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CONTROL THEORY

Necessary Conditions for Optimality in Problems of Optimal Control of Systems with Discontinuous Right-Hand Side

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Abstract—We consider the problems of optimal control of a dynamical system whose right-hand side is discontinuous in the state variable and is linear in the control with sufficiently smooth coefficients in each of the half-spaces into which the space is divided by the switching hyperplane. The main attention is paid to the situation where there exist intervals on which the optimal trajectory lies on the switching surface. New nondegenerate necessary conditions for optimality are stated and proved in the maximum principle form. The obtained optimality conditions are compared with the already known conditions.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

In the present paper, we study the problems of optimal control of the solutions of systems with discontinuous dynamics, i.e., of systems whose velocity vector determining their dynamics can be a discontinuous function of the variable determining the state of the system. Dynamical systems with discontinuous dynamics arise in the mathematical description of real systems (e.g., see [1–5] and the references therein).

In the present paper, we consider the following problem of optimal control with discontinuous dynamics:

$$\min_{(x,u)} F_0(x(T)),$$
$$\dot{x}(t) = \begin{cases} F_+(x(t),u(t)) & \text{if } r(x(t)) > 0, \\ F_-(x(t),u(t)) & \text{if } r(x(t)) < 0, \\ F_+(x(t),u(t)) \vee F_-(x(t),u(t)) & \text{if } r(x(t)) = 0, \end{cases} \tag{1.1}$$
$$u(t) \in [-1,1], \qquad t \in \mathcal{T}, \qquad x(0) = x_0, \qquad h(x(T)) = 0,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $t \in \mathcal{T} = [0,T]$, $r(x) = d'x$, $d \in \mathbb{R}^n$ is a given vector, the symbols "$\vee$" and "$'$" respectively denote the logic "or" and the transposition operation, and the functions $F_\pm(x,u) = a^\pm(x) + b^\pm(x)u$, $h(x) \in \mathbb{R}^m$, $F_0(x) \in \mathbb{R}$, $a^\pm(x) \in \mathbb{R}^n$, and $b^\pm(x) \in \mathbb{R}^n$ are sufficiently smooth. Here and below, the sufficient smoothness of a function is understood as follows: the function itself and all of its partial derivatives used in the paper exist and are continuous.

Since the right-hand side is discontinuous, it may happen that the classical solution of this system of differential equations does not exist and problem (1.1) must be restated. There exist several methods for completing the definition of the solutions for systems of differential equations with discontinuous right-hand side. The most popular of them is the redefinition by the Filippov rule [3]. The use of this rule results in the generalized problem which, in terms of differential inclusions, is stated as follows:

$$\min_x F_0(x(T)), \qquad \dot{x}(t) \in U(x(t)), \qquad x(0) = x_0, \qquad h(x(T)) = 0, \tag{1.2}$$

where the mapping $U(x)$, $x \in \mathbb{R}^n$, is defined by the relations

$$\begin{aligned} &U(x) := \{v \in \mathbb{R}^n : v = F_+(x,u),\ u \in [-1,1]\} \quad \text{if} \ \ r(x) > 0, \\ &U(x) := \{v \in \mathbb{R}^n : v = F_-(x,u),\ u \in [-1,1]\} \quad \text{if} \ \ r(x) < 0, \\ &U(x) := \{v \in \mathbb{R}^n : v = \alpha F_+(x,u) + (1-\alpha)F_-(x,u),\ \alpha \in [0,1],\ u \in [-1,1]\} \quad \text{if} \ \ r(x) = 0. \end{aligned} \tag{1.3}$$

Note that, in the general case, the set $U(x)$ in definition (1.3) is nonconvex.

There are few results of studies of the optimality conditions for problems of optimal control of systems of differential equations with discontinuous right-hand side. As a rule (see, e.g., [6–11]), the original problem (1.1) is replaced by the generalized problem (1.2) and attention is mainly paid to the cases where there are no parts of the trajectories lying on the switching surface $r(x) = 0$, $x \in \mathbb{R}^n$ [6, 7, 11].

It was shown in [12] that, in the case where there are parts of trajectories lying on the switching surface, the necessary conditions for optimality stated in [6–11] degenerate and become noninformative. In this paper, the degeneration of necessary conditions for optimality is understood as follows: they are satisfied for admissible controls. Such a phenomenon is known for optimal control problems with state constraints (see, e.g., [13]).

In the general case, in problem (1.2), the set of velocities $U(x)$ is nonconvex. The nonconvexity of the set of velocities leads to the following difficulties: first, in the problem (1.2) under study, the optimal control need not exist, and second, even if it exists, the problem itself can be ill posed, which means that arbitrarily small variations in the problem conditions may result in significant variations in the solution and the optimal value of the performance criterion. Therefore, along with problem (1.2), one usually (see [14]) considers the weakened problem where the set $U(x)$ is replaced by its convex hull $\operatorname{conv} U(x)$

$$\min_x F_0(x(T)), \qquad \dot{x}(t) \in \operatorname{conv} U(x(t)), \quad x(0) = x_0, \qquad h(x(T)) = 0. \tag{1.4}$$

Note that now, in problem (1.4), the set of velocities $\operatorname{conv} U(x)$ is *convex* for a fixed $x \in \mathbb{R}^n$ but the mapping $x \mapsto \operatorname{conv} U(x)$, $x \in \mathbb{R}^n$, does not satisfy the Lipschitz conditions introduced in [15–17]. Therefore, the results in these papers cannot be used to study the optimality conditions in problem (1.4).

The goal on this paper is to obtain nondegenerate (informative) necessary conditions for optimality for weakened problem (1.4) and construct a control process in original problem (1.1) based on the use of the solution of the weakened problem.

Note that all results given below can be generalized to the case where the switching surface in problem (1.1) is given by a sufficiently smooth scalar function $r(x)$, $x \in \mathbb{R}^n$. To simplify the calculations, we here assume that this function is linear.

2. MAXIMUM PRINCIPLE

Consider the set $U(x)$ defined by relations (1.3). One can readily show that

$$\operatorname{conv} U(x) = \{v \in \mathbb{R}^n : v = f(x,\alpha,u_+,u_-),\ \alpha \in [0,1],\ |u_+| \le \alpha,\ |u_-| \le 1-\alpha\},$$

where

$$\begin{aligned} f(x,\alpha,u_+,u_-) &:= \alpha a^+(x) + (1-\alpha)a^-(x) + u_+ b^+(x) + u_- b^-(x) \\ &= \Delta a(x)\alpha + a^-(x) + b^+(x)u_+ + b^-(x)u_-, \qquad \Delta a(x) := a^+(x) - a^-(x). \end{aligned} \tag{2.1}$$

Then the weakened problem (1.4) can be written as the following optimal control problem:

$$\begin{aligned}&\min_{\alpha,u_+,u_-} F_0(x(T)),\\ &\dot x(t)=f(x(t),\alpha(t),u_+(t),u_-(t)),\qquad x(0)=x_0,\qquad h(x(T))=0,\\ &|u_+(t)|\le\alpha(t),\quad |u_-(t)|\le 1-\alpha(t),\quad \alpha(t)r(x(t))\ge 0,\quad (1-\alpha(t))r(x(t))\le 0,\quad t\in\mathcal{T}.\end{aligned}\tag{2.2}$$

Note that the equations of dynamics in problem (2.2) are linear in the control (α,u_+,u_-).

Assume that $(\alpha^0(t),u^0_+(t),u^0_-(t))$ and $x^0(t)$, where $t\in\mathcal{T}$, are the optimal control and the corresponding trajectory in problem (2.2). On the control interval, we distinguish three sets,

$$\mathcal{T}_a=\{t\in\mathcal{T}:d'x^0(t)=0\},\qquad \mathcal{T}^+=\{t\in\mathcal{T}:d'x^0(t)>0\},\qquad \mathcal{T}^-=\{t\in\mathcal{T}:d'x^0(t)<0\}.$$

In the general case, if the set $\mathcal{T}_a$ (i.e., the set of $t\in\mathcal{T}$ for which the trajectory $x^0(t)$ lies on the switching surface) is nonempty, it consists of intervals and/or isolated points whose number may be infinite. If $\tau\in\mathcal{T}_a$ is an isolated point, then one says that the trajectory of the system intersects the switching surface at this point or is tangent to it. If $[\tau,\bar\tau]\subset\mathcal{T}_a$, $\tau<\bar\tau$, then one says that the trajectory lies on the switching surface on the interval $[\tau,\bar\tau]$. Such an interval is called a singular arc [10]. The cases where the set $\mathcal{T}_a$ is either empty or consists only of isolated points have been sufficiently well studied in the literature. Therefore, to simplify the presentation, we (without loss of generality) assume in what follows that the set $\mathcal{T}_a$ consists only of singular arcs.

Further, we assume that the following assumptions are satisfied.

(A) $\operatorname{rank}\dfrac{\partial h(x^0(T))}{\partial x}=\bar m$.

(B) The optimal control $(\alpha^0(t),u^0_+(t),u^0_-(t))$, $t\in\mathcal{T}$, is piecewise smooth.

(C) The set $\mathcal{T}_a=\{t\in\mathcal{T}:d'x^0(t)=0\}$ consists of finitely many singular arcs,

$$\mathcal{T}_a=\mathcal{T}_1\bigcup\mathcal{T}_2\bigcup\ldots\bigcup\mathcal{T}_p,\qquad \mathcal{T}_k=[\tau_k,\tau^k];\qquad 0<\tau_1<\tau^1<\tau_2<\cdots<\tau^p<T,$$

and the following inequalities are satisfied: $d'\dot x^0(\tau_k-0)\ne 0$, $d'\dot x^0(\tau^k+0)\ne 0$, $k=1,\ldots,p$.

(D) For each singular arc $\mathcal{T}_k=[\tau_k,\tau^k]$, there exists a number $\varepsilon=\varepsilon(k)>0$, time instants $\mu_i(k)$, $i=1,\ldots,p(k)$, $p(k)\ge 2$, satisfying the inequalities

$$\tau_k=\mu_1(k)<\mu_2(k)<\cdots<\mu_{p(k)-1}(k)<\mu_{p(k)}(k)=\tau^k,$$

and a partition of the set $I(k)=\{1,2,\ldots,p(k)-1\}$ into nonintersecting subsets (some of them can be empty) $I^+(k)$, $I^-(k)$, $I^*(k)$, $I^*_+(k)$, $I^*_-(k)$ such that, on the subintervals $\mathcal{T}_i(k):=(\mu_i(k),\mu_{i+1}(k))$, $i=1,\ldots,p(k)-1$, the following relations[1] are satisfied:

$$\begin{aligned}&\alpha^0(t)=1,\quad |u^0_+(t)|\le 1-\varepsilon,\quad |d'b^+(x^0(t))|>\varepsilon,\quad t\in\mathcal{T}_i(k),\quad i\in I^+(k),\\ &\alpha^0(t)=0,\quad |u^0_-(t)|\le 1-\varepsilon,\quad |d'b^-(x^0(t))|>\varepsilon,\quad t\in\mathcal{T}_i(k),\quad i\in I^-(k),\\ &\alpha^0(t)\in[\varepsilon,1-\varepsilon],\quad |u^0_+(t)|+|u^0_-(t)|=1,\quad |d'b^*(x^0(t),u^0_+(t),u^0_-(t))|>\varepsilon,\quad t\in\mathcal{T}_i(k),\quad i\in I^*(k),\\ &\alpha^0(t)\in[\varepsilon,1-\varepsilon],\quad |u^0_+(t)|\le\alpha^0(t)-\varepsilon,\quad |d'b^+(x^0(t))|>\varepsilon,\quad t\in\mathcal{T}_i(k),\quad i\in I^*_+(k),\\ &\alpha^0(t)\in[\varepsilon,1-\varepsilon],\quad |u^0_-(t)|\le(1-\alpha^0(t))-\varepsilon,\quad |d'b^-(x^0(t))|>\varepsilon,\quad t\in\mathcal{T}_i(k),\quad i\in I^*_-(k),\end{aligned}\tag{2.3}$$

where

$$b^*(x,u_+,u_-):=\Delta a(x)+b^+(x)\operatorname{sgn}u_+-b^-(x)\operatorname{sgn}u_-.\tag{2.4}$$

Note that the function $(u^0_+(t),u^0_-(t))$ is continuous on the intervals $\mathcal{T}_i(k)$, $i\in I^\pm(k)$. If the function $(u^0_+(t),u^0_-(t))$ is discontinuous at a certain point $\bar t\in\mathcal{T}_i(k)$, $i\in I^*(k)\bigcup I^*_\pm(k)$, then relations (2.3) are assumed to be satisfied at this point if they are satisfied for $\bar t-0$ and $\bar t+0$.

[1] Such a partition of the set $\mathcal{T}_k$ into subsets can be nonunique. Therefore, we further consider the partition for which the number $p(k)$ is minimal.

Theorem 1 (maximum principle). *Let assumptions* (A)–(D) *be satisfied for the optimal control* $(\alpha^0(t), u_+^0(t), u_-^0(t))$, $t \in \mathcal{T}$, *and the corresponding trajectory* $x^0(t)$, $t \in \mathcal{T}$, *of problem* (2.2). *Then there exists a vector* $y \in \mathbb{R}^{\bar{m}}$, *numbers* $y_0 \geq 0$, γ_k, γ^k, $k = 1, \dots, p$, *a piecewise continuous function* $\xi(t)$, $t \in \mathcal{T}$, *and a piecewise continuous solution* $\psi(t)$, $t \in \mathcal{T}$, *of the adjoint system*

$$\begin{aligned} \dot{\psi}'(t) &= -\psi'(t)\frac{\partial f(x^0(t), \alpha^0(t), u_+^0(t), u_-^0(t))}{\partial x} + d'\xi(t), \\ \psi'(T) &= -y'\frac{\partial h(x^0(T))}{\partial x} - y_0\frac{\partial F_0(x^0(T))}{\partial x} \end{aligned} \tag{2.5}$$

with the jump conditions

$$\psi(\tau_k - 0) = \psi(\tau_k + 0) + d\gamma_k, \qquad \psi(\tau^k - 0) = \psi(\tau^k + 0) + d\gamma^k, \qquad k = 1, \dots, p, \tag{2.6}$$

such that the following conditions are satisfied:

$$\sum_{k=1}^{p}(|\gamma_k| + |\gamma^k|) + y_0 + \|y\| > 0, \tag{2.7}$$

$$\psi'(t-0)\dot{x}^0(t-0) = \psi'(t+0)\dot{x}^0(t+0), \quad t = \tau_k, \quad t = \tau^k, \quad k = 1, \dots, p, \tag{2.8}$$

$$\begin{aligned} &\xi(t) \leq 0 \quad \text{if } d'x^0(t) = 0, \quad \alpha^0(t) = 0, \\ &\xi(t) \geq 0 \quad \text{if } d'x^0(t) = 0, \quad \alpha^0(t) = 1, \\ &\xi(t) = 0 \quad \text{if } d'x^0(t) \neq 0, \quad t \in \mathcal{T}; \end{aligned} \tag{2.9}$$

$$\psi'(t)f(x^0(t), \alpha^0(t), u_+^0(t), u_-^0(t)) = \max_{\substack{\alpha \\ |u_+| \leq \alpha \\ |u_-| \leq 1-\alpha}} \psi'(t)f(x^0(t), \alpha, u_+, u_-) \quad \text{for a.a. } t \in \mathcal{T}_a, \tag{2.10}$$

$$\psi'(t)F_\pm(x^0(t), u_\pm^0(t)) = \max_{|u| \leq 1}\psi'(t)F_\pm(x^0(t), u) \quad \text{for a.a. } t \in \mathcal{T}^\pm. \tag{2.11}$$

To verify the above-formulated optimality conditions, it is necessary to know a finite set of parameters, namely, the numbers γ_k, γ^k, $k = 1, \dots, p$, $y_0 \geq 0$ and the vector y, as well as the function $\xi(t)$, $t \in \mathcal{T}$. We restate the assertions of Theorem 1 so as not to use the functions $\xi(t)$, $t \in \mathcal{T}$, explicitly in them. To this end, we analyze the maximum conditions (2.10) on singular arcs.

We write

$$a_1 = a_1(t) := \psi'(t)\Delta a(x^0(t)), \qquad a_2 = a_2(t) := \psi'(t)b^+(x^0(t)), \qquad a_3 = a_3(t) := \psi'(t)b^-(x^0(t)).$$

Then the maximum condition (2.10) can be written as the linear programming problem

$$\max_{\alpha, u_+, u_-}(a_1\alpha + a_2u_+ + a_3u_-), \qquad \alpha \in [0,1], \qquad |u_+| \leq \alpha, \qquad |u_-| \leq 1 - \alpha. \tag{2.12}$$

Let (α^0, u_+^0, u_-^0) be an optimal solution of problem (2.12). The optimality of this solution implies the relations

$$-a_1 - |a_2| + |a_3| \leq 0 \quad \text{if } \alpha^0 = 1; \qquad -a_1 - |a_2| + |a_3| \geq 0 \quad \text{if } \alpha^0 = 0;$$

$$-a_1 - |a_2| + |a_3| = 0 \quad \text{if } \alpha^0 \in (0,1);$$

$$\text{if } |a_2| > 0, \quad \text{then } |u_+^0| = \alpha^0; \quad \text{if } |a_2| = 0, \quad \text{then } |u_+^0| \leq \alpha^0;$$

$$\text{if } |a_3| > 0, \quad \text{then } |u_-^0| = 1 - \alpha^0; \quad \text{if } |a_3| = 0, \quad \text{then } |u_-^0| \leq 1 - \alpha^0.$$

Taking into account these relations and the maximum conditions (2.10), we conclude that the following assertions hold for a.a. $t \in \mathcal{T}_a$.

1. If $\alpha^0(t) = 1$, then

$$\psi'(t)\Delta a(x^0(t)) \ge -|\psi'(t)b^+(x^0(t))| + |\psi'(t)b^-(x^0(t))|, \qquad |u^0_+(t)| \le 1, \qquad u^0_-(t) = 0.$$

Moreover, if $|u^0_+(t)| < 1$, then $\psi'(t)b^+(x^0(t)) = 0$.

2. If $\alpha^0(t) = 0$, then

$$\psi'(t)\Delta a(x^0(t)) \le -|\psi'(t)b^+(x^0(t))| + |\psi'(t)b^-(x^0(t))|, \qquad |u^0_-(t)| \le 1, \qquad u^0_+(t) = 0. \tag{2.13}$$

Moreover, if $|u^0_-(t)| < 1$, then $\psi'(t)b^-(x^0(t)) = 0$.

3. If $\alpha^0(t) \in (0, 1)$, then

$$\psi'(t)\Delta a(x^0(t)) = -|\psi'(t)b^+(x^0(t))| + |\psi'(t)b^-(x^0(t))|, \quad |u^0_+(t)| \le \alpha^0(t), \quad |u^0_-(t)| \le 1 - \alpha^0(t).$$

Moreover, if $|u^0_+(t)| < \alpha^0(t)$, then $\psi'(t)b^+(x^0(t)) = 0$, and if $|u^0_-(t)| < 1 - \alpha^0(t)$, then

$$\psi'(t)b^-(x^0(t)) = 0.$$

These relations, assumptions (A)–(D), and the assertions of Theorem 1 imply that

$$\begin{aligned} \psi'(t)b^\pm(x^0(t)) = 0, &\qquad t \in \mathcal{T}_i(k), \qquad i \in I^\pm(k) \bigcup I^*_\pm(k); \\ \psi'(t)b^*(x^0(t), u^0_+(t), u^0_-(t)) = 0, &\qquad t \in \mathcal{T}_i(k), \qquad i \in I^*(k), \qquad k = 1, \dots, p. \end{aligned}$$

We differentiate the last relations with respect to t with (2.2) and (2.5) taken into account and obtain the following relations for the function $\xi(t)$, $t \in \mathcal{T}_a$:

$$\begin{aligned} \xi(t) = \psi'(t)q^0_\pm(t), &\qquad t \in \mathcal{T}_i(k), \qquad i \in I^\pm(k) \bigcup I^*_\pm(k); \\ \xi(t) = \psi'(t)q^0_*(t), &\qquad t \in \mathcal{T}_i(k), \qquad i \in I^*(k), \qquad k = 1, \dots, p. \end{aligned} \tag{2.14}$$

Here the function $b^*(x, u_+, u_-)$ is defined by formula (2.4), and

$$\begin{aligned} &q_\pm(x, \alpha, u_+, u_-) = \frac{1}{d'b^\pm(x)}\left[\frac{\partial f(x, \alpha, u_+, u_-)}{\partial x} b^\pm(x) - \frac{\partial b^\pm(x)}{\partial x} f(x, \alpha, u_+, u_-)\right], \\ &q_*(x, \alpha, u_+, u_-) = \frac{1}{d'b^*(x, u_+, u_-)}\left[\frac{\partial f(x, \alpha, u_+, u_-)}{\partial x} b^*(x, u_+, u_-) - \frac{\partial b^*(x, u_+, u_-)}{\partial x} f(x, \alpha, u_+, u_-)\right], \\ &q^0_\pm(t) = q_\pm(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)), \quad q^0_*(t) = q_*(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)), \quad t \in \mathcal{T}_a. \end{aligned} \tag{2.15}$$

Note that, by assumptions (A) and (D), we have the inequalities

$$\begin{aligned} d'b^\pm(x^0(t)) \ne 0, &\qquad t \in \mathcal{T}_i(k), \qquad i \in I^\pm(k) \bigcup I^*_\pm(k); \\ d'b^*(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)) \ne 0, &\qquad t \in \mathcal{T}_i(k), \qquad i \in I^*(k), \qquad k = 1, \dots, p. \end{aligned}$$

It follows from relations (2.14) that the adjoint system (2.5) can be written as

$$\dot\psi'(t) = -\psi'(t)S^0(t), \qquad t \in \mathcal{T}, \qquad \psi'(T) = -y'\frac{\partial h(x^0(T))}{\partial x} - y_0\frac{\partial F_0(x^0(T))}{\partial x}, \tag{2.16}$$

where

$$S^0(t) = \begin{cases} Q(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)), & t \in \mathcal{T} \setminus \mathcal{T}_a, \\ Q_*(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)), & t \in \mathcal{T}_i(k), \quad i \in I^*(k), \\ Q_\pm(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)), & t \in \mathcal{T}_i(k), \quad i \in I^\pm(k) \bigcup I^*_\pm(k), \end{cases} \qquad k = 1, \dots, p, \tag{2.17}$$

$$\begin{aligned} &Q(x, \alpha, u_+, u_-) = \frac{\partial f(x, \alpha, u_+, u_-)}{\partial x}, \quad Q_*(x, \alpha, u_+, u_-) = Q(x, \alpha, u_+, u_-) - q_*(x, \alpha, u_+, u_-)d', \\ &Q_\pm(x, \alpha, u_+, u_-) = Q(x, \alpha, u_+, u_-) - q_\pm(x, \alpha, u_+, u_-)d'. \end{aligned} \tag{2.18}$$

With the resulting relations taken into account, we can state Theorem 1 as follows.

Theorem 2. *Let the optimal control* $(\alpha^0(t), u_+^0(t), u_-^0(t))$, $t \in \mathcal{T}$, *and the corresponding trajectory* $x^0(t)$, $t \in \mathcal{T}$, *of problem* (2.2) *satisfy assumptions* (A)–(D). *Then there exists a vector* $y \in \mathbb{R}^{\bar m}$, *numbers* $y_0 \geq 0$, γ_k, γ^k, $k = 1, \dots, p$, *satisfying condition* (2.7), *and a piecewise continuous solution* $\psi(t)$, $t \in \mathcal{T}$, *of the adjoint system* (2.16) *with the jump conditions* (2.6) *such that relations* (2.8), (2.10), (2.11) *and the following conditions are satisfied*:

$$\psi'(t) q_+^0(t) \leq 0, \qquad t \in \mathcal{T}_i(k), \qquad i \in I^+(k);$$
$$\psi'(t) q_-^0(t) \geq 0, \qquad t \in \mathcal{T}_i(k), \qquad i \in I^-(k), \qquad k = 1, \dots, p.$$

Proof. To simplify the computations, we assume that

$$\begin{gathered} p = 1, \qquad \mathcal{T}_a(1) = [\tau_1(1), \tau^1(1)], \qquad 0 < \tau_1 = \tau_1(1) < \tau^1 = \tau^1(1) < T; \\ p(1) = 4, \quad I^+(1) = \varnothing, \quad I^-(1) = \{1\}, \quad I^*(1) = \{2\}, \quad I_+^*(1) = \varnothing, \quad I_-^*(1) = \{3\}. \end{gathered} \tag{2.19}$$

It follows from (2.19) and assumptions (A)–(D) that, for some μ_i, $i = 0, \dots, 5$, and $\varepsilon > 0$, the following relations hold:

$$\begin{gathered} \mu_0 = 0 < \mu_1 = \tau_1 < \mu_2 < \mu_3 < \mu_4 = \tau^1 < T = \mu_5, \qquad \mathcal{T}_+ := [\mu_0, \mu_1), \qquad \mathcal{T}_- := (\mu_4, \mu_5], \\ d'x^0(t) > 0, \quad t \in \mathcal{T}_+; \quad d'x^0(t) < 0, \quad t \in \mathcal{T}_-; \qquad d'x^0(t) = 0, \quad t \in \mathcal{T}_a := [\mu_1, \mu_4]; \\ \alpha^0(t) = 0, \qquad |u_+^0(t)| = 0, \qquad u_-^0(t) \leq 1 - \varepsilon, \qquad t \in (\mu_1, \mu_2); \\ \varepsilon \leq \alpha^0(t) \leq 1 - \varepsilon, \qquad |u_+^0(t)| + |u_-^0(t)| = 1, \qquad t \in (\mu_2, \mu_3); \\ \varepsilon \leq \alpha^0(t) \leq 1 - \varepsilon, \qquad |u_-^0(t)| \leq 1 - \alpha^0(t) - \varepsilon, \qquad t \in (\mu_3, \mu_4); \\ d'\dot x^0(\mu_1 - 0) < 0, \quad d'\dot x^0(\mu_4 + 0) < 0. \end{gathered} \tag{2.20}$$

We introduce the vector of parameters

$$\gamma = (y_0, y, \gamma_1, \gamma^1), \tag{2.21}$$

where $y_0 \geq 0$, $\gamma_1, \gamma^1 \in \mathbb{R}$, $y \in \mathbb{R}^{\bar m}$, and let $\psi(t \mid \gamma)$, $t \in \mathcal{T}$, denote the piecewise continuous solution of the system

$$\begin{gathered} \dot\psi'(t) = \begin{cases} -\psi'(t) Q(x^0(t), \alpha^0(t), u_+^0(t), u_-^0(t)), & t \in [\mu_0, \mu_1) \bigcup (\mu_4, \mu_5]; \\ -\psi'(t) Q_-(x^0(t), \alpha^0(t), u_+^0(t), u_-^0(t)), & t \in [\mu_1, \mu_2) \bigcup [\mu_3, \mu_4), \\ -\psi'(t) Q_*(x^0(t), \alpha^0(t), u_+^0(t), u_-^0(t)), & t \in [\mu_2, \mu_3), \end{cases} \\ \psi'(T) = -y' \frac{\partial h(x^0(T))}{\partial x} - y_0 \frac{\partial F_0(x^0(T))}{\partial x} \end{gathered} \tag{2.22}$$

with the jump conditions

$$\psi(\mu_1 - 0) = \psi(\mu_1 + 0) + d\gamma_1, \qquad \psi(\mu_4 - 0) = \psi(\mu_4 + 0) + d\gamma^1. \tag{2.23}$$

The following lemma is proved in Appendix B (see below).

Lemma. *Assume that* $(\alpha^0(t), u_+^0(t), u_-^0(t))$ *and* $x^0(t)$, $t \in \mathcal{T}$, *are an optimal control and a trajectory of problem* (2.2) *for which assumptions* (A)–(D) *are satisfied and relations* (2.20) *hold. Then for any* $m \in \mathbb{N}$, $m \geq 2$, *and any set of points* $\bar t_1, \bar t_2, \dots, \bar t_{2m+1}$ *satisfying the inequalities*

$$\mu_1 = \bar t_1 < \bar t_2 < \dots < \bar t_{2m-1} = \mu_2 < \bar t_{2m} = \mu_4 < \bar t_{2m+1} = \mu_5,$$

there exists a vector γ, $\|\gamma\| = 1$, *of the form* (2.21) *such that the following relations are satisfied along the corresponding solution* $\psi(t) = \psi(t|\gamma)$, $t \in \mathcal{T}$, *of system* (2.22), (2.23):

$$\psi'(t-0)\dot{x}^*(t-0) = \psi'(t+0)\dot{x}^*(t+0), \qquad t = \mu_1, \qquad t = \mu_4,$$

$$\psi'(t)b^{\pm}(x^0(t))u^0_{\pm}(t) = \max_{|u|\le 1} \psi'(t)b^{\pm}(x^0(t))u \quad \textit{for a.a. } t \in \mathcal{T}_{\pm},$$

$$\psi'(t)f(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)) = \max_{\alpha\in[0,1],|u_+|\le\alpha,|u_-|\le 1-\alpha} \psi'(t)f(x^0(t), \alpha, u_+, u_-)$$

$$\text{for a.a. } t \in [\bar{t}_{2i-1}, \bar{t}_{2i}], \qquad i = 1, \ldots, m;$$

$$\psi'(t)q^0_-(t) \ge 0, \qquad \psi'(t)b^-(x^0(t))u^0_-(t) = \max_{|u|\le 1} \psi'(t)b^-(x^0(t))u$$

$$\text{for a.a. } t \in [\bar{t}_{2i}, \bar{t}_{2i+1}], \qquad i = 1, \ldots, m-1. \tag{2.24}$$

For each $m \in \mathbb{N}$, $m \ge 2$, consider the set of points

$$t^{(m)}_i = \tau_1 + (i-1)\frac{\tau_0 - \tau_1}{2m-2}, \qquad i = 1, \ldots, 2m-1, \qquad t^{(m)}_{2m} = \tau^1. \tag{2.25}$$

Recall that, by assumption, $\alpha^0(t) = 0$, $t \in (\mu_1, \mu_2)$. Then it follows from Lemma 1 with regard to assumption (D) and relations (2.13) that there exists a vector

$$\gamma(m) = (y_0(m), y(m), \gamma_1(m), \gamma^1(m)), \qquad \|\gamma(m)\| = 1,$$

such that the following relations are satisfied:

$$\psi'(t-0|\gamma(m))\dot{x}^*(t-0) = \psi'(t+0|\gamma(m))\dot{x}^*(t+0), \qquad t = \mu_1, \qquad t = \mu_4, \tag{2.26}$$

$$\psi'(t|\gamma(m))b^{\pm}(x^0(t))u^0_{\pm}(t) = \max_{|u|\le 1} \psi'(t|\gamma(m))b^{\pm}(x^0(t))u \quad \text{for a.a. } t \in \mathcal{T}_{\pm}, \tag{2.27}$$

$$\begin{aligned}\psi'(t|\gamma(m))&f(x^0(t), \alpha^0(t), u^0_+(t), u^0_-(t)) \\ &= \max_{\alpha,|u_+|\le\alpha,|u_-|\le 1-\alpha} \psi'(t|\gamma(m))f(x^0(t), \alpha, u_+, u_-), \quad t \in [\mu_2, \mu_4],\end{aligned} \tag{2.28}$$

$$\psi'(t|\gamma(m))b^-(x^0(t)) = 0, \qquad t \in [\mu_1, \mu_2]; \tag{2.29}$$

$$\begin{aligned}\psi'(t|\gamma(m))\Delta a(x^0(t)) \le -|\psi'(t|\gamma(m))b^+(x^0(t))|, \qquad t \in [t^{(m)}_{2i-1}, t^{(m)}_{2i}], \\ \psi'(t|\gamma(m))q^0_-(t) \ge 0, \qquad t \in [t^{(m)}_{2i}, t^{(m)}_{2i+1}], \qquad i = 1, \ldots, m-1.\end{aligned} \tag{2.30}$$

Consider the sequence of vectors $\gamma(m)$, $m = 2, 3, \ldots$ Since $\|\gamma(m)\| = 1$, $m = 2, 3, \ldots$, we can choose a converging subsequence from this sequence. Without loss of generality, we assume that the sequence $\gamma(m)$, $m = 2, 3, \ldots$, itself converges. We write $\gamma^* = \lim_{m\to\infty} \gamma(m)$. Obviously, $\|\gamma^*\| = 1$.

It follows from definition (2.25) that, for any point $t \in [\mu_1, \mu_2]$, there exists a sequence of indices $i(m) = i(m \mid t) \in \{2, 3, \ldots, 2m-1\}$, $m = 2, 3, 4, \ldots$, such that $t^{(m)}_{i(m)} \to t$ as $m \to \infty$. Bearing this in mind, we pass to the limit as $m \to \infty$ in relations (2.30). As a result, we obtain

$$\psi'(t|\gamma^*)\Delta a(x^0(t)) \le -|\psi'(t|\gamma^*)b^+(x^0(t))|, \qquad \psi'(t|\gamma^*)q^0_-(t) \ge 0, \qquad t \in [\mu_1, \mu_2].$$

As a result of passing to the limit as $m \to \infty$, the last inequalities and the relations obtained from (2.26)–(2.29) coincide with the assertions of Theorem 2 for the considered type of the structure of solution (2.19) of original problem (2.2).

For other types of the structure of the solution, the theorem can be proved in a similar way. The proof of the theorem is complete.

3. CONSTRUCTION OF CONTROL ACTIONS IN THE ORIGINAL PROBLEM (1.1) BASED ON SOLUTION OF WEAKENED PROBLEM (2.2)

Let us show how the solution of the weakened problem (2.2) can be used to construct a control in the original problem (1.1), where the velocity vector on the switching surface can have the form $F_+(x(t),u(t))$ or $F_-(x(t),u(t))$.

Let $(\alpha^0(t),u^0_+(t),u^0_-(t))$ and $x^0(t)$, $t\in\mathcal{T}$, be the optimal control and the trajectory of problem (2.2) satisfying assumptions (A)–(D). By construction, $|u^0_+(t)|\le 1$, $u^0_-(t)=0$ if $t\in\bar{\mathcal{T}}_1=\{t\in\mathcal{T}:\alpha^0(t)=1\}$, $u^0_+(t)=0$, $|u^0_-(t)|\le 1$ if $t\in\bar{\mathcal{T}}_0=\{t\in\mathcal{T}:\alpha^0(t)=0\}$, and $|u^0_+(t)|\le\alpha^0(t)$, $|u^0_-(t)|\le 1-\alpha^0(t)$ if $t\in\mathcal{T}_*=\mathcal{T}\setminus(\bar{\mathcal{T}}_0\bigcup\bar{\mathcal{T}}_1)$. We set

$$u_1(t)=u^0_+(t),\quad u_2(t)=0,\quad t\in\bar{\mathcal{T}}_1;\quad u_1(t)=0,\quad u_2(t)=u^0_-(t),\quad t\in\bar{\mathcal{T}}_0,$$
$$u_1(t)=u^0_+(t)/\alpha^0(t),\quad u_2(t)=u^0_-(t)/(1-\alpha^0(t)),\quad t\in\mathcal{T}_*.$$

Obviously, the functions $u_1(t)$, $u_2(t)$, $t\in\mathcal{T}$, are piecewise continuous and $|u_1(t)|\le 1$, $|u_2(t)|\le 1$, $t\in\mathcal{T}$. We write $g_1(t,x)=a^+(x)+b^+(x)u_1(t)$, $g_2(t,x)=a^-(x)+b^-(x)u_2(t)$, $t\in\mathcal{T}$, and consider the multivalued mappings $\mathcal{F}(t,x):\mathcal{T}\times\mathbb{R}^n\to\mathbb{R}^n$ and $\operatorname{conv}\mathcal{F}(t,x):\mathcal{T}\times\mathbb{R}^n\to\mathbb{R}^n$ defined by the relations

$$\mathcal{F}(t,x)=\{g_1(t,x),g_2(t,x)\},$$
$$\operatorname{conv}\mathcal{F}(t,x)=\{z\in\mathbb{R}^n:\text{there exists }\ \alpha\in[0,1]\ \text{ such that }\ z=\alpha g_1(t,x)+(1-\alpha)g_2(t,x)\}.$$

The mapping $\mathcal{F}(t,x):\mathcal{T}\times\mathbb{R}^n\to\mathbb{R}^n$ is piecewise continuous in t and satisfies assumptions (A1) and (A2) in [15]. Therefore, by Theorem 4.1 in [15], the set of absolutely continuous solutions of the system of differential inclusions

$$x(t)\in\mathcal{F}(t,x(t)),\qquad t\in\mathcal{T},\qquad x(0)=x_0, \tag{3.1}$$

is everywhere dense with respect to the norm of the space $C(\mathcal{T},\mathbb{R}^n)$ in the set of absolutely continuous solutions of the system of differential inclusions

$$x(t)\in\operatorname{conv}\mathcal{F}(t,x(t)),\qquad t\in\mathcal{T},\qquad x(0)=x_0. \tag{3.2}$$

Thus, for any absolutely continuous solution $\bar{x}(t)$, $t\in\mathcal{T}$, of the inclusion (3.2), there exists a sequence of absolutely continuous solutions $x^s(t)$, $t\in\mathcal{T}$, of the inclusion (3.1), $s=1,2,\dots$, such that $\lim_{s\to\infty}\max_{t\in\mathcal{T}}\|\bar{x}(t)-x^s(t)\|=0$.

By construction, the optimal trajectory $x^0(t)$, $t\in\mathcal{T}$, of problem (2.2) satisfies the system of differential inclusions (3.2). Therefore, there exists a sequence of measurable functions $\alpha^s(t)\in\{0,1\}$, $t\in\mathcal{T}$, $s=1,2,\dots$, such that, for the corresponding trajectories $x^s(t)$, $t\in\mathcal{T}$, of the systems of differential equations

$$\dot{x}^s(t)=\alpha^s(t)g_1(t,x^s(t))+(1-\alpha^s(t))g_2(t,x^s(t)),\qquad x^s(0)=x_0, \tag{3.3}$$

we have

$$\lim_{s\to\infty}\max_{t\in\mathcal{T}}\|x^0(t)-x^s(t)\|=0. \tag{3.4}$$

Assume that the functions $\alpha^s(t)$, $t\in\mathcal{T}$, $s=1,2,\dots$, are piecewise continuous. For a fixed s, the control process in the original problem (1.1) can be realized by the control actions $u^s(t)$, $t\in\mathcal{T}$:

$$u^s(t)=u^0_+(t),\qquad t\in\bar{\mathcal{T}}_+(s),\qquad u^s(t)=u^0_+(t),\qquad t\in\bar{\mathcal{T}}_-(s),$$

where $\bar{\mathcal{T}}_+(s)=\{t\in\mathcal{T}:\alpha^s(t)=1\}$, $\bar{\mathcal{T}}_-(s)=\mathcal{T}\backslash\bar{\mathcal{T}}_+(s)$. The corresponding trajectory is a continuous solution of the system of differential equations

$$\dot{x}^s(t)=F_+(x^s(t),u^s(t)),\quad t\in\bar{\mathcal{T}}_+(s);\quad \dot{x}^s(t)=F_-(x^s(t),u^s(t)),\quad t\in\bar{\mathcal{T}}_-(s),\quad x^s(0)=x_0. \tag{3.5}$$

By construction, the solutions of systems (3.3) and (3.5) coincide.

Note that, in the general case for a fixed s, the constructed control and the trajectory $u^s(t)$, $x^s(t)$, $t \in \mathcal{T}$, are not admissible in problem (1.1), because in system (3.5) the switching from the function $F_+(x,u)$ to the function $F_-(x,u)$ can be realized in a neighborhood of the switching surface $d'x = 0$. But by (3.4), as $s \to \infty$, the trajectory $x^s(t)$, $t \in \mathcal{T}$, converges to an admissible trajectory of problem (2.2). Therefore, for sufficiently large $s \gg 1$, in system (3.5), the switching from the function $F_+(x,u)$ to the function $F_-(x,u)$ can be realized in the $\varepsilon(s)$-neighborhood of the switching surface, where $\varepsilon(s) \to 0$ as $s \to \infty$. This allows one to call the constructed control $u^s(t)$ and the trajectory $x^s(t)$, $t \in \mathcal{T}$, $\varepsilon(s)$-admissible controls in the original problem (1.1).

4. COMPARISON OF THE RESULTS WITH THE KNOWN ONES. EXAMPLE

The specific case of problem (1.1) where $b^-(x) = b^+(x)$ was studied in [18]. In this case, we have $\operatorname{conv} U(x) = U(x)$, and hence it is unnecessary to consider the weakened problem (1.4). Note that, in [18], the condition $b^-(x) = b^+(x)$ is significant, and hence the results of this paper cannot be used in problem (1.1) with $b^-(x) \neq b^+(x)$.

When considering optimal control problems for dynamical systems with discontinuous right-hand side, the Filippov rule was used in [7–9, 12] to redefine the solution on the switching surface, which leads to problem (1.2). In [10], the behavior of the dynamical system on the discontinuity surface is described by an arbitrary prescribed function.

It was shown in [12] that, in the case where there are parts of the trajectory on the switching surface, the optimality conditions stated in [7–10] are noninformative (degenerate), because they are realized for any admissible control. The main distinction of this paper from [12] is that, in [12], the control of original system (1.1) is constructed based on the solution of problem (1.2), and in the present paper, based on the solution of problem (2.2). The use of the solution of problem (2.2) instead of the solution of problem (1.2) has several significant advantages. We point out two of them:

(a) in problem (1.2), the dynamic equations are nonlinear in the control (α, u), and in problem (2.2), the dynamic equations are linear in the control (α, u_+, u_-);

(b) in the general case, the problem (1.2) may have no solution, and in the case where it has an optimal control, the optimal value of the performance criterion in this problem cannot be better than in problem (2.2).

Let us illustrate this with an example of problem (1.1) with the following data:

$$x \in \mathbb{R}^3, \qquad F_\pm(x,u) = Ax + b^\pm u, \qquad h(x) = x_1 - x_2 + 1, \qquad x(0) = (2,1,0), \qquad T = 4,$$

$$d = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad b^+ = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \qquad b^- = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{4.1}$$

For this problem, the weakened problem (2.2) becomes

$$\begin{gathered} x_3(4) \to \min, \\ \dot{x}(t) = Ax(t) + b^+u_+(t) + b^-u_-(t), \qquad x(0) = (2,1,0)', \qquad x_1(4) - x_2(4) = -1, \\ |u_+(t)| \le \alpha(t), \qquad |u_-(t)| \le 1 - \alpha(t), \qquad \alpha(t) \in [0,1], \\ \alpha(t)\, d'x(t) \ge 0, \qquad (1-\alpha(t))\, d'x(t) \le 0, \qquad t \in [0,4]. \end{gathered} \tag{4.2}$$

In problem (4.2), there exist many admissible controls, for example, the controls of the form

$$\begin{aligned} \alpha^*(t) &= 1, & u_+^*(t) &= 1, & u_-^*(t) &= 0, & t &\in [0,1), \\ \alpha^*(t) &= 1, & u_+^*(t) &= 0, & u_-^*(t) &= 0, & t &\in [1,3), \\ \alpha^*(t) &= 0, & u_+^*(t) &= 0, & u_-^*(t) &= 1, & t &\in [3,4], \end{aligned} \tag{4.3}$$

$$\begin{aligned} \bar{\alpha}(t) &= 1, & \bar{u}_+(t) &= 1, & \bar{u}_-(t) &= 0, & t &\in [0,1), \\ \bar{\alpha}(t) &= 0, & \bar{u}_+(t) &= 0, & \bar{u}_-(t) &= 1, & t &\in [1,3), \\ \bar{\alpha}(t) &= 0, & \bar{u}_+(t) &= 0, & \bar{u}_-(t) &= -1, & t &\in [3,4]. \end{aligned} \tag{4.4}$$

One can show the following.

1. In problem (4.2), the optimal control and the trajectory have the form

$$
\begin{aligned}
&\alpha^0(t)=1, \quad u_+^0(t)=1, \quad u_-^0(t)=0, \quad t\in[0,1),\\
&\alpha^0(t)=0.5, \quad u_+^0(t)=0.5, \quad u_-^0(t)=-0.5, \quad t\in[1,3),\\
&\alpha^0(t)=0, \quad u_+^0(t)=0, \quad u_-^0(t)=1, \quad t\in[3,4],\\
&x_1^0(t)=2-t, \quad x_2^0(t)=1, \quad t\in[0,1], \quad x_1^0(t)=1.5-t/2, \quad x_2^0(t)=1.5-t/2, \quad t\in[1,3],\\
&x_1^0(t)=0, \quad x_2^0(t)=-3+t, \quad t\in[3,4], \quad x_3^0(t)=\int\limits_0^t (x_1^0(\tau)+x_2^0(\tau))\,d\tau, \quad t\in[0,4],
\end{aligned}
\tag{4.5}
$$

and the optimal value of the performance criterion is $F_0(x^0(T))=5$.

2. The optimal control (4.5) in problem (4.2) satisfies the necessary (nondegenerate) optimality conditions stated in Theorem 2 with the following data:

$$
\begin{aligned}
&y_0=1, \quad y=2, \quad \gamma^1=2, \quad \gamma_1=0, \quad S^0(t)=A, \quad t\in[0,4],\\
&\psi(t)=(t-4-y,t-4+y,-1)', \quad t\in(3,4], \qquad \psi(t)=(t-4,t-4,-1)', \quad t\in[0,3].
\end{aligned}
$$

3. The admissible but nonoptimal controls (4.3), (4.4) in problem (4.2) do not satisfy the necessary conditions for optimality stated in Theorem 2.

4. For problem (1.1) with the data (4.1), necessary conditions for optimality given in [7–10] are degenerate, because they are satisfied for any admissible control provided that there are trajectory parts lying on the switching surface. One can readily verify that these conditions are, in particular, satisfied by all controls (4.3)–(4.5).

5. For problem (1.1) with the data (4.1) and the corresponding problem (1.2), the optimal control and the trajectory $x^*(t)$, $t\in\mathcal{T}$, exist and can readily be constructed from the control (4.3) admissible in problem (4.2). In this case, the optimal value of the performance criterion in problems (1.1) and (1.2) is $F_0(x^*(T))=9$, i.e., is greater by four units than the optimal value of the performance criterion $F_0(x^0(T))=5$ in the corresponding problem (2.2).

6. For each $s=1,2,\ldots$, using the optimal control (4.5) of weakened problem (4.2) and the rules described in the preceding section, we can readily construct $\varepsilon(s)$-admissible controls $u^s(t)$ and the trajectories $x^s(t)$, $t\in\mathcal{T}$, for which

$$
\lim_{s\to\infty}\varepsilon(s)=0, \qquad \lim_{s\to\infty}F_0(x^s(T))=F_0(x^0(T))=5<F_0(x^*(T))=9.
$$

Obviously, from the practical standpoint, it is preferable to use (for sufficiently large $s>0$) the $\varepsilon(s)$-admissible control $u^s(t)$ and the trajectory $x^s(t)$, $t\in\mathcal{T}$, than the admissible optimal solution $u^*(t)$, $x^*(t)$, $t\in\mathcal{T}$, but with a significantly worse performance criterion.

The above arguments show that the use of problem (2.2) as a problem approximating the original problem (1.1) is justified, and the necessary conditions for optimality obtained in this paper in the form of the nondegenerate maximum principle lead to some advance in studying and solving this problem.

5. APPENDIX A. NONDEGENERATE MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS

First, we consider several auxiliary assertions necessary to prove the above-formulated lemmas.

Consider the optimal control problem with state constraints-equalities and constraints-inequalities

$$
\begin{aligned}
&\min c(z(1)),\\
&\dot z(t)=f(z(t),v(t)), \quad \Phi(z(0),z(1))=0,\\
&G_0(z(t))=0, \qquad G_*(z(t))\le 0, \qquad \mathcal{D}v(t)\le\tilde b, \qquad t\in[0,1].
\end{aligned}
\tag{5.1}
$$

Here $z \in \mathbb{R}^{\bar n}$ is the state vector, v is the control vector, $c(z) \in \mathbb{R}$, $G_0(z) \in \mathbb{R}^{m_0}$, $G_*(z) \in \mathbb{R}^{m_*}$, $f(z,v) \in \mathbb{R}^{\bar n}$, and $\Phi(z,y) \in \mathbb{R}^m$ are given sufficiently smooth functions, the function $f(z,v)$ is linear in v; $\mathcal{D} = (d(i), i \in I)'$, $d(i) \in \mathbb{R}^s$, $i \in I$; $\tilde b = (\tilde b_i, i \in I)'$.

A pair $(v(t), z(t))$, $t \in [0,1]$, consisting of a control and the corresponding trajectory is called an *admissible process* in problem (5.1) if they satisfy all restrictions of this problem. We say that the admissible process $(v^0(t), z^0(t))$, $t \in [0,1]$, realizes a strong local minimum in problem (5.1) if there exists an $\varepsilon > 0$ such that, for any admissible process $(v(t), z(t))$, $t \in [0,1]$, satisfying the condition $\max_{t\in[0,1]} \|z^0(t) - z(t)\| \le \varepsilon$, the inequality $c(z^0(1)) \le c(z(1))$ holds.

Assume that the process $(v^0(t), z^0(t))$, $t \in [0,1]$, realizes a strong local minimum in problem (5.1). By some causes related to the proof of the lemma, it is necessary to consider a *degenerate* situation, namely, the situation where all constraints-inequalities of problem (5.1) are active on the trajectory $z^0(t)$, $t \in [0,1]$. In this situation, the conditions of the classical maximum principle degenerate and become noninformative.

The aim of this section is to show that, in the situation under study, for problem (5.1) one can prove the nondegenerate maximum principle, using Theorems 3 and 5 in [13].

Assume that the following conditions are satisfied for the process $(v^0(t), z^0(t))$, $t \in [0,1]$.

(I) The control $v^0(t)$, $t \in [0,1]$, is a piecewise smooth function.

(II) For all $t \in [0,1]$, $G_*(z^0(t)) = 0$ and

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial G_0(z^0(t))}{\partial z} B(z^0(t)) \\ \dfrac{\partial G_*(z^0(t))}{\partial z} B(z^0(t)) \\ \mathcal{D}_a(v) \end{pmatrix} = m_0 + m_* + |I_a(v)| \quad \text{for a.a.} \;\; v \in \mathcal{V}^*(t),$$

where $\mathcal{V}^*(t) = \{v^0(t-0), v^0(t+0)\}$, $t \in (0,1)$, $\mathcal{V}^*(0) = \{v^0(+0)\}$, $\mathcal{V}^*(1) = \{v^0(1-0)\}$, and

$$I_a(v) = \{i \in I : d'(i)v = \tilde b_i\}, \qquad \mathcal{D}_a(v) = \begin{pmatrix} d'(i) \\ i \in I_a(v) \end{pmatrix}.$$

(III) The following relations hold for the terminal and state constraints:

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial \Phi(z^0(0), z^0(1))}{\partial z(0)} & \dfrac{\partial \Phi(z^0(0), z^0(1))}{\partial z(1)} \\ \dfrac{\partial G_0(z^0(0))}{\partial z} & \mathbb{O} \\ \mathbb{O} & \dfrac{\partial G_0(z^0(1))}{\partial z} \end{pmatrix} = m + 2m_0, \qquad \operatorname{rank}\begin{pmatrix} \dfrac{\partial G_0(z^0(t))}{\partial z} \\ \dfrac{\partial G_*(z^0(t))}{\partial z} \end{pmatrix} = m_0 + m_*,$$

where $t \in [0,1]$, and there exists a number $\varepsilon_0 > 0$ such that

$$\begin{aligned} &\{(z,y) \in \mathbb{R}^{2\bar n} : \Phi(z,y) = 0, \;\; G_0(z) = G_0(y) = 0, \;\; \|(z,y) - (z^0(0), z^0(1))\| \le \varepsilon_0\} \\ &\qquad \subset \{(z,y) \in \mathbb{R}^{2\bar n} : G_*(z) \le 0, \;\; G_*(y) \le 0\}. \end{aligned}$$

Obviously, problem (5.1) is equivalent to the problem

$$\begin{aligned} &\min c(z(1)), \\ &\dot z(t) = f(z(t), v(t)), \qquad \tilde\Phi(z(0), z(1)) = 0, \\ &G_0(z(t)) = 0, \qquad G_*(z(t)) \le 0, \qquad \mathcal{D}v(t) \le \tilde b, \qquad t \in [0,1], \end{aligned} \tag{5.2}$$

where $\tilde\Phi(z,y) = (\Phi'(z,y), G_0'(z), G_0'(y))'$, $z \in \mathbb{R}^{\bar n}$, and $y \in \mathbb{R}^{\bar n}$.

It follows from assumptions (I)–(III) that all assumptions of Theorems 3 and 5 in [13] are satisfied for problem (5.2) and $\operatorname{cl} U_R(z^0(t)) = U(z^0(t))$ for any $t \in [0,1]$, where

$$U(z) := \left\{ v : \mathcal{D}v \le \tilde{b}, \quad \frac{\partial G_0(z)}{\partial z} f(z,v) = 0 \right\}$$

and $U_R(z)$ is the set of regular points of the set $U(z)$ (see Definition 4 in [13]).

The above considerations and Theorems 3 and 5 in [13] imply the following theorem.

Theorem 3. *Assume that the process* $(v^0(t), z^0(t))$, $t \in [0,1]$, *realizes a strong local minimum in problem* (5.1) *and conditions* (I)–(III) *are satisfied for it. Then there exists a number* $y_0 \ge 0$, *vectors* $\theta \in \mathbb{R}^m$, $\rho^* \in \mathbb{R}^{m_0}$, $\rho_* \in \mathbb{R}^{m_0}$, *and piecewise continuous functions* $\nu_0(t) \in \mathbb{R}^{m_0}$, $\nu_*(t) \in \mathbb{R}^{m_*}$, $\nu_*(t) \ge 0$, $t \in [0,1]$, *such that, along a solution* $\phi(t) \in \mathbb{R}^{\bar{n}}$, $t \in [0,1]$, *of the boundary value problem*

$$\begin{aligned}
\dot{\phi}'(t) &= -\phi'(t)\frac{\partial f(z^*(t), v^*(t))}{\partial z} - \nu_0'(t)\frac{\partial G_0(z^*(t))}{\partial z} + \nu_*'(t)\frac{\partial G_*(z^*(t))}{\partial z},\\
\phi'(1) &= -y_0\frac{\partial c(z^*(1))}{\partial z} - \theta'\frac{\partial \Phi(z^*(0), z^*(1))}{\partial z(1)} - \rho^{*\prime}\frac{\partial G_0(z^*(1))}{\partial z},\\
\phi'(0) &= \theta'\frac{\partial \Phi(z^*(0), z^*(1))}{\partial z(0)} + \rho_*'\frac{\partial G_0(z^*(0))}{\partial z},
\end{aligned} \tag{5.3}$$

the following conditions are satisfied:

$$y_0 > 0 \quad \text{or} \quad \mathcal{M}(\{t \in [0,1] : \|\phi(t)\| > 0\}) > 0, \tag{5.4}$$

$$\max_{\mathcal{D}v \le \tilde{b}} \phi'(t) f(z^0(t), v) = \phi'(t) f(z^0(t), v^0(t)) = 0 \quad \text{for a.a. } t \in [0,1]. \tag{5.5}$$

Here and below, $\mathcal{M}(\mathcal{A})$ denotes the Lebesgue measure of the set $\mathcal{A} \subset \mathbb{R}$.

6. APPENDIX B. PROOF OF THE LEMMA

For a fixed $m \ge 2$, we introduce the parameter vector $(t_1, t_2, \ldots, t_{2m})$ and consider the optimal control problem for the hybrid system

$$\min_{\alpha(\cdot), u_+(\cdot), u_-(\cdot), t_1, t_2, \ldots, t_{2m}} F_0(x(T)),$$

$$\begin{gathered}
\dot{x}(t) = f(x(t), 1, u_+(t), 0), \quad |u_+(t)| \le 1, \quad d'x(t) \ge 0, \quad t \in [0, t_1[,\\
\dot{x}(t) = f(x(t), \alpha(t), u_+(t), u_-(t)), \quad |u_+(t)| \le \alpha(t), \quad |u_-(t)| \le 1 - \alpha(t), \quad d'x(t) = 0, \quad t \in [t_{2i-1}, t_{2i}[,\\
\dot{x}(t) = f(x(t), 0, 0, u_-(t)), \quad |u_-(t)| \le 1, \quad d'x(t) \le 0, \quad t \in [t_{2i}, t_{2i+1}[, \quad i = 1, \ldots, m,\\
x(0) = x_0, \quad h(x(T)) = 0, \quad 0 = t_0 \le t_1 \le \cdots \le t_{2m} \le t_{2m+1} = T.
\end{gathered} \tag{6.1}$$

Let $t_1^0, t_2^0, \ldots, t_{2m+1}^0$ be the set of points satisfying the inequalities

$$\mu_1 = t_1^0 < t_2^0 < \cdots < t_{2m-1}^0 = \mu_2 < t_{2m}^0 = \mu_4 < t_{2m+1}^0 = \mu_5 \tag{6.2}$$

and let $(\alpha^0(t), u_+^0(t), u_-^0(t))$ and $x^0(t)$, $t \in \mathcal{T}$, be an optimal control and a trajectory of problem (2.2). One can readily show that $(t_1^0, t_2^0, \ldots, t_{2m+1}^0)$, $\alpha^0(\cdot) := (\alpha^0(t), t \in \mathcal{T})$, $u_+^0(\cdot) := (u_+^0(t), t \in \mathcal{T})$, $u_-^0(\cdot) := (u_-^0(t), t \in \mathcal{T})$, and $x^0(\cdot) := (x^0(t), t \in \mathcal{T})$ form on optimal solution of problem (6.1).

Let us now describe the main steps of the rather "technical" proof.

Step 1. We reduce problem (6.1) to the form (5.1). To this end, we divide the control interval $[0,T]$ into subintervals by points $t_0 \le t_1 \le \cdots \le t_{2m} \le t_{2m+1} = T$.

We write $J=\{1,2,\dots,2m+1\}$, $J_*=\{2i+1: i=1,\dots,m\}$; $J_\beta = J\setminus (J_*\cup\{1\})$,

$$\begin{aligned} z_i(\tau)&=x(t_{i-1}+\tau(t_i-t_{i-1})),\quad i\in J;\quad v_i(\tau)=u_+(t_{i-1}+\tau(t_i-t_{i-1})),\quad i\in J\setminus J_*;\\ w_i(\tau)&=u_-(t_{i-1}+\tau(t_i-t_{i-1})),\quad i\in J\setminus\{1\};\quad \beta_i(\tau)=\alpha(t_{i-1}+\tau(t_i-t_{i-1})),\quad i\in J_\beta; \end{aligned}\tag{6.3}$$

and form the extended state vector

$$Z(\tau)=(z_i(\tau),\ i=1,\dots,2m+1;\ t_i(\tau),\ i=1,\dots,2m)\in\mathbb{R}^{n(2m+1)+2m},\quad \tau\in[0,1],\tag{6.4}$$

and the extended control vector

$$\begin{aligned} V(\tau)&=(v_1(\tau),w_i(\tau),i\in J_*,V_i(\tau),i\in J_\beta)\in\mathbb{R}^{1+|J_*|+3|J_\beta|},\\ V_i(\tau)&=(v_i(\tau),w_i(\tau),\beta_i(\tau))\in\mathbb{R}^3,\qquad i\in J_\beta,\qquad \tau\in[0,1]. \end{aligned}\tag{6.5}$$

We use this notation to rewrite problem (6.1) as

$$\begin{gathered} \min_{Z,V} F_0(z_{2m+1}(1)),\\ \dot Z(\tau)=\mathcal{F}(Z(\tau),V(\tau)),\qquad \Phi(Z(0),Z(1))=0,\\ G_1'Z(\tau)\ge 0,\quad G_{2i}'Z(\tau)=0,\quad G_{2i+1}'Z(\tau)\le 0,\quad i=1,\dots,m;\\ |v_1(\tau)|\le 1,\quad |w_i(\tau)|\le 1,\quad i\in J_*;\quad DV_i(\tau)\le b,\quad i\in J_\beta,\\ t_i(\tau)\le t_{i+1}(\tau),\quad i=0,\dots,2m,\quad \tau\in[0,1]. \end{gathered}\tag{6.6}$$

Here $t_0(\tau)\equiv 0$, $t_{2m+1}(\tau)\equiv T$,

$$\begin{aligned} \mathcal{F}'(Z,V)=((t_1-t_0)f^+(z_1,v_1),(t_{2i}-t_{2i-1})f(z_{2i},\beta_{2i},v_{2i},w_{2i}),\\ (t_{2i+1}-t_{2i})f^-(z_{2i+1},w_{2i+1}),\ i=1,\dots,m,\ \underbrace{0,\dots,0}_{2m}), \end{aligned}$$

$$f^+(z,v):=(a^+(z)+b^+(z)v)=f(z,1,v,0),\qquad f^-(z,w):=a^-(z)+b^-(z)w=f(z,0,0,w),$$

the function $f(z,\beta,v,w)$ is defined by formula (2.1),

$$\Phi(Z(0),Z(1))=\begin{pmatrix} -z_1(0)+x_0\\ z_i(1)-z_{i+1}(0),\quad i=1,\dots,2m\\ h(z_{2m+1}(1)), \end{pmatrix},\quad D=\begin{pmatrix} 1&0&-1\\ -1&0&-1\\ 0&1&1\\ 0&-1&1 \end{pmatrix},\quad b=\begin{pmatrix}0\\0\\1\\1\end{pmatrix},$$

$$G_i'=(\underbrace{\mathbb{O}',\dots,\mathbb{O}'}_{i-1},d',\underbrace{\mathbb{O}',\dots,\mathbb{O}'}_{2m+1-i},\underbrace{0,\dots,0}_{2m}),\qquad i=1,\dots,2m+1,$$

where $\mathbb{O}\in\mathbb{R}^n$ is a zero vector.

We consider the set of points $t_1^0,t_2^0,\dots,t_{2m+1}^0$ satisfying inequalities (6.2) and let $Z^0(\tau)$, $V^0(\tau)$, $\tau\in[0,1]$, denote the functions (6.3)–(6.5) constructed by using this set and the optimal control $(\alpha^0(t),u_+^0(t),u_-^0(t))$ and the optimal trajectory $x^0(t)$, $t\in\mathcal{T}$, of problem (2.2).

It is obvious that $V^0(\tau)$ and $Z^0(\tau)$, $\tau\in[0,1]$, are the optimal control and the optimal trajectory of problem (6.6). By assumption (see (2.20)), we have

$$\begin{aligned} G_1'Z^0(\tau)>0,\qquad \tau\in[0,1),\qquad G_1'Z^0(1)=0,\qquad G_1'\dot Z^0(1-0)\ne 0,\\ G_{2m+1}'Z^0(\tau)<0,\qquad \tau\in(0,1],\qquad G_{2m+1}'Z^0(0)=0,\qquad G_{2m+1}'\dot Z^0(+0)\ne 0. \end{aligned}$$

Therefore, the process $(V^0(\tau),Z^0(\tau))$, $\tau\in[0,1]$, also realizes a strong local minimum in the problem obtained from problem (6.6) by eliminating the state constraints $G_1'Z(\tau)\ge 0$, $G_{2m+1}'Z(\tau)\le 0$,

$\tau \in [0,1]$, and the constraints $t_i(\tau) \le t_{i+1}(\tau)$, $i = 0, \ldots, 2m$, $\tau \in [0,1]$:

$$\begin{gathered} \min F_0(z_{2m+1}(1)), \\ \dot Z(\tau) = \mathcal{F}(Z(\tau), V(\tau)), \qquad \Phi(Z(0), Z(1)) = 0, \\ G'_{2i}(Z(\tau)) = 0, \qquad G'_{2i+1}Z(\tau) \le 0, \qquad i = 1, \ldots, m-1; \\ G'_{2m}(Z(\tau)) = 0, \qquad \mathcal{D}V(\tau) \le \tilde b, \qquad \tau \in [0,1]. \end{gathered} \tag{6.7}$$

Step 2. Problem (6.7) is a special case of problem (5.1). By assumptions (A)–(D) and relations (2.20), the process $(V^0(\tau), Z^0(\tau))$, $\tau \in [0,1]$, satisfies conditions (I)–(III). Therefore, Theorem 3 (see Appendix A) also holds for problem (6.7) and this process. By this theorem, there exists a number $y_0 \ge 0$, vectors $\theta = (\rho_1, \rho_1, \ldots, \rho_{2m+1}, y)$, $\rho_* = (\rho_{*i}, i = 1, \ldots, m)$, $\rho^* = (\rho^{*i}, i = 1, \ldots, m)$, $\rho_i \in \mathbb{R}^n$, $y \in \mathbb{R}^{\bar m}$, $\rho_{*i}, \rho^*_i \in \mathbb{R}$, and functions $\eta_i(\tau)$, $i = 2, \ldots, 2m$, $\tau \in [0,1]$, and

$$\phi(\tau) = (\phi_i(\tau) \in \mathbb{R}^n,\ i = 1, \ldots, 2m+1;\ \phi^*_i(\tau) \in \mathbb{R},\ i = 1, \ldots, 2m), \qquad \tau \in [0,1],$$

such that relations (5.3)–(5.5) are satisfied and

$$\eta_{2i+1}(\tau) \ge 0, \qquad i = 1, \ldots, m-1, \qquad \tau \in [0,1]. \tag{6.8}$$

Set

$$\begin{gathered} \beta^0_1(\tau) \equiv 1, \quad v^0_1(\tau) \equiv 0, \quad \beta^0_{2i+1}(\tau) \equiv 0, \quad w^0_{2i+1}(\tau) \equiv 0, \quad i = 1, \ldots, m, \quad \tau \in [0,1], \\ f^0_i(\tau) := f(z^0_i(\tau), \beta^0_i(\tau), v^0_i(\tau), w^0_i(\tau)), \qquad i = 1, \ldots, 2m+1, \qquad \tau \in [0,1], \end{gathered}$$

take the specific features of the functions $\mathcal{F}(Z, V)$ and $\Phi(Z(0), Z(1))$ into account, and write relations (5.3)–(5.5) as

$$\begin{gathered} \dot\phi'_1(\tau) = -\phi'_1(\tau)\frac{\partial f^+(z^0_1(\tau), v^0_1(\tau))}{\partial z}(t^0_1 - t^0_0), \\ \dot\phi'_{2m+1}(\tau) = -\phi'_{2m+1}(\tau)\frac{\partial f^-(z^0_{2m+1}(\tau), w^0_{2m+1}(\tau))}{\partial z}(t^0_{2m+1} - t^0_{2m}), \\ \dot\phi'_{2i}(\tau) = -\phi'_{2i}(\tau)\frac{\partial f(z^0_{2i}(\tau), \beta^0_{2i}(\tau), v^0_{2i}(\tau), w^0_{2i}(\tau))}{\partial z}(t^0_{2i} - t^0_{2i-1}) + d'\eta_{2i}(\tau), \quad i = 1, \ldots, m, \\ \dot\phi'_{2i+1}(\tau) = -\phi'_{2i+1}(\tau)\frac{\partial f^-(z^0_{2i+1}(\tau), w^0_{2i+1}(\tau))}{\partial z}(t^0_{2i+1} - t^0_{2i}) + d'\eta_{2i+1}(\tau), \qquad i = 1, \ldots, m-1, \\ \dot\phi^*_i(\tau) = -\phi'_i(\tau)f^0_i(\tau) + \phi'_{i+1}(\tau)f^0_{i+1}(\tau), \qquad i = 1, \ldots, 2m, \\ \phi^*_i(0) = \phi^*_i(1) = 0, \quad i = 1, \ldots, 2m; \end{gathered} \tag{6.9}$$

$$\phi_{2i-1}(0) = -\rho_{2i-1}, \quad \phi_{2i-1}(1) = -\rho_{2i}; \quad \phi_{2i}(0) = -\rho_{2i} + \rho_{*i}d, \quad \phi_{2i}(1) = -\rho_{2i+1} - \rho^*_i d, \quad i = 1, \ldots, m;$$

$$\phi_{2m+1}(0) = -\rho_{2m+1}, \quad \phi_{2m+1}(1) = -y_0\frac{\partial F'_0(z^0_{2m+1}(1))}{\partial x} - \frac{\partial h'(z^0_{2m+1}(1))}{\partial x}y, \tag{6.10}$$

$$\begin{gathered} \max_{|v|\le 1} \phi'_1(\tau)b^+(z^0_1(\tau))v = \phi'_1(\tau)b^+(z^0_1(\tau))v^0_1(\tau) \quad \text{for a.a. } \tau \in [0,1], \\ \max_{|v|\le\beta, |w|\le 1-\beta} \phi'_i(\tau)f(z^0_i(\tau)), \beta, v, w) = \phi'_i(\tau)f^0_i(\tau) \quad \text{for a.a. } \tau \in [0,1], \quad i \in J_\beta, \\ \max_{|w|\le 1} \phi'_i(\tau)b^-(z^0_i(\tau))w = \phi'_i(\tau)b^-(z^0_i(\tau))w^0_i(\tau) \quad \text{for a.a. } \tau \in [0,1], \quad i \in J_*, \end{gathered} \tag{6.11}$$

$$y_0 > 0 \quad \text{or} \quad \mathcal{M}\left(\left\{t \in [0,1] : \sum_{i=1}^{2m+1} \|\phi_i(\tau)\| + \sum_{i=1}^{2m} |\phi^*_i(\tau)| > 0\right\}\right) > 0. \tag{6.12}$$

Step 3. Analyzing relations (6.7), (6.9)–(6.11) with regard to (2.3) and (2.20), we can prove that

$$\phi_i^*(\tau) \equiv 0, \quad \tau \in [0,1]; \quad \phi_i'(1) f_i(1) = \phi_{i+1}'(0) f_{i+1}(0), \quad i = 1, \ldots, 2m;$$

$$\rho_{*i} = 0, \quad i = 2, \ldots, m; \quad \rho_i^* = 0, \quad i = 1, \ldots, m-1; \tag{6.13}$$

$$\eta_i(\tau) = \phi_i'(\tau) q_-^0(i,\tau)(t_i^0 - t_{i-1}^0), \quad \tau \in [0,1], \quad i = 2, \ldots, 2m-1,$$

$$\eta_{2m}(\tau) = \phi_{2m}'(\tau) q_*^0(2m,\tau)(t_{2m}^0 - t_{2m-1}^0), \quad \tau \in [0, \tau^*[,$$

$$\eta_{2m}(\tau) = \phi_{2m}'(\tau) q_-^0(2m,\tau)(t_{2m}^0 - t_{2m-1}^0), \quad \tau \in [\tau^*, 1], \tag{6.14}$$

$$\dot{\phi}_i'(\tau) = -\phi_i'(\tau) \bar{Q}_i^0(\tau)(t_i^0 - t_{i-1}^0), \quad i = 1, \ldots, 2m+1, \quad \tau \in [0,1],$$

$$\phi_2(0) = \phi_1(1) + \rho_{*1}, \quad \phi_{2m}(1) = \phi_{2m+1}(0) - \rho_m^* d, \quad \phi_i(1) = \phi_{i+1}(0), \quad i = 2, \ldots, 2m-1,$$

$$\phi_{2m+1}(1) = -y_0 \frac{\partial F_0'(z_{2m+1}^0(1))}{\partial x} - \frac{\partial h'(z_{2m+1}^0(1))}{\partial x} y. \tag{6.15}$$

Here $q_-^0(i,\tau) = q_-(z_i^0(\tau), \beta_i^0(\tau), v_i^0(\tau), w_i^0(\tau))$, $q_*^0(i,\tau) = q_*(z_i^0(\tau), \beta_i^0(\tau), v_i^0(\tau), w_i^0(\tau))$,

$$\begin{aligned}
\bar{Q}_1^0(\tau) &:= Q_+(z_1^0(\tau), \beta_1^0(\tau), v_1^0(\tau), w_1^0(\tau)),\\
\bar{Q}_i^0(\tau) &:= Q_-(z_i^0(\tau), \beta_i^0(\tau), v_i^0(\tau), w_i^0(\tau)), \quad i = 2, \ldots, 2m-1 \quad \text{and} \quad i = 2m+1, \quad \tau \in [0,1],\\
\bar{Q}_{2m}^0(\tau) &:= Q_*(z_{2m}^0(\tau), \beta_{2m}^0(\tau), v_{2m}^0(\tau), w_{2m}^0(\tau)), \quad \tau \in [0, \tau^*[,\\
\bar{Q}_{2m}^0(\tau) &:= Q_-(z_{2m}^0(\tau), \beta_{2m}^0(\tau), v_{2m}^0(\tau), w_{2m}^0(\tau)), \quad \tau \in [\tau^*, 1],
\end{aligned}$$

and the functions $q_-(z,\beta,v,w)$, $q_*(z,\beta,v,w)$, $Q_\pm(z,\beta,v,w)$, $Q_*(z,\beta,v,w)$ are defined by formulas (2.15) and (2.18).

Let us show that

$$y_0 + |\rho_{*1}| + |\rho_m^*| + \|y\| \neq 0.$$

Assume the contrary:

$$y_0 + |\rho_{*1}| + |\rho_m^*| + \|y\| = 0.$$

Then $y_0 = 0$, $\rho_{*1} = 0$, $\rho_m^* = 0$, $y = 0$ and relations (6.15) imply that $\phi_i(\tau) \equiv 0$, $\tau \in [0,1]$, $i = 1, \ldots, 2m+1$. But these identities and the identities in (6.13) contradict condition (6.12). The obtained contradiction proves that $y_0 + |\rho_{*1}| + |\rho_m^*| + \|y\| \neq 0$. Without loss of generality, we can assume that

$$y_0 + |\rho_{*1}| + |\rho_m^*| + \|y\| = 1. \tag{6.16}$$

Step 4. At Step 3, we have shown that if the process $(V^0(\tau), Z^0(\tau))$, $\tau \in [0,1]$, realizes a strong local minimum in problem (6.7), then there exist numbers y_0, ρ_{*1}, ρ_m^* and a vector y satisfying condition (6.16) such that relations (6.11), (6.13), (6.8), and (6.14) are satisfied along the solution $\phi_i(\tau)$, $i = 1, \ldots, 2m+1$, of the adjoint system (6.15).

We write

$$\psi(t) := \phi_i\left(\frac{t - t_{i-1}^0}{t_i^0 - t_{i-1}^0}\right), \quad t \in [t_{i-1}^0, t_i^0), \quad i = 1, \ldots, 2m+1; \qquad \gamma_1 = \rho_{*1}, \quad \gamma^1 = \rho_m^*.$$

It follows from (6.15) and (6.16) that the function $\psi(t)$, $t \in \mathcal{T}$, is a solution of system (2.21), (2.22) corresponding to the set of parameters $\gamma = (y_0, y, \gamma_1, \gamma^1)$, $\|\gamma\| = 1$. Relations (6.11), (6.13) and (6.8), (6.14) imply that conditions (2.24) are satisfied along this solution. The proof of the lemma is complete.

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