le k_{\alpha\alpha}(u) \le c_2 \quad \text{for any} \quad u \in [m_1, m_2], \quad \alpha = 1,2.$$

We assume in what follows that there exists a unique solution of problem (6)–(8) and all coefficients in Eq. (6) and the desired function have continuous bounded derivatives of order that is required as the presentation proceeds.

Let $Q_{t_1} = \{(x,t) \in Q_T : t \le t_1\}$. Then the following assertion holds.

Theorem 1 [10, p. 22]. *The following two-sided estimates hold for the solution $u(x,t)$ of problem* (6)–(8) *at any point $(x,t_1) \in \bar{Q}_T$:*

$$u(x,t_1) \ge m_1 = \sup_{\lambda>0} \min\left\{0, \min_{Q_{t_1}}\{\mu(x,t), u_0(x)\}e^{\lambda(t_1-t)}, \lambda^{-1}\min_{Q_{t_1}}(f(x,t)e^{\lambda(t_1-t)})\right\}, \tag{9}$$

$$u(x,t_1) \le m_2 = \inf_{\lambda>0} \max\left\{0, \max_{Q_{t_1}}\{\mu(x,t), u_0(x)\}e^{\lambda(t_1-t)}, \lambda^{-1}\max_{Q_{t_1}}(f(x,t)e^{\lambda(t_1-t)})\right\}. \tag{10}$$

Remark 1. In the finite-element method, the estimates of solutions in terms of the functions depending on the minimization or maximization with respect to auxiliary functions of some functionals containing the input data were widely developed by Repin (e.g., see [22]).

3. FINITE-DIFFERENCE SCHEME

On the interval $[0,T]$, we introduce the grid $\bar{\omega}_\tau = \{t_n = n\tau : n = 0,\dots,N_0,\ \tau N_0 = T\} = \omega_\tau \bigcup T$ uniform in time with step τ, and in the rectangle $\bar{G}$ we introduce the grid $\bar{\omega}_h = \omega_h \bigcup \gamma_h$, where γ_h is the set of boundary nodes uniform in each direction x_α,

$$\bar{\omega}_h = \{x_i = (x_1^{(i_1)}, x_2^{(i_2)}) : x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha,\ i_\alpha = 0,\dots,N_\alpha,\ h_\alpha N_\alpha = l_\alpha,\ \alpha = 1,2\}.$$

For simplicity, we use the notation without indices for the independent variables $x = x_i$, $x_\alpha = x_\alpha^{i_\alpha}$, $t = t_n$, $\hat{t} = t_{n+1}$ and for the grid functions:

$$g = g(x_1^{i_1}, x_2^{i_2}, t_n) = g(x,t), \qquad g^{\pm 1_1} = g_{i_1 \pm 1, i_2}, \qquad g^{\pm 1_2} = g_{i_1, i_2 \pm 1},$$

$$\hat{g} = g^{n+1} = g(x, t_{n+1}), \qquad g_{\bar{x}_\alpha} = \frac{g - g^{(-1_\alpha)}}{h_\alpha}, \qquad g_{x_\alpha} = \frac{g^{(+1_\alpha)} - g}{h_\alpha}.$$

On the uniform grid $\omega = \omega_h \times \omega_\tau$, we approximate the differential problem (6) by the purely implicit finite-difference scheme

$$y_t = \sum_{\alpha=1}^{2} \Lambda_{\alpha\alpha}\hat{y} + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2} \Lambda_{\alpha\beta}\hat{y} + \varphi, \tag{11}$$

$$y(x,0) = u_0(x), \qquad x \in \omega_h, \qquad \hat{y}|_{\gamma_h} = \mu(x,t), \qquad x \in \gamma_h, \tag{12}$$

where

$$\Lambda_{\alpha\alpha}\hat{y} = (a_{\alpha\alpha}(y)\hat{y}_{\bar{x}_\alpha})_{x_\alpha} = \frac{a_{\alpha\alpha}^{(+1_\alpha)}(y)(\hat{y}^{(+1_\alpha)} - y) - a_{\alpha\alpha}(y)(y - \hat{y}^{(-1_\alpha)})}{h_\alpha^2},$$

$$\Lambda_{\alpha\beta}\hat{y} = 0.5[(k_{\alpha\beta}^-(y)\hat{y}_{\bar{x}_\beta})_{x_\alpha} + (k_{\alpha\beta}^-(y)\hat{y}_{x_\beta})_{\bar{x}_\alpha} + (k_{\alpha\beta}^+(y)\hat{y}_{x_\beta})_{x_\alpha} + (k_{\alpha\beta}^+(y)\hat{y}_{\bar{x}_\beta})_{\bar{x}_\alpha}], \qquad \alpha \neq \beta,$$

$$a_{\alpha\alpha}^{(+1_\alpha)}(y) = \frac{k_{\alpha\alpha}(y^{(+1_\alpha)}) + k_{\alpha\alpha}(y)}{2}, \qquad a_{\alpha\alpha}(y) = \frac{k_{\alpha\alpha}(y^{(-1_\alpha)}) + k_{\alpha\alpha}(y)}{2},$$

$$k_{\alpha\beta}^+ = 0.5(k_{\alpha\beta} + |k_{\alpha\beta}|) \geq 0, \qquad k_{\alpha\beta}^- = 0.5(k_{\alpha\beta} - |k_{\alpha\beta}|) \leq 0, \qquad \alpha \neq \beta,$$

$$k_{\alpha\beta}^+ + k_{\alpha\beta}^- = k_{\alpha\beta}, \qquad k_{\alpha\beta}^+ - k_{\alpha\beta}^- = |k_{\alpha\beta}|, \qquad \varphi = \hat{f}, \qquad y_t = (y^{n+1} - y^n)/\tau.$$

Approximation error. The approximation error of the scheme (11), (12) is calculated by the formula

$$\psi = -u_t + \sum_{\alpha=1}^{2} \Lambda_{\alpha\alpha}\hat{u} + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2} \Lambda_{\alpha\beta}\hat{u} + \varphi. \tag{13}$$

With regard to the relations

$$u_t = \frac{\partial \hat{u}}{\partial t} + O(\tau), \qquad a_{\alpha\alpha}^{(+1_\alpha)}(u)\hat{u}_{x_\alpha} = k_{\alpha\alpha}(\hat{u})\frac{\partial \hat{u}}{\partial x_\alpha} + \frac{h_\alpha}{2}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\alpha}(\hat{u})\frac{\partial \hat{u}}{\partial x_\alpha}\right) + O(h_\alpha^2 + \tau),$$

$$a_{\alpha\alpha}(u)\hat{u}_{\bar{x}_\alpha} = k_{\alpha\alpha}(\hat{u})\frac{\partial \hat{u}}{\partial x_\alpha} - \frac{h_\alpha}{2}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\alpha}(\hat{u})\frac{\partial \hat{u}}{\partial x_\alpha}\right) + O(h_\alpha^2 + \tau),$$

we have

$$\Lambda_{\alpha\alpha}\hat{u} = (a_{\alpha\alpha}(u)\hat{u}_{\bar{x}_\alpha})_{x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\alpha}(\hat{u})\frac{\partial \hat{u}}{\partial x_\alpha}\right) + O(h_\alpha^2 + \tau). \tag{14}$$

Now let us show that the operator $\Lambda_{\alpha\beta}\hat{u}$, $\alpha \neq \beta$ has the second order of approximation. Consider the term $(k^-_{\alpha\beta}(u)\hat{u}_{\bar{x}_\beta})_{x_\alpha}$. Substituting the expansions

$$u_{\bar{x}_\beta} = \frac{\partial u}{\partial x_\beta} - \frac{h_\beta}{2}\frac{\partial^2 u}{\partial x_\beta^2} + \frac{h_\beta^2}{6}\frac{\partial^3 u}{\partial x_\beta^3}(x_\alpha, \bar{x}_\beta), \quad \bar{x}_\beta \in [x_\beta - h_\beta, x_\beta], \quad v_{x_\alpha} = \frac{\partial v}{\partial x_\alpha} + \frac{h_\alpha}{2}\frac{\partial^2 v}{\partial x_\alpha^2} + O(h_\alpha^2)$$

into it for $v = k^-_{\alpha\beta}(u)\hat{u}_{\bar{x}_\beta}$, we obtain

$$(k^-_{\alpha\beta}(u)\hat{u}_{\bar{x}_\beta})_{x_\alpha} = L^-_{\alpha\beta}\hat{u} + \frac{h_\alpha}{2}\frac{\partial}{\partial x_\alpha}L^-_{\alpha\beta}\hat{u} - \frac{h_\beta}{2}L^-_{\alpha\beta}\frac{\partial \hat{u}}{\partial x_\beta} + O(h_1^2 + h_2^2 + \tau).$$

In a similar way, we obtain

$$(k^-_{\alpha\beta}(u)\hat{u}_{x_\beta})_{\bar{x}_\alpha} = L^-_{\alpha\beta}\hat{u} - \frac{h_\alpha}{2}\frac{\partial}{\partial x_\alpha}L^-_{\alpha\beta}\hat{u} + \frac{h_\beta}{2}L^-_{\alpha\beta}\frac{\partial \hat{u}}{\partial x_\beta} + O(h_1^2 + h_2^2 + \tau),$$
$$(k^+_{\alpha\beta}(u)\hat{u}_{x_\beta})_{x_\alpha} = L^+_{\alpha\beta}\hat{u} + \frac{h_\alpha}{2}\frac{\partial}{\partial x_\alpha}L^+_{\alpha\beta}\hat{u} + \frac{h_\beta}{2}L^+_{\alpha\beta}\frac{\partial \hat{u}}{\partial x_\beta} + O(h_1^2 + h_2^2 + \tau),$$
$$(k^+_{\alpha\beta}(u)\hat{u}_{\bar{x}_\beta})_{\bar{x}_\alpha} = L^+_{\alpha\beta}\hat{u} - \frac{h_\alpha}{2}\frac{\partial}{\partial x_\alpha}L^+_{\alpha\beta}\hat{u} - \frac{h_\beta}{2}L^+_{\alpha\beta}\frac{\partial \hat{u}}{\partial x_\beta} + O(h_1^2 + h_2^2 + \tau).$$

It follows that

$$\Lambda_{\alpha\beta}\hat{u} = L^+_{\alpha\beta}\hat{u} + L^-_{\alpha\beta}\hat{u} + O(h_1^2 + h_2^2 + \tau) = L_{\alpha\beta}\hat{u} + O(|h|^2 + \tau), \qquad \alpha \neq \beta. \tag{15}$$

Then we have $\psi = O(|h|^2 + \tau)$ from (13)–(15). Thus, we have proved the following theorem.

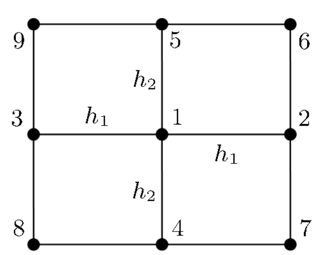
Theorem 2. *The finite-difference scheme* (11), (12) *has the second order of approximation with respect to the spatial variables and the first order with respect to the time variable.*

4. MONOTONICITY, TWO-SIDED ESTIMATES, AND A PRIORI ESTIMATES

To apply the maximum principle, we reduce the scheme (11) to the canonical form (1) and verify the sufficient conditions (2), (3) on the coefficients.

In the case of sign alternating coefficients $k_{\alpha\beta}(u)$, the grid has a 9-point stencil and consists of the nodes shown in the figure. The stencil nodes are numbered as shown in the figure. Then for the scheme (11) we have

$$\sum_{\xi\in\mathcal{M}'(x)} B^n(x,\xi) = \sum_{j=2}^{9} B^n_j.$$



9-point stencil.

To write the coefficients A^n, B^n, F^n, it is necessary to write the scheme (11) in index form. After elementary transformations, we obtain

$$B_2^n = \tau\left(\frac{k_{11}(y_1^n)+k_{11}(y_2^n)}{2h_1^2} - \frac{|k_{12}(y_2^n)|+|k_{21}(y_1^n)|}{2h_1h_2}\right),$$
$$B_3^n = \tau\left(\frac{k_{11}(y_1^n)+k_{11}(y_3^n)}{2h_1^2} - \frac{|k_{12}(y_3^n)|+|k_{21}(y_1^n)|}{2h_1h_2}\right),$$
$$B_4^n = \tau\left(\frac{k_{22}(y_1^n)+k_{22}(y_4^n)}{2h_2^2} - \frac{|k_{12}(y_1^n)|+|k_{21}(y_4^n)|}{2h_1h_2}\right),$$
$$B_5^n = \tau\left(\frac{k_{22}(y_1^n)+k_{22}(y_5^n)}{2h_2^2} - \frac{|k_{12}(y_1^n)|+|k_{21}(y_5^n)|}{2h_1h_2}\right), \quad B_6^n = \tau\frac{k_{12}^+(y_2^n)+k_{21}^+(y_5^n)}{2h_1h_2} \ge 0,$$
$$B_7^n = -\tau\frac{k_{12}^-(y_2^n)+k_{21}^-(y_4^n)}{2h_1h_2} \ge 0, \quad B_8^n = \tau\frac{k_{12}^+(y_3^n)+k_{21}^+(y_4^n)}{2h_1h_2} \ge 0, \quad B_9^n = -\tau\frac{k_{12}^-(y_3^n)+k_{21}^-(y_5^n)}{2h_1h_2} \ge 0,$$
$$A^n = 1+\tau\left(k^n + \frac{k_{11}(y_1^n)}{h_1^2} - \frac{|k_{12}(y_1^n)|+|k_{21}(y_1^n)|}{h_1h_2} + \frac{k_{22}(y_1^n)}{h_2^2}\right) = 1+\sum_{j=2}^{9} B_j^n,$$
$$k^n = \frac{k_{11}(y_2^n)+k_{11}(y_3^n)}{2h_1^2} + \frac{k_{22}(y_4^n)+k_{22}(y_5^n)}{2h_2^2}, \quad D^n = A^n - \sum_{j=2}^{9} B_j^n = 1, \quad F^n = y_1^n + \tau\varphi.$$

Assume that the following conditions are satisfied for the grid steps h_1 and h_2 which are expressed in terms of input data of the problem:

$$\frac{c_3+c_4}{2c_1} \le \frac{h_1}{h_2} \le \frac{2c_1}{c_3+c_4}, \qquad c_3 = \max_{u\in\bar{D}_u} |k_{21}(u)|, \qquad c_4 = \max_{u\in\bar{D}_u} |k_{12}(u)|. \tag{16}$$

Let us prove that $y_{i_1i_2}^n \in [m_1, m_2]$ for all $i_\alpha = 1,\ldots,N-1$, $\alpha = 1,2$, $n = 0,\ldots,N_0$. We take an auxiliary grid function $z(x,t) = z_{i_1i_2}^n = y_{i_1i_2}^n e^{-\lambda t_n}$, $\lambda \ne 0$. The function $z(x,t)$ satisfies the finite-difference equation

$$\frac{\hat{z}e^{\lambda\tau} - z}{\tau} = e^{\lambda\tau}\sum_{\alpha=1}^{2}\Lambda_{\alpha\alpha}\hat{z} + e^{\lambda\tau}\sum_{\substack{\alpha,\beta=1\\ \alpha\ne\beta}}^{2}\Lambda_{\alpha\beta}\hat{z} + e^{-\lambda t_n}\hat{f}.$$

We write this equation in the canonical form (1),

$$A_{(*)}^n z_1^{n+1} = \sum_{j=2}^{9} B_{(*)j}^n z_j^{n+1} + K_1^n z_1^n + F_{(*)}^n, \tag{17}$$

where

$$B_{(*)j}^n = e^{\lambda\tau}B_j^n, \quad j = 2,\ldots,9, \quad K_1^n = 1, \quad A_{(*)}^n = e^{\lambda\tau} + \sum_{j=2}^{9} B_{(*)j}^n, \quad F_{(*)}^n = \tau f^{n+1}e^{-\lambda t_n}.$$

We introduce the coefficients $D_{(*)}^n$ as follows:

$$D_{(*)}^n = A_{(*)}^n - \sum_{j=2}^{9} B_{(*)j}^n - K_1^n = e^{\lambda\tau} - 1 > 0 \quad \text{for all} \quad \lambda\tau > 0.$$

Take an arbitrary $t_n \in \omega_\tau$. The following three cases are possible for the function $z(x,t)$.

1. $\max_{\omega_{t_n}} z(x,t)$ is nonpositive (i.e., $z(x,t) \le 0$, $(x,t) \in \omega_{t_n}$).

2. $\max_{\omega_{t_n}} z(x,t)$ is located on the base $t = 0$ or on the boundary (i.e., the inequality $z(x,t) \le \max_{\omega_{t_n}} e^{-\lambda t}\{\mu(x,t), u_0(x)\}$, $(x,t) \in \omega_{t_n}$, holds).

3. A positive maximum is attained at some interior point (x^0, t^0), $z(x,t) \le z(x^0,t^0) = \max\limits_{\omega_{t_n}} z(x,t)$.

Obviously, for $n = 0$ we have $y^0_{i_1 i_2} = u^0_{0 i_1 i_2} \in [m_1, m_2]$ for all $i_\alpha = 1, \dots, N-1$, $\alpha = 1, 2$. Assume that, for an arbitrary n, the inclusion $y^n_{i_1 i_2} \in [m_1, m_2]$ is also true. Then we have $B^n_j \ge 0$, $j = 6, \dots, 9$, $k^n > 0$, and

$$A^n_+ = 1 + \tau\left(k^n + \frac{k_{11}(y^n_1)}{h_1^2} - \frac{k_{12}(y^n_1) + k_{21}(y^n_1)}{h_1 h_2} + \frac{k_{22}(y^n_1)}{h_2^2}\right), \qquad k_{\alpha\beta} \ge 0, \qquad \alpha \ne \beta,$$
$$A^n_- = 1 + \tau\left(k^n + \frac{k_{11}(y^n_1)}{h_1^2} + \frac{k_{12}(y^n_1) + k_{21}(y^n_1)}{h_1 h_2} + \frac{k_{22}(y^n_1)}{h_2^2}\right), \qquad k_{\alpha\beta} \le 0, \qquad \alpha \ne \beta.$$

Using the ellipticity condition (8) and assuming that $\xi^- = (1/h_1, 1/h_2)$ and $\xi^+ = (-1/h_1, 1/h_2)$, we see that the coefficients $A^n_\pm$ are positive. The other coefficients B^n_j, $j = 2, \dots, 5$, are positive if the following conditions are satisfied:

$$\max_{m=4,5} \frac{|k_{12}(y^n_1)| + |k_{21}(y^n_m)|}{k_{22}(y^n_1) + k_{22}(y^n_m)} \le \frac{h_1}{h_2} \le \min_{m=2,3} \frac{k_{11}(y^n_1) + k_{11}(y^n_m)}{|k_{21}(y^n_1)| + |k_{12}(y^n_m)|}. \tag{18}$$

Indeed, since

$$\min_{m=2,3} \frac{k_{11}(y^n_1) + k_{11}(y^n_m)}{|k_{21}(y^n_1)| + |k_{12}(y^n_m)|} \ge \frac{2\min\limits_{u\in\bar D_u} k_{11}(u)}{\max\limits_{u\in\bar D_u} |k_{21}(u)| + \max\limits_{u\in\bar D_u} |k_{12}(u)|} = \frac{2c_1}{c_3 + c_4},$$
$$\max_{m=4,5} \frac{|k_{12}(y^n_1)| + |k_{21}(y^n_m)|}{k_{22}(y^n_1) + k_{22}(y^n_m)} \le \frac{\max\limits_{u\in\bar D_u} |k_{21}(u)| + \max\limits_{u\in\bar D_u} |k_{12}(u)|}{2\min\limits_{u\in\bar D_u} k_{11}(u)} = \frac{c_3 + c_4}{2c_1},$$

we see that the system of inequalities (18) is satisfied; i.e., $B^n_j \ge 0$, $j = 2, \dots, 5$. Then at the point of maximum (x^0, t^0), by the estimate (4) and Eq. (17), we obtain

$$z(x,t) \le z(x^0, t^0) \le \frac{\tau}{e^{\lambda\tau} - 1} f(x^0, t^0) e^{-\lambda t^0} \le \frac{\tau}{e^{\lambda\tau} - 1} \max_{\omega_{t_n}} f(x,t) e^{-\lambda t}, \qquad \lambda > 0.$$

Then, in all cases 1–3, the function $z(x,t)$ satisfies the estimate

$$z(x,t) \le \max\left\{0, \max_{\omega_{t_{n+1}}} e^{-\lambda t}\{\mu(x,t), u_0(x)\}, \frac{\tau}{e^{\lambda\tau} - 1} \max_{\omega_{t_{n+1}}} f(x,t) e^{-\lambda t}\right\},$$

which implies that

$$y(x, t_{n+1}) \le m_2^{n+1} = \inf_{\lambda > 0} \max\left\{0, \max_{\omega_{t_{n+1}}} e^{\lambda(t_{n+1} - t)}\{\mu(x,t), u_0(x)\}, \frac{\tau}{e^{\lambda\tau} - 1} \max_{\omega_{t_{n+1}}} f(x,t) e^{\lambda(t_{n+1} - t)}\right\}. \tag{19}$$

In a similar way, we obtain the lower bound

$$y(x, t_{n+1}) \ge m_1^{n+1} = \sup_{\lambda > 0} \min\left\{0, \min_{\omega_{t_{n+1}}} e^{\lambda(t_{n+1} - t)}\{\mu(x,t), u_0(x)\}, \frac{\tau}{e^{\lambda\tau} - 1} \min_{\omega_{t_{n+1}}} f(x,t) e^{\lambda(t_{n+1} - t)}\right\}. \tag{20}$$

Since

$$\frac{\tau}{e^{\lambda\tau} - 1} \le \frac{1}{\lambda} \quad \text{for all} \quad \lambda, \tau > 0,$$

we see that the estimates (9), (10) and (19), (20) imply the inequalities

$$m_1 \le m_1^{n+1}, \qquad m_2^{n+1} \le m_2;$$

i.e., $y^{n+1}_{i_1 i_2} \in [m_1, m_2]$, $i_\alpha = 1, \dots, N-1$, $\alpha = 1, 2$. In this sense, the finite-difference estimates inherit the properties of the differential problem.

Thus, we have proved the following theorem.

Theorem 3. *Assume that conditions* (16) *are satisfied. Then the finite-difference scheme* (11), (12) *is unconditionally monotone* (*without constraints on the steps* τ *and* h_α, $\alpha = 1, 2$), *and its solution satisfies the two-sided estimate* (19), (20) *at any point* $(x, t_{n+1}) \in \omega$.

Remark 2. The upper and lower bounds of the two-sided estimate (19), (20) are independent of the diffusion coefficients $k_{\alpha\beta}(u)$, $\alpha, \beta = 1, 2$.

Remark 3. If the coefficient matrix of Eq. (6) has diagonal predomination with respect to rows and columns: $k_{\alpha\alpha}(u) \geq |k_{\alpha\beta}(u)|$ for any $u \in [m_1, m_2]$, $\alpha, \beta = 1, 2$, $\alpha \neq \beta$, then one can set $h_1 = h_2 = h$, and then condition (16) is always satisfied.

Based on the maximum principle, we in a standard way obtain the following important a priori estimate in the strong C-norm.

Theorem 4. *Assume that conditions* (16) *are satisfied. Then the solution of the finite-difference scheme* (11), (12) *satisfies the following a priori estimate for any* $t_n \in \omega_\tau$:

$$\|y(t_{n+1})\|_{\bar{C}} \leq \max\left\{\|u_0\|_{\bar{C}}, \max_{1\leq k\leq n+1}\|\mu(t_k)\|_{C_\gamma}\right\} + t_{n+1}\max_{1\leq k\leq n+1}\|f(t_k)\|_C. \tag{21}$$

Proof. Since the assumptions of the corollary are satisfied, we have

$$\|y(t_{n+1})\|_{\bar{C}} \leq \max\{\|\mu(t_{n+1})\|_{C_\gamma}, \|F^n\|_C\}$$

based on the a priori estimate (5). Note that

$$\|F^n\|_C = \|y(t_n) + \tau f(t_{n+1})\|_C \leq \|y(t_n)\|_{\bar{C}} + \tau\|f(t_{n+1})\|_C.$$

Substituting this estimate into the preceding inequality, we obtain the chain of relations

$$\begin{aligned}
&\|y(t_{n+1})\|_{\bar{C}} \leq \max\{\|\mu(t_{n+1})\|_{C_\gamma}, \|y(t_n)\|_{\bar{C}} + \tau\|f(t_{n+1})\|_C\} \\
&\quad\leq \max\{\|\mu(t_{n+1})\|_{C_\gamma}, \|\mu(t_n)\|_{C_\gamma} + \tau\|f(t_{n+1})\|_C, \|y(t_{n-1})\|_{\bar{C}} + \tau(\|f(t_n)\|_C + \|f(t_{n+1})\|_C)\} \leq \cdots \\
&\quad\leq \max\left\{\max_{1\leq k\leq n+1}\|\mu(t_k)\|_{C_\gamma} + \sum_{k=0}^{n}\tau\|f(t_{k+1})\|_C, \|u_0\|_{\bar{C}} + \sum_{k=0}^{n}\tau\|f(t_{k+1})\|_C\right\}.
\end{aligned}$$

Taking into account the inequality

$$\sum_{k=0}^{n}\tau\|f(t_{k+1})\|_C \leq \sum_{k=0}^{n}\tau\max_{1\leq k\leq n+1}\|f(t_k)\|_C = t_{n+1}\max_{1\leq k\leq n+1}\|f(t_k)\|_C,$$

we obtain the desired estimate (21). The proof of the theorem is complete.

Remark 4. The results obtained above can naturally be generalized to p-dimensional parabolic equations with mixed derivatives where $p \geq 2$ is an arbitrary integer.

5. CONVERGENCE OF THE FINITE-DIFFERENCE SCHEME IN THE GRID L_2-NORM

If one succeeds in obtaining a two-sided estimate of the solution of the finite-difference scheme, then the study of the convergence reduces to linear computational algorithms.

For simplicity, consider the case in which $k_{\alpha\beta}(u) \leq 0$, $u \in \bar{D}_u$, and $\alpha \neq \beta$. Then the finite-difference scheme (11) becomes

$$y_t = \sum_{\alpha,\beta=1}^{2}\Lambda_{\alpha\beta}\hat{y} + \varphi, \quad y(x,0) = u_0(x), \quad x \in \omega_h, \quad \hat{y}|_{\gamma_h} = \mu(x,t), \quad x \in \gamma_h, \tag{22}$$

where

$$\Lambda_{\alpha\beta}\hat{y} = 0.5[(k_{\alpha\beta}(y)\hat{y}_{\bar{x}_\beta})_{x_\alpha} + (k_{\alpha\beta}(y)\hat{y}_{x_\beta})_{\bar{x}_\alpha}].$$

Here we note that

$$\Lambda_{\alpha\alpha}\hat{y} = 0.5[(k_{\alpha\alpha}(y)\hat{y}_{\bar{x}_\alpha})_{x_\alpha} + (k_{\alpha\alpha}(y)\hat{y}_{x_\alpha})_{\bar{x}_\alpha}] = (a_{\alpha\alpha}(y)\hat{y}_{\bar{x}_\alpha})_{x_\alpha},$$

which implies an equation of the following form for the discrepancy:

$$u_t = \sum_{\alpha,\beta=1}^{2} \Lambda_{\alpha\beta}\hat{u} + \varphi - \psi. \tag{23}$$

Subtracting the corresponding equation (22) from Eq. (23), we obtain the following problem for the method error $z = y - u$:

$$z_t = \sum_{\alpha,\beta=1}^{2} (\Lambda_{\alpha\beta}\hat{y} - \Lambda_{\alpha\beta}\hat{u}) + \psi, \qquad z(x,0) = 0, \qquad x \in \omega_h, \qquad \hat{z}|_{\gamma_h} = 0, \qquad x \in \gamma_h. \tag{24}$$

We define the following inner products and the corresponding norms:

$$(u,v) = \sum_{i_1=1}^{N_1-1}\sum_{i_2=1}^{N_2-1} h_1 h_2 u_{i_1 i_2} v_{i_1 i_2}, \qquad \|u\| = \sqrt{(u,u)},$$

$$(u,v]_\alpha = \sum_{i_\alpha=1}^{N_\alpha}\sum_{i_{3-\alpha}=1}^{N_{3-\alpha}-1} h_1 h_2 u_{i_1 i_2} v_{i_1 i_2}, \qquad ||u]|_\alpha = \sqrt{(u,u]_\alpha},$$

$$[u,v)_\alpha = \sum_{i_\alpha=0}^{N_\alpha-1}\sum_{i_{3-\alpha}=1}^{N_{3-\alpha}-1} h_1 h_2 u_{i_1 i_2} v_{i_1 i_2}, \qquad |[u||_\alpha = \sqrt{[u,u)_\alpha}, \qquad \alpha = 1,2.$$

The following assertion holds.

Theorem 5. *Assume that conditions* (16) *are satisfied. Then the solution of the finite-difference scheme* (22) *converges to the exact solution of differential problem* (6), *and the following estimate of the method accuracy holds*:

$$\|\hat{z}\| \le C(h_1^2 + h_2^2 + \tau), \qquad C = \text{const} > 0.$$

Proof. Taking the inner product of Eq. (24) by $2\tau\hat{z}$, we obtain

$$2\tau(z_t,\hat{z}) = 2\tau\left(\hat{z}, \sum_{\alpha,\beta=1}^{2} (\Lambda_{\alpha\beta}\hat{y} - \Lambda_{\alpha\beta}\hat{u})\right) + 2\tau(\hat{z},\psi). \tag{25}$$

Taking into account the fact that

$$\begin{aligned}\Lambda_{\alpha\beta}\hat{y} - \Lambda_{\alpha\beta}\hat{u} = 0.5[&(k_{\alpha\beta}(y)\hat{z}_{\bar{x}_\beta})_{x_\alpha} + ((k_{\alpha\beta}(y) - k_{\alpha\beta}(u))\hat{u}_{\bar{x}_\beta})_{x_\alpha}\\ &+ (k_{\alpha\beta}(y)\hat{z}_{x_\beta})_{\bar{x}_\alpha} + ((k_{\alpha\beta}(y) - k_{\alpha\beta}(u))\hat{u}_{x_\beta})_{\bar{x}_\alpha}],\end{aligned}$$

and applying the summation by parts formula [2, p. 98] to the first term on the right-hand side in (25), we obtain

$$LHS \equiv 2\tau\left(\hat{z}, \sum_{\alpha,\beta=1}^{2} (\Lambda_{\alpha\beta}\hat{y} - \Lambda_{\alpha\beta}\hat{u})\right) = -\tau(\theta + \eta),$$

where

$$\theta = \sum_{\alpha,\beta=1}^{2} \left(\hat{z}_{\bar{x}_\alpha}, k_{\alpha\beta}(y)\hat{z}_{\bar{x}_\beta}\right]_\alpha + \sum_{\alpha,\beta=1}^{2} \left[\hat{z}_{x_\alpha}, k_{\alpha\beta}(y)\hat{z}_{x_\beta}\right)_\alpha,$$

$$\eta = \sum_{\alpha,\beta=1}^{2} \left(\hat{z}_{\bar{x}_\alpha}, (k_{\alpha\beta}(y) - k_{\alpha\beta}(u))\hat{u}_{\bar{x}_\beta}\right]_\alpha + \sum_{\alpha,\beta=1}^{2} \left[\hat{z}_{x_\alpha}, (k_{\alpha\beta}(y) - k_{\alpha\beta}(u))\hat{u}_{x_\beta}\right)_\alpha.$$

Since $\sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}(y)\xi_\alpha\xi_\beta \geq c_1 \sum_{\alpha=1}^{2} \xi_\alpha^2$ for all $y \in \bar{D}_u$ and any vector $\xi = (\xi_1, \xi_2)$, we have the inequality

$$\sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}(y)\hat{z}_{\bar{x}_\alpha}\hat{z}_{\bar{x}_\beta} \geq c_1 \sum_{\alpha=1}^{2} (\hat{z}_{\bar{x}_\alpha})^2,$$

which implies that

$$\sum_{i_\alpha=1}^{N_\alpha} \sum_{i_{3-\alpha}=1}^{N_{3-\alpha}-1} h_1 h_2 \sum_{\alpha,\beta=1}^{2} k_{\alpha\beta}(y_{i_1 i_2})\hat{z}_{\bar{x}_\alpha, i_1 i_2}\hat{z}_{\bar{x}_\beta, i_1 i_2} \geq c_1 \sum_{i_\alpha=1}^{N_\alpha} \sum_{i_{3-\alpha}=1}^{N_{3-\alpha}-1} h_1 h_2 \sum_{\alpha=1}^{2} (\hat{z}_{\bar{x}_\alpha, i_1 i_2})^2.$$

In the last inequality, we replace the outer summation symbols by the inner product and arrive at the estimate

$$\sum_{\alpha,\beta=1}^{2} \left(\hat{z}_{\bar{x}_\alpha}, k_{\alpha\beta}(y)\hat{z}_{\bar{x}_\beta}\right]_\alpha \geq c_1 \sum_{\alpha=1}^{2} \left(\hat{z}_{\bar{x}_\alpha}, \hat{z}_{\bar{x}_\alpha}\right]_\alpha = c_1 \sum_{\alpha=1}^{2} ||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2.$$

In a similar way, we obtain

$$\sum_{\alpha,\beta=1}^{2} \left[\hat{z}_{x_\alpha}, k_{\alpha\beta}(y)\hat{z}_{x_\beta}\right)_\alpha \geq c_1 \sum_{\alpha=1}^{2} \left[\hat{z}_{x_\alpha}, \hat{z}_{x_\alpha}\right)_\alpha = c_1 \sum_{\alpha=1}^{2} |[\hat{z}_{x_\alpha}||_\alpha^2,$$

which implies that

$$\theta \geq c_1 \sum_{\alpha=1}^{2} (||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2 + |[\hat{z}_{x_\alpha}||_\alpha^2) = 2c_1 \sum_{\alpha=1}^{2} ||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2.$$

Further, for the functions $k_{\alpha\beta}$, $\alpha, \beta = 1, 2$, there exist positive constants $L_{\alpha\beta}$, $\alpha, \beta = 1, 2$, such that the estimates $|k_{\alpha\beta}(y) - k_{\alpha\beta}(u)| \leq L_{\alpha\beta}|z|_{\alpha,(0.5)}$ are satisfied, where

$$|z_{i_1 i_2}|_{1,(0.5)} = \frac{|z_{i_1 i_2}| + |z_{i_1-1, i_2}|}{2}, \qquad |z_{i_1 i_2}|_{2,(0.5)} = \frac{|z_{i_1 i_2}| + |z_{i_1, i_2-1}|}{2}.$$

It follows that

$$\sum_{\alpha,\beta=1}^{2} \left(|\hat{z}_{\bar{x}_\alpha}|, |k_{\alpha\beta}(y) - k_{\alpha\beta}(u)||\hat{u}_{\bar{x}_\beta}|\right]_\alpha \leq \sum_{\alpha,\beta=1}^{2} L_{\alpha\beta}\left(|\hat{z}_{\bar{x}_\alpha}|, |z|_{\alpha,(0.5)}|\hat{u}_{\bar{x}_\beta}|\right]_\alpha.$$

The solution of problem (6), (7) is sufficiently smooth, and hence

$$|\hat{u}_{\bar{x}_\alpha}| \leq \frac{1}{h_\alpha} \int\limits_{x_\alpha^{i_\alpha - 1}}^{x_\alpha^{i_\alpha}} \left|\frac{\partial \hat{u}}{\partial x_\alpha}\right| dx_\alpha \leq c_5, \qquad \alpha = 1, 2.$$

Then, applying the ε-inequality, we obtain the estimate

$$\sum_{\alpha,\beta=1}^{2} L_{\alpha\beta}\left(|\hat{z}_{\bar{x}_\alpha}|, |z|_{\alpha,(0.5)}|\hat{u}_{\bar{x}_\beta}|\right]_\alpha \leq c_5 \sum_{\alpha,\beta=1}^{2} \left(\varepsilon_{\alpha\beta} L_{\alpha\beta}||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2 + \frac{L_{\alpha\beta}}{4\varepsilon_{\alpha\beta}}\|z\|^2\right);$$

i.e.,

$$\sum_{\alpha,\beta=1}^{2}\left(|\hat{z}_{\bar{x}_\alpha}|,|k_{\alpha\beta}(y)-k_{\alpha\beta}(u)||\hat{u}_{\bar{x}_\beta}|\right]_\alpha \le c_5\sum_{\alpha,\beta=1}^{2}\left(\varepsilon_{\alpha\beta}L_{\alpha\beta}||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2+\frac{L_{\alpha\beta}}{4\varepsilon_{\alpha\beta}}\|z\|^2\right). \tag{26}$$

Here and below, $\varepsilon_{\alpha\beta}>0$, $\alpha,\beta=1,2$, $\varepsilon_i>0$, $i=3,4$. In a similar way, we obtain the estimate

$$\sum_{\alpha,\beta=1}^{2}\left[|\hat{z}_{x_\alpha}|,|k_{\alpha\beta}(y)-k_{\alpha\beta}(u)||\hat{u}_{x_\beta}|\right)_\alpha \le c_5\sum_{\alpha,\beta=1}^{2}\left(\varepsilon_{\alpha\beta}L_{\alpha\beta}|[\hat{z}_{x_\alpha}\||_\alpha^2+\frac{L_{\alpha\beta}}{4\varepsilon_{\alpha\beta}}\|z\|^2\right). \tag{27}$$

The estimates (26) and (27) imply that

$$|\eta|\le 2c_5\sum_{\alpha,\beta=1}^{2}\varepsilon_{\alpha\beta}L_{\alpha\beta}||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2+c_5\sum_{\alpha,\beta=1}^{2}\frac{L_{\alpha\beta}}{2\varepsilon_{\alpha\beta}}\|z\|^2.$$

Note that $LHS\le-\tau\min\theta+\tau\max|\eta|$. Therefore, for the first term on the right-hand side in (25), we have the estimate

$$LHS\le-2\tau\sum_{\alpha=1}^{2}\left(c_1-c_5\sum_{\beta=1}^{2}\varepsilon_{\alpha\beta}L_{\alpha\beta}\right)||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2+\tau c_5\sum_{\alpha,\beta=1}^{2}\frac{L_{\alpha\beta}}{2\varepsilon_{\alpha\beta}}\|z\|^2, \tag{28}$$

and the second term satisfies the following estimate:

$$2\tau(\hat{z},\psi)=2\tau(\tau z_t+z,\psi)\le 2\tau^2\varepsilon_3\|z_t\|^2+\frac{\tau^2}{2\varepsilon_3}\|\psi\|^2+2\tau\varepsilon_4\|z\|^2+\frac{\tau}{2\varepsilon_4}\|\psi\|^2. \tag{29}$$

On the other hand, using the identity $\hat{z}=0.5(\hat{z}+z)+0.5\tau z_t$, we write the left-hand side of (25) as

$$2\tau(z_t,\hat{z})=\|\hat{z}\|^2-\|z\|^2+\tau^2\|z_t\|^2. \tag{30}$$

Thus, from the results (25) and (28)–(30) we derive the estimate

$$\|\hat{z}\|^2+\tau^2(1-2\varepsilon_3)\|z_t\|^2+2\tau\sum_{\alpha=1}^{2}\left(c_1-c_5\sum_{\beta=1}^{2}\varepsilon_{\alpha\beta}L_{\alpha\beta}\right)||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2$$
$$\le\left(1+\tau\left(2\varepsilon_4+c_5\sum_{\alpha,\beta=1}^{2}\frac{L_{\alpha\beta}}{2\varepsilon_{\alpha\beta}}\right)\right)\|z\|^2+\tau\left(\frac{1}{2\varepsilon_3}+\frac{1}{2\varepsilon_4}\right)\|\psi\|^2.$$

We take the values ε_3, $\varepsilon_{\alpha\beta}$, $\alpha,\beta=1,2$, sufficiently small, namely, such that the inequalities $1-2\varepsilon_3>0$ and $c_1-c_5\sum_{\beta=1}^{2}\varepsilon_{\alpha\beta}L_{\alpha\beta}>0$, $\alpha=1,2$, are satisfied. Then we arrive at the final estimate

$$\|\hat{z}\|^2\le(1+\tau c_6)\|z\|^2+\tau c_7(h_1^2+h_2^2+\tau)^2\le e^{\tau c_6}\|z\|^2+\tau c_7(h_1^2+h_2^2+\tau)^2.$$

Applying the finite-difference analog of the Gronwall lemma [2, p. 273] to the last inequality, we obtain the desired estimate. The proof of the theorem is complete.

6. FINITE-DIFFERENCE SCHEMES FOR EQUATIONS OF GENERAL FORM

To complete the discussion of the above results, it is necessary to consider problems with lower-order derivatives. We again consider the initial–boundary value problem (6) with operator L of the form

$$Lu=\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(u)\frac{\partial u}{\partial x_\beta}\right)+\sum_{\alpha=1}^{2}r_\alpha(u)\frac{\partial u}{\partial x_\alpha}-q(x)u, \tag{31}$$

where $q(x) \geq c_0 > 0$ and the coefficients $k_{\alpha\beta}(u)$, $\alpha, \beta = 1, 2$, satisfy the ellipticity conditions (8). Then, by the results of [10, p. 22], we have the following assertion.

Theorem 6 [10, p. 22]. *The solution $u(x,t)$ of problem* (6), (31), (8) *at any point $(x, t_1) \in \bar{Q}_T$ satisfies the two-sided estimate*

$$u(x,t_1) \geq m_3 = \sup_{\lambda>\lambda_0} \min\left\{0, \min_{Q_{t_1}}\{\mu(x,t), u_0(x)\}e^{\lambda(t_1-t)}, \min_{Q_{t_1}} \frac{f(x,t)e^{\lambda(t_1-t)}}{\lambda+q(x)}\right\}, \tag{32}$$

$$u(x,t_1) \leq m_4 = \inf_{\lambda>\lambda_0} \max\left\{0, \max_{Q_{t_1}}\{\mu(x,t), u_0(x)\}e^{\lambda(t_1-t)}, \max_{Q_{t_1}} \frac{f(x,t)e^{\lambda(t_1-t)}}{\lambda+q(x)}\right\}, \tag{33}$$

where $\lambda_0 = \max_{x\in\bar{G}}\{-q(x)\} = -\min_{x\in\bar{G}} q(x)$.

To construct the corresponding monotone scheme of the second order of local approximation with respect to the spatial variables $O(h_1^2 + h_2^2 + \tau)$ for the equations containing lower-order derivatives, we use an idea due to Samarskii [2]. In the operator (31), we replace the derivatives on the uniform grid $\omega = \omega_h \times \omega_\tau$ by the finite-difference relations

$$\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\alpha}(\hat{u})\frac{\partial \hat{u}}{\partial x_\alpha}\right) + r_\alpha(\hat{u})\frac{\partial \hat{u}}{\partial x_\alpha} = \kappa_\alpha(u)(a_{\alpha\alpha}(u)\hat{u}_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha^+(u)\hat{u}_{x_\alpha} + b_\alpha^-(u)\hat{u}_{\bar{x}_\alpha} + O(h_\alpha^2 + \tau),$$
$$\kappa_\alpha(u) = (1 + R_\alpha(u))^{-1}, \qquad R_\alpha = 0.5|r_\alpha(u)|h_\alpha/k_{\alpha\alpha}(u), \qquad b_\alpha^\pm(u) = r_\alpha^\pm(u)/k_{\alpha\alpha}(u),$$
$$r_\alpha^\pm(u) = 0.5(r_\alpha(u) \pm |r_\alpha(u)|), \qquad \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2} \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(\hat{u})\frac{\partial \hat{u}}{\partial x_\beta}\right) = \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2} \Lambda_{\alpha\beta}\hat{u} + O(h_1^2 + h_2^2 + \tau).$$

As a result, we obtain the following finite-difference scheme of the second order of approximation with respect to the spatial variables:

$$y_t = \sum_{\alpha=1}^{2} (\kappa_\alpha(y)(a_{\alpha\alpha}(y)\hat{y}_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha^+(y)\hat{y}_{x_\alpha} + b_\alpha^-(y)\hat{y}_{\bar{x}_\alpha}) + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{2} \Lambda_{\alpha\beta}\hat{y} - dy + \varphi, \tag{34}$$

$$y(x,0) = u_0(x), \qquad x \in \omega_h, \qquad \hat{y}|_{\gamma_h} = \mu(x,t), \qquad x \in \gamma_h, \tag{35}$$

where d and φ are some stencil functionals which, in particular, can be taken in the form $d(x) = q(x)$, $\varphi(x,t) = f(x,t)$, $(x,t) \in \omega$. For the scheme (34), (35) to be monotone (and hence satisfy the maximum principle), it suffices to require that the following condition be satisfied: the double inequality

$$\frac{(c_3+c_4)(1+R_{\max,2})}{2c_1} \leq \frac{h_1}{h_2} \leq \frac{2c_1}{(c_3+c_4)(1+R_{\max,1})}, \qquad R_{\max,\alpha} = 0.5h_\alpha \max_{u\in\bar{D}_u} \frac{|r_\alpha(u)|}{k_\alpha(u)}, \tag{36}$$

must hold for all $x \in \omega_h$ and $t \in \omega_\tau$.

The following theorem can be proved in a similar way.

Theorem 7. *Assume that conditions* (36) *are satisfied for all $x \in \omega_h$ and $t \in \omega_\tau$. Then the finite-difference scheme* (34), (35) *is unconditionally monotone* (*without restrictions on the steps τ and h_α, $\alpha = 1, 2$*), *and its solution satisfies the two-sided estimate*

$$y(x,t_n) \geq m_3^n = \sup_{\lambda>0} \min\left\{0, \min_{\omega_{t_n}} e^{\lambda(t_n-t)}\{\mu(x,t), u_0(x)\}, \min_{\omega_{t_n}} \frac{\tau f(x,t)e^{\lambda(t_n-t)}}{(1+\tau q)e^{\lambda\tau}-1}\right\}, \tag{37}$$

$$y(x,t_n) \leq m_4^n = \inf_{\lambda>0} \max\left\{0, \max_{\omega_{t_n}} e^{\lambda(t_n-t)}\{\mu(x,t), u_0(x)\}, \max_{\omega_{t_n}} \frac{\tau f(x,t)e^{\lambda(t_n-t)}}{(1+\tau q)e^{\lambda\tau}-1}\right\} \tag{38}$$

at any point $(x, t_n) \in \omega$.

Remark 5. Since

$$\frac{\tau}{(1+\tau q)e^{\lambda\tau}-1} \le \frac{1}{\lambda+q} \quad \text{for all} \quad \lambda, \tau > 0,$$

we see that the estimates (32), (33) and (37), (38) imply that $m_3 \le m_3^n$ and $m_4^n \le m_4$, and in this sense, one can say that the finite-difference estimates inherit the properties of the differential problem.

Remark 6. In the case without mixed derivatives, the estimates (19)–(21) and (37), (38) are satisfied without restrictions (16), (36) on the grid steps h_1 and h_2.

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