ISSN 0012-2661, Differential Equations, 2018, Vol. 54, No. 12, pp. 1535–1544. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © E.A. Barabanov, V.V. Bykov, M.V. Karpuk, 2018, published in Differentsial'nye Uravneniya, 2018, Vol. 54, No. 12, pp. 1579–1588.

ORDINARY DIFFERENTIAL EQUATIONS

Complete Description of the Lyapunov Spectra of Families of Linear Differential Systems Whose Dependence on the Parameter Is Continuous Uniformly on the Time Semiaxis

E. A. Barabanov[1*], V. V. Bykov[2], and M. V. Karpuk[1***]**

[1] *Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, 220072 Belarus*
[2] *Lomonosov Moscow State University, Moscow, 119991 Russia*
*e-mail: *bar@im.bas-net.by, **vvbykov@gmail.com, ***m.vasilitch@gmail.com*

Received April 25, 2018

Abstract—We consider families of n-dimensional ($n \geq 2$) linear differential systems on the time semiaxis with parameter varying in a metric space. For such families continuously depending on the parameter in the sense of uniform convergence on the time semiaxis, we completely describe the spectra of their Lyapunov exponents as vector functions of the parameter.

DOI: 10.1134/S0012266118120017

1. INTRODUCTION. STATEMENT OF THE PROBLEM

Let M be a metric space. For a given $n \in \mathbb{N}$, consider a family

$$\dot{x} = A(t,\mu)x, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}_+ \equiv [0,+\infty), \tag{1}$$

of linear differential systems with parameter $\mu \in M$ such that for each $\mu \in M$ the matrix-valued function $A(\cdot\,,\mu)\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ defined on the time semiaxis $\mathbb{R}_+$ is continuous and bounded (in general, by a constant depending on μ). Therefore, by taking any fixed $\mu \in M$ in the family (1), we obtain a linear differential system with continuous coefficients bounded on the semiaxis. We denote its Lyapunov exponents [1, p. 27; 2, p. 63] by $\lambda_1(\mu;A) \leq \cdots \leq \lambda_n(\mu;A)$, and hence we obtain a function $\lambda_k(\cdot\,;A)\colon M \to \mathbb{R}$ for each $k = 1,\ldots,n$, which we call the *kth Lyapunov exponent of the family* (1), and the vector function $\Lambda(\cdot\,;A)\colon M \to \mathbb{R}^n$ defined by the relation $\Lambda(\mu;A) = (\lambda_1(\mu;A),\ldots,\lambda_n(\mu;A))^{\mathrm{T}}$ which we call the *Lyapunov spectrum of the family* (1). (In this notation, the argument "A" indicates the dependence of the exponents on the choice of the family (1).)

If the mapping $A : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is not required to have any additional properties, then for an arbitrary function $M \to \mathbb{R}$ there obviously exists a family (1) whose exponent $\lambda_k(\cdot\,;A)$ coincides with that function. Therefore, it is meaningless to consider such families (1) without additional assumptions from the viewpoint of mathematics as well as practical applications, where these families in particular arise as the variational systems of nonlinear families continuously depending on a parameter in a certain sense.

One usually considers a family of matrix-valued mappings $A(\cdot\,,\mu)$, $\mu \in M$ (each of which is continuous and bounded on the semiaxis) under one of the following two natural assumptions: this family is continuous in either **(a)** the compact-open topology or **(b)** the uniform topology. Condition **(a)** is equivalent to saying that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points in M converges to a point μ_0, then the sequence of functions $A(\cdot\,,\mu_k)$ converges as $k \to +\infty$ to the function $A(\cdot\,,\mu_0)$ uniformly on each closed interval of the time semiaxis $\mathbb{R}_+$, and condition **(b)** means that this convergence is uniform on the whole semiaxis $\mathbb{R}_+$. The class of families (1) with matrix-valued function $A(\cdot\,,\cdot)$ jointly continuous in all the variables (and hence continuous in the above-mentioned sense in the compact-open topology) will be denoted by $\mathcal{C}^n(M)$, and the class of such functions continuous in the uniform topology will be denoted by $\mathcal{U}^n(M)$. The inclusion $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$ is obvious. Further, we identify the family (1) with the matrix-valued function $A(\cdot\,,\cdot)$ determining it, and hence we write $A \in \mathcal{C}^n(M)$ or $A \in \mathcal{U}^n(M)$.

Very simple examples show that, starting from $n = 1$, the functions $\lambda_k(\cdot\,; A)$, $k = 1, \dots, n$, for a bounded mapping $A \in \mathcal{C}^n([0,1])$, can be everywhere discontinuous. This is impossible for families $A \in \mathcal{U}^1(M)$; namely, the Lyapunov exponent of such a family is a continuous function of the parameter $\mu \in M$. Perron [3] (see also [4, pp. 13–15]) presented an example of a mapping $A \in \mathcal{U}^2([0,1])$ such that the function $\lambda_2(\cdot\,; A)$ is not upper semicontinuous. For the families introduced above, Millionshchikov [5] posed the problem of describing their Lyapunov exponents as functions of the parameter, in other words, the problem of describing the function classes

$$\Lambda_k(M; n, \mathcal{C}) = \{\lambda_k(\cdot\,; A) : A \in \mathcal{C}^n(M)\} \quad \text{and} \quad \Lambda_k(M; n, \mathcal{U}) = \{\lambda_k(\cdot\,; A) : A \in \mathcal{U}^n(M)\} \tag{2}$$

for any $n \in \mathbb{N}$, $k = 1, \dots, n$, and any metric space M. Millionshchikov showed that the natural language for such a description is the language of the Baire theory of discontinuous functions. Let us recall the following definition [6, p. 294].

The *Baire classes* with finite indices on a metric space M are defined by induction as follows. The zeroth Baire class is the set of continuous functions $M \to \mathbb{R}$. If the classes with numbers less than $k \in \mathbb{N}$ have already been defined, then the kth Baire class is the set of functions $M \to \mathbb{R}$ representable as the pointwise limit of a sequence of functions of the $(k-1)$st class.

Millionshchikov made a fundamentally important step towards the solution of the problem of describing the classes (2); namely, he proved [7] that for any metric space M and any family $A \in \mathcal{C}^n(M)$, each of the Lyapunov exponents $\lambda_k(\cdot\,; A)$ can be represented as the limit of a decreasing sequence of functions of the first Baire class and is in particular a function of the second Baire class on this space. This assertion was proved by Millionshchikov for substantially more general objects, i.e., for the Lyapunov exponents (introduced in [7]) of families of morphisms of metrized vector bundles. (The latter present a wide generalization of families of the class $\mathcal{C}^n(M)$.) Rakhimberdiev [8] proved that the second Baire class in this assertion cannot be replaced by the first Baire class even for families of the class $\mathcal{U}^n(M)$.

At the same time, the problem of complete description of the classes (2) remained unsolved until recently, and its solution was obtained in [9] and [10]. Before we present the corresponding results, recall that a function $f\colon M \to \mathbb{R}$ is called a function of the class $({}^*, G_\delta)$ [6, p. 267] if for each $r \in \mathbb{R}$ the preimage $f^{-1}([r, +\infty))$ of the half-interval $[r, +\infty)$ is a G_δ-set in the metric space M. Below, following [6, p. 264], we denote the set $f^{-1}([r, +\infty))$ by $[f \geq r]$.

Let $n \in \mathbb{N}$ and $k = 1, \dots, n$ be given. It was proved in [9] that a function $f\colon M \to \mathbb{R}$ belong to the class $\Lambda_k(M; n, \mathcal{C})$ if and only if it satisfies the following two conditions: (i) it is a function of the class $({}^*, G_\delta)$, and (ii) it has an upper semicontinuous minorant. (In [9], just as in [7], this assertion was proved for the Lyapunov exponents of a family of morphisms of metrized vector bundles.) It was proved in [10] that a function $f\colon M \to \mathbb{R}$ belongs to the class $\Lambda_k(M; n, \mathcal{U})$ for $n \geq 2$ if and only if it satisfies condition (i) and the following condition (ii′): it has a continuous minorant and a continuous majorant. We see from the statements given above that the descriptions of the classes $\Lambda_k(M; n, \mathcal{C})$ and $\Lambda_k(M; n, \mathcal{U})$ differ only slightly, but at the level of the proofs they are substantially different.

The description of the classes (2) follows from the somewhat more general problem of describing the classes

$$\Lambda(M; n, \mathcal{C}) = \{\Lambda(\cdot\,; A) : A \in \mathcal{C}^n(M)\} \quad \text{and} \quad \Lambda(M; n, \mathcal{U}) = \{\Lambda(\cdot\,; A) : A \in \mathcal{U}^n(M)\} \tag{3}$$

of vector functions for any $n \in \mathbb{N}$ and any metric space M. The class $\Lambda(M; n, \mathcal{C})$ was described in [9] for any $n \in \mathbb{N}$ and any metric space M. The problem of describing the class $\Lambda(M; n, \mathcal{U})$ for $n \geq 2$, which is precisely the problem studied in the present paper, was explicitly stated in [10]. For $n = 1$, such a description is obvious; namely, the class $\Lambda(M; 1, \mathcal{U}) = \Lambda_1(M; 1, \mathcal{U})$ consists of all continuous functions $M \to \mathbb{R}$. We point out that the description of the class $\Lambda(M; n, \mathcal{C})$ for any $n \in \mathbb{N}$ automatically follows from the description of the class $\Lambda(M; 1, \mathcal{C})$. Such a reduction to the case of

$n = 1$ is impossible for the class $\Lambda(M; n, \mathcal{U})$, because, as was already mentioned, the class $\Lambda(M; 1, \mathcal{U})$ coincides with the class of continuous real-valued functions on M. Therefore, the description of the classes $\Lambda(M; n, \mathcal{U})$, $n \geq 2$, is a separate, independent problem.

2. PRELIMINARIES

In this section, we construct a special linear differential system defined on an interval and depending on three positive real parameters b, c, and σ and one positive integer parameter k. Such systems will be used in the proof of the theorem to construct the desired family (1). In what follows, we denote the zero and identity $n \times n$ matrices by O_n and E_n, respectively.

We recursively define a sequence $(T_k)_{k \in \mathbb{N}_0}$ of integers by the relations $T_0 = 0$ and

$$T_{6m+i} = T_{6m+i-1} + \begin{cases} 2^{m+1} & \text{if} \quad i = 1, 3, 4, 6, \\ 1 & \text{if} \quad i = 2, 5, \end{cases} \qquad m \in \mathbb{N}_0.$$

The relation $T_{6k} = 4(2^{k+1} - 2) + 2k$, $k \in \mathbb{N}_0$, is obvious. To simplify further notation, we write $\Delta_k = [T_{6k}, T_{6(k+1)}]$ and $\Delta_k^i = [T_{6k+i-1}, T_{6k+i}]$, $k \in \mathbb{N}_0$, $i = 1, \ldots, 6$.

We fix some positive numbers c and b ($c > b$) and define the two-dimensional diagonal system

$$\dot{x} = B(t)x, \qquad x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2, \qquad t \in \Delta_k, \tag{4}$$

on the interval Δ_k with coefficient matrix

$$B(t) = \begin{cases} \operatorname{diag}[b, -c] & \text{for} \quad t \in \Delta_k^1 \bigsqcup \Delta_k^4, \\ \operatorname{diag}[-b, c] & \text{for} \quad t \in \Delta_k^3 \bigsqcup \Delta_k^6, \\ O_2 & \text{for the other} \quad t \in \Delta_k. \end{cases} \tag{5}$$

Let $x^1(\cdot)$ be the solution of system (4), (5) with the initial vector $x^1(T_{6k}) = (1, 0)^{\mathrm{T}}$, and let $x^2(\cdot)$ be the solution with the initial vector $x^2(T_{6k}) = (0, 1)^{\mathrm{T}}$. Since system (4), (5) is diagonal, one can readily verify the following properties. The norm $\|x^2(t)\|$, $t \in \Delta_k$, does not exceed 1, and it is equal to 1 only at the points $t = T_{6k}$, $t = T_{6k+3}$, and $t = T_{6k+6}$. The norm $\|x^1(t)\|$, $t \in \Delta_k$, is equal to $e^{b(t - T_{6k})}$ for $t \in \Delta_k^1$; on the interval Δ_k^2, it is constant and equal to $e^{2^{k+1}b}$; and on the interval Δ_k^3, it monotonically decreases to 1. On the intervals Δ_k^{j+3}, the behavior of the norm $\|x^1(t)\|$ is the same as on the intervals Δ_k^j, $j = 1, 2, 3$, respectively; namely, it is equal to $e^{b(t - T_{6k+3})}$ for $t \in \Delta_k^4$, is constant and equal to $e^{2^{k+1}b}$ on the interval Δ_k^5, and monotonically decreases to 1 on the interval Δ_k^6. In particular, if $X(t, \tau)$ is the Cauchy matrix of system (4), (5), then $X(T_{6k+3}, T_{6k}) = X(T_{6k+6}, T_{6k}) = E_2$, and

$$\max_{t \in \Delta_k} \frac{1}{t} \ln \|x^1(t)\| = \frac{2^{k+1}b}{T_{6k+1}} = \frac{2^{k+1}b}{4(2^{k+1} - 2) + 2k + 2^{k+1}} \underset{k \to +\infty}{\longrightarrow} \frac{b}{5} \quad \text{and} \quad \max_{t \in \Delta_k} \frac{1}{t} \ln \|x^2(t)\| = 0.$$

In what follows, we denote the matrix of system (4), (5) by $B[b, c; k](t)$.

Consider a system that is a perturbation of system (4), (5) by special exponentially small additional terms. We fix a positive number σ and define the perturbation matrix $Q[\sigma; k](\cdot)$ as

$$Q[\sigma; k](t) = \begin{cases} \begin{pmatrix} 0 & 0 \\ (-1)^i \exp\{-\sigma_i T_{6k+i}\} & 0 \end{pmatrix} & \text{for } t \in \Delta_k^i, \ i = 2, 5, \\ O_2 & \text{for the other } t \in \Delta_k, \end{cases}$$

where $\sigma_2 = \sigma$ and $\sigma_5 = \sigma T_{6k+2}/T_{6k+5}$. Obviously, for $k \geq 5$ we have the inequality

$$\sigma_5 = \sigma \frac{T_{6k+2}}{T_{6k+5}} = \sigma \frac{4(2^{k+1} - 2) + 2k + 2^{k+1} + 1}{4(2^{k+1} - 2) + 2k + 3 \cdot 2^{k+1} + 2} \geq \frac{5}{7}\sigma.$$

Let us study the behavior of solutions of the perturbed system

$$\dot{y} = (B[b,c;k](t) + Q[\sigma;k](t))y, \qquad y = (y_1, y_2)^{\mathrm{T}} \in \mathbb{R}^2, \qquad t \in \Delta_k. \tag{6}$$

Let $y^1(\cdot)$ be the solution of system (6) with the initial vector $y^1(T_{6k}) = (1,0)^{\mathrm{T}}$, and let $y^2(\cdot)$ be the solution with the initial vector $y^2(T_{6k}) = (0,1)^{\mathrm{T}}$. Obviously, $y^2(t) \equiv x^2(t)$ for all $t \in \Delta_k$.

Consider the behavior of the solution $y^1(t)$ on each of the intervals $t \in \Delta_k^i$, $i = 1,\dots,6$. For $t \in \Delta_k^1$, the identity $y^1(t) \equiv x^1(t)$ is obvious. For $t \in \Delta_k^2$, we have

$$y^1(t) = \begin{pmatrix} 1 & 0 \\ (t - T_{6k+1})e^{-\sigma T_{6k+2}} & 1 \end{pmatrix} \begin{pmatrix} e^{2^{k+1}b} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ (t - T_{6k+1})e^{-\sigma T_{6k+2}} \end{pmatrix} e^{2^{k+1}b}.$$

Therefore,

$$y^1(T_{6k+2}) = (1, e^{-\sigma T_{6k+2}})^{\mathrm{T}} e^{2^{k+1}b}.$$

Then for $t \in \Delta_k^3$ we have

$$y^1(t) = \begin{pmatrix} e^{-b(t-T_{6k+2})} & 0 \\ 0 & e^{c(t-T_{6k+2})} \end{pmatrix} \begin{pmatrix} 1 \\ e^{-\sigma T_{6k+2}} \end{pmatrix} e^{2^{k+1}b} = \begin{pmatrix} e^{-b(t-T_{6k+2})} \\ e^{-\sigma T_{6k+2} + c(t-T_{6k+2})} \end{pmatrix} e^{2^{k+1}b}.$$

Therefore,

$$y^1(T_{6k+3}) = (1, e^{-\sigma T_{6k+2} + (b+c)2^{k+1}})^{\mathrm{T}}.$$

Further for $t \in \Delta_k^4$ we have

$$y_1(t) = \begin{pmatrix} e^{b(t-T_{6k+3})} & 0 \\ 0 & e^{-c(t-T_{6k+3})} \end{pmatrix} \begin{pmatrix} 1 \\ e^{-\sigma T_{6k+2} + 2^{k+1}(b+c)} \end{pmatrix} = \begin{pmatrix} e^{b(t-T_{6k+3})} \\ e^{-\sigma T_{6k+2} + 2^{k+1}(b+c) - c(t-T_{6k+3})} \end{pmatrix}.$$

Therefore,

$$y^1(T_{6k+4}) = (1, e^{-\sigma T_{6k+2}})^{\mathrm{T}} e^{2^{k+1}b}.$$

For $t \in \Delta_k^5$, we have

$$\begin{aligned} y^1(t) &= \begin{pmatrix} 1 & 0 \\ -(t - T_{6k+4})e^{-\sigma_5 T_{6k+5}} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ e^{-\sigma T_{6k+2}} \end{pmatrix} e^{2^{k+1}b} \\ &= \begin{pmatrix} 1 \\ -(t - T_{6k+4})e^{-\sigma_5 T_{6k+5}} + e^{-\sigma T_{6k+2}} \end{pmatrix} e^{2^{k+1}b}. \end{aligned}$$

Therefore, $y^1(T_{6k+5}) = (1,0)^{\mathrm{T}} e^{2^{k+1}b}$. Finally, for $t \in \Delta_k^6$ we have

$$y^1(t) = \begin{pmatrix} e^{-b(t-T_{6k+5})} & 0 \\ 0 & e^{c(t-T_{6k+5})} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2^{k+1}b} = \begin{pmatrix} e^{-b(t-T_{6k+5})} \\ 0 \end{pmatrix} e^{2^{k+1}b}.$$

Therefore, $y^1(T_{6k+6}) = (1,0)^{\mathrm{T}}$.

Thus, if $Y(t,\tau)$ is the Cauchy matrix of system (6), then

$$Y(T_{6k+6}, T_{6k}) = E_2, \qquad \max_{t \in \Delta_k} t^{-1} \ln \|y^2(t)\| = 0,$$

and if $\sigma \le (5c-b)/25$, then

$$\lim_{k\to+\infty}\max_{t\in\Delta_k}\frac{1}{t}\ln\|y^1(t)\| = \lim_{k\to+\infty}\frac{1}{T_{6k+3}}\ln\|y^1(T_{6k+3})\|$$
$$= \lim_{k\to+\infty}\frac{(b+c)2^{k+1}-\sigma(4(2^{k+1}-2)+2k+2^{k+1}+1)}{4(2^{k+1}-2)+2k+2\cdot 2^{k+1}+1} = \frac{b+c-5\sigma}{6}. \quad (7)$$

Let us prove relation (7). Since its left-hand side is independent of the choice of the norm, we take the maximum of absolute values of its components for the norm. Let $y^1(t) = (y_1^1(t), y_2^1(t))^{\mathrm{T}}$. It follows from the above calculations that any of the functions $\varphi_m(t) \stackrel{\text{def}}{=} t^{-1}\ln|y_m^1(t)|$, $m=1,2$, is a monotone linear fractional function on each interval Δ_k^i, $i=1,3,4,6$, and hence its maximum is attained at one of the endpoints of this interval. Calculating the values of the functions $\varphi_m(t)$, $m=1,2$, at the points T_{6k+j}, $j=1,\dots,6$, and taking the maximum of these values, we see that since $\sigma\le(5c-b)/25$, it follows that the limit of these maximum values is equal to the right-hand side of (7). It remains to take into account the fact that the values of the functions $\varphi_m(t)$ on the intervals Δ_k^i, $i=2,5$, cannot change this limit, because the ratios T_{6k+2}/T_{6k+1} and T_{6k+5}/T_{6k+4} tend to 1 as $k\to+\infty$.

3. MAIN RESULT

A complete description of each of the classes $\Lambda(M;n,\mathcal{U})$, $n\ge 2$, is given by the following theorem announced in the report [11].

Theorem. *For any positive integer $n\ge 2$ and any metric space M, the vector function $f(\cdot) = (f_1(\cdot),\dots,f_n(\cdot))^{\mathrm{T}}\colon\ M\to\mathbb{R}^n$ is the Lyapunov spectrum of some family $A\in\mathcal{U}^n(M)$ if and only if it satisfies the following conditions.*

(i) *The inequalities $f_1(\mu)\le\dots\le f_n(\mu)$ hold for each $\mu\in M$.*

(ii) *The function $f_i(\cdot)\colon M\to\mathbb{R}$ is a function of the class $({}^*,G_\delta)$, $i=1,\dots,n$.*

(iii) *The function $f_1(\cdot)$ has a continuous minorant, and the function $f_n(\cdot)$ has a continuous majorant.*

To prove the theorem, we, in particular, need the following assertion proved in [12, 13] and independently in [9].

Lemma. *Let $(P,\prec)$ be an at most countable linearly ordered set, and let $\{A_\rho : \rho\in P\}$ be a collection of G_δ-sets in a topological space τ such that $A_\lambda\subset A_\mu$ if $\mu\prec\lambda$. Then there exists a representation $A_\rho=\bigcap_{k\in\mathbb{N}}G_k^\rho$ of each of the sets A_ρ, where G_k^ρ is an open subset of τ such that every $a\notin A_\lambda$ belongs to at most finitely many sets in the collection $\{G_k^\rho : k\in\mathbb{N},\ \rho\succeq\lambda\}$.*

One can readily find examples showing that this lemma is not true in general if the set P is uncountable.

Proof of the theorem. Necessity of the assertion of the theorem was proved in [10].

Sufficiency. 1. We denote a continuous minorant of the function $f_1(\cdot)$ by $\underline{m}(\cdot)$ and a continuous majorant of the function $f_n(\cdot)$ by $\overline{m}(\cdot)$. We define functions $F_k(\cdot)\colon M\to\mathbb{R}$, $k=1,\dots,n$, by the formulas $F_k(\mu)=f_k(\mu)-\underline{m}(\mu)+1$ for all $\mu\in M$. Then, as is easy to see, the vector function $F(\cdot)=(F_1(\cdot),\dots,F_n(\cdot))^{\mathrm{T}}\colon M\to\mathbb{R}^n$ satisfies conditions (i)–(iii). Indeed, the inequalities $F_1(\mu)\le\dots\le F_n(\mu)$ are obvious for any $\mu\in M$; the fact that the function $F_i(\cdot)\colon M\to\mathbb{R}$ is a function of the class $({}^*,G_\delta)$ for each $i=1,\dots,n$ follows from [6, p. 267]; moreover, the function $F_1(\cdot)$ has a continuous minorant $m_*(\cdot)$ (identically equal to 1), and the function $F_n(\cdot)$ has a continuous majorant $m^*(\cdot)$ (equal to $\overline{m}(\cdot)-\underline{m}(\cdot)+1$).

In part 4 of the proof of sufficiency, we construct a family in $\mathcal{U}^n(M)$ whose Lyapunov spectrum coincides with the vector function $F(\cdot)$. Then we apply a standard method (see part 5) to this family to obtain the desired family.

2. In this part of the proof of sufficiency, we present some auxiliary constructions to be used below.

By $\mathbb{Q}_1$ we denote the set of rational numbers on the half-interval $[1,+\infty)$. We arbitrarily enumerate the elements of the set $\mathbb{Q}_1 : r_1,\ r_2,\ \ldots,$ and fix this enumeration. We assume that the natural order "<" is used to arrange the elements of the set $\mathbb{Q}_1$. Since the function $F_k(\cdot)$ is a function of the class $({}^*,G_\delta)$, we see that for each $m\in\mathbb{N}$ the set $G_{k,m}\overset{\text{def}}{=}[F_k\ge r_m]$ is a (possibly, empty) G_δ-set. By the above lemma, for nonempty sets $G_{k,m}$ there exist representations

$$G_{k,m}=\bigcap_{i\in\mathbb{N}}G^i_{k,m},\qquad k=1,\ldots,n,\qquad m\in\mathbb{N},$$

where the $G^i_{k,m}$, $(k,m,i)\in\{1,\ldots,n\}\times\mathbb{N}^2$, are open sets such that every point $\mu\in M\setminus G_{k,m}$ belongs to at most finitely many sets $G^i_{k,p}$ with $r_p>r_m$.

In turn, each set $G^i_{k,m}$ can be represented as the union

$$G^i_{k,m}=\bigcup_{l\in\mathbb{N}}J^i_{k,m}(l),$$

where $J^i_{k,m}(l)$ is the set of points $\mu\in G^i_{k,m}$ whose distance from the boundary $\operatorname{Fr}G^i_{k,m}$ of the set $G^i_{k,m}$ is not less than 1 for $l=1$ and belongs to the interval $[l^{-1},(l-1)^{-1}]$ for $l>1$. Note that for each pair (k,m) there exist infinitely many $l\in\mathbb{N}$ such that $J^i_{k,m}(l)$ is nonempty. Since the distance from a subset of a metric space is a continuous function on that space [14, p. 209], we see that the sets $J^i_{k,m}(l)$ are closed as the preimages of closed sets. Since the set $G_{k,m}$ is open, it follows that the distance from any of its points to the boundary $\operatorname{Fr}G_{k,m}$ is positive, and hence the family $J^i_{k,m}(l)$, $l\in\mathbb{N}$, is a cover of the set $G^i_{k,m}$; by construction, each point of the set $G^i_{k,m}$ is covered at most twice. In addition, let $\widetilde{J}^i_{k,m}(l)$ be the set of points $\mu\in G^i_{k,m}$ whose distance from the boundary $\operatorname{Fr}G^i_{k,m}$ is greater than 0.5 for $l=1$ and belongs to the interval $((l+0.5)^{-1},(l-1.5)^{-1})$ for $l>1$. By construction, the sets $\widetilde{J}^i_{k,m}(l)$ are open, each point $\mu\in G^i_{k,m}$ belongs to at most three of them, and the inclusions $J^i_{k,m}(l)\subset\widetilde{J}^i_{k,m}(l)\subset G^i_{k,m}$ hold.

We separately note the case in which the boundary $\operatorname{Fr}G^i_{k,m}$ of the set $G^i_{k,m}$ is empty. This means that the set $G^i_{k,m}$ is open and closed simultaneously, and in this case we define the sets $J^i_{k,m}(l)$ and $\widetilde{J}^i_{k,m}(l)$ by the relations $J^i_{k,m}(1)=\widetilde{J}^i_{k,m}(1)=G^i_{k,m}$ and $J^i_{k,m}(l)=\widetilde{J}^i_{k,m}(l)=\varnothing$ for $l>1$. By Ind we denote the set of quadruples $(k,m,i,l)\in\{1,\ldots,n\}\times\mathbb{N}^3$ such that $J^i_{k,m}(l)\ne\varnothing$.

Let $(k,m,i,l)\in\mathrm{Ind}$. If $\operatorname{Fr}G^i_{k,m}\ne\varnothing$, then the set $J^i_{k,m}(l)$ is nonempty and closed. By the Urysohn lemma [14, p. 126], there exists a continuous function $\varphi^{i,l}_{k,m}\colon M\to\mathbb{R}$ equal to 1 on the closed set $J^i_{k,m}(l)$ and zero on the closed set $M\setminus\widetilde{J}^i_{k,m}(l)$ and taking intermediate values on the difference $\widetilde{J}^i_{k,m}(l)\setminus J^i_{k,m}(l)$. In the case of empty boundary $\operatorname{Fr}G^i_{k,m}$, we put the function $\varphi^{i,1}_{k,m}$ to be equal to 1 on the set $J^i_{k,m}(1)=G^i_{k,m}$ and zero on the set $M\setminus G^i_{k,m}$. This function is continuous, because $G^i_{k,m}$ is an open-closed set.

3. Fix an arbitrary bijection $o\colon\mathbb{N}\to\mathrm{Ind}$.

First, we construct a family

$$\dot{x}=C(t,\mu)x,\qquad x=(x_1,\ldots,x_n)^{\mathrm{T}}\in\mathbb{R}^n,\qquad t\ge 0,\tag{8}$$

such that the family itself is bounded and piecewise continuous for each fixed $\mu\in M$, all points of discontinuity are contained in the set $\{T_{6k+j}:k\in\mathbb{N},\ j=1,\ldots,6\}$, and the Lyapunov spectrum $\Lambda(\cdot;C)$ coincides with the vector function $F(\cdot)$. We construct the family (8) by induction. At the qth inductive step ($q\in\mathbb{N}_0$), the family (8) is constructed on the half-interval $[T_{6k},T_{6(k+1)})$, and it does not change on this half-interval at the subsequent steps. To have the base of induction, we take the zeroth step at which we set $C(t,\mu)\equiv O_n$ for $t\in[0,T_6]$. Assume that we have taken $q-1$ steps; i.e., the family (8) has been defined for all $t\in[0,T_{6q}]$.

Let us take the qth step. Let $o(q) = (k, m, i, l)$. Then for $t \in [T_{6q}, T_{6(q+1)})$ the family (8) coincides with the family

$$\dot{x}_j = 0 \quad \text{for} \quad j \notin \{k, k+1\}, \quad \text{and} \quad \begin{pmatrix} \dot{x}_k \\ \dot{x}_{k+1} \end{pmatrix} = (B[1, 6m^*(\mu); q](t) + Q[\sigma_q(\mu), q](t)) \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}, \quad (9)$$

where $\sigma_q(\mu) = (1 + 6m^*(\mu) - 6\varphi_{k,m}^{i,l}(\mu) \min\{r_m, m^*(\mu)\})/5$ and we also assume that $x_{n+1} = x_1$. The qth step is complete. The family (8) has been constructed.

In other words, definition (9) can be stated as follows: let $C_0(t, \mu)$ and $Q(t, \mu)$ denote $n \times n$ matrices such that, for $t \in [T_{6q}, T_{6(q+1)})$ and $q \in \mathbb{N}$, their 2×2 submatrices located on the intersection of the kth and $(k+1)$st rows and columns (the $(n+1)$th row and column are the first row and column) coincide with the matrices $B[1, 6m^*(\mu); q](t)$ and $Q[\sigma_q(\mu); q](t)$, respectively, and their other entries are zero; then the identity $C(t, \mu) \equiv C_0(t, \mu) + Q(t, \mu)$ holds for $(t, \mu) \in [T_6, +\infty) \times M$. We set $C_0(t, \mu) = Q(t, \mu) = O_n$ for $t \in [0, T_6)$ and find that this identity holds for all $(t, \mu) \in \mathbb{R}_+ \times M$.

Note that the numbers $\sigma_q(\mu)$ are separated from zero by the number $1/5$ for all $q \in \mathbb{N}$ and $\mu \in M$. Indeed, $\sigma_q(\mu) = (1 + 6m^*(\mu) - 6\varphi_{k,m}^{i,l}(\mu) \min\{r_m, m^*(\mu)\})/5 \geq (1 + 6m^*(\mu) - 6m^*(\mu))/5 = 1/5$. Therefore,

$$\|Q(t, \mu)\| \leq \exp(-t/7) \quad \text{for all} \quad t \geq 0. \quad (10)$$

Let us calculate the Lyapunov exponents of system (8). By e_k we denote the vector in $\mathbb{R}^n$ whose kth component is equal to 1 and the other components are zero. For $k \in \{1, \ldots, n\}$, by N_k we denote the set of positive integers q for which the first element in the quadruple $o(q)$ is equal to k. Let $\Delta^{(k)} = \{[T_{6q}, T_{6(q+1)}] : q \in N_k\}$. We will calculate the Lyapunov exponent of the solution $x^k(\cdot)$ issuing at time $t = 0$ from the vector e_k. If $q \notin N_k$, then the inequality $\|x^k(t)\| \leq 1$ holds on the entire interval Δ_q, and therefore,

$$t^{-1} \ln \|x^k(t)\| \leq 0 \quad \text{for} \quad t \in \bigcup_{s \in \{1,\ldots,n\} \setminus \{k\}} \Delta^{(s)}.$$

Let $t \in \Delta^{(k)}$. We fix $\mu \in M$ and an arbitrary strictly increasing sequence of rational numbers $(r_{m_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to+\infty} r_{m_j} = F_k(\mu)$. Then $\mu \in [F_k \geq r_{m_j}]$ for all $j \in \mathbb{N}$, and hence for each $j \in \mathbb{N}$ there exists a pair (i_j, l_j) such that $\mu \in J_{k,m_i}^{i_j}(l_j)$, whence for q_j such that $o(q_j) = (k, m_j, i_j, l_j)$ we obtain, by (7) and the definition of system (8),

$$\sup_{t\in\Delta_{q_j}} t^{-1} \ln \|x^k(t)\| = \varphi_{k,m_j}^{i_j,l_j}(\mu) \min\{r_{m_j}, m^*(\mu)\} + \varepsilon_j = r_{m_j} + \varepsilon_j,$$

where $\varepsilon_j \to 0$ as $j \to +\infty$. Therefore, $\lambda[x^k] \geq F_k(\mu)$.

Let us prove that the last inequality is in fact an equality. Assume the contrary: $\lambda[x^k] > F_k(\mu)$ for a point $\mu \in M$. This means that there exists a rational number $r > F_k(\mu)$ and sequences $(m_j)_{j\in\mathbb{N}}$, $(i_j)_{j\in\mathbb{N}}$, and $(l_j)_{j\in\mathbb{N}}$ of positive integers such that $r_{m_j} \geq r$ and $\mu \in \widetilde{J}_{k,m_j}^{i_j}(l_j)$ for all $j \in \mathbb{N}$. But this is impossible. Indeed, by the lemma, the point μ can only belong to finitely many sets $G_{k,m}$, $m \in \mathbb{N}$, $r_m > r$; in turn, each point of the set $G_{k,m}$ belongs to at most three sets $\widetilde{J}_{k,m}^{i}(l)$, $l \in \mathbb{N}$. The proof of the equality is complete.

Let us show that the basis $(x^1(\cdot), \ldots, x^n(\cdot))$ of solutions of system (8) is normal. To this end, note that if T is an unbounded subset of the time semiaxis $\mathbb{R}_+$, then the functional

$$\chi[x] = \varlimsup_{T\ni t\to+\infty} t^{-1} \ln \|x(t)\|$$

defined on the linear space $\{x\}$ of solutions of the linear differential system is a Lyapunov exponent on this space in the sense of [2, Sec. 2.1] and, in particular, does not increase when taking linear combinations. For each $k = 1, \ldots, n$, let $\chi_k[f]$ be the characteristic exponent of the restriction of a function $f : \mathbb{R}^+ \to \mathbb{R}^n$ to the set $\Delta^{(k)}$, i.e., the number

$$\chi_k[f] \stackrel{\text{def}}{=} \varlimsup_{\substack{t\to+\infty \\ t\in\Delta^{(k)}}} t^{-1} \ln \|f(t)\|.$$

Then, as was stated above, the functional $\chi_k[\cdot]$ is a Lyapunov exponent on the space of solutions of system (8) in the sense of [2, Sec. 2.1] and does not increase when taking linear combinations. By construction, $\chi_k[x^k] = F_k(\mu) \ge 1$ and $\chi_k[x^i] = 0$ for $i \in \{1, \dots, n\}\setminus\{k\}$. Therefore, for each set of numbers $\alpha_1, \dots, \alpha_k$, $\alpha_k \ne 0$ we obtain

$$\lambda\left[\sum_{i=1}^{k} \alpha_i x^i\right] \ge \chi_k\left[\sum_{i=1}^{k} \alpha_i x^i\right] = \chi_k[x^k] = F_k(\mu) = \lambda[x^k],$$

which implies that no subset of the set of solutions $\{x^1(\cdot), \dots, x^k(\cdot)\}$ admits a decreasing combination. By [2, Sec. 2.3.10], the basis $(x^1(\cdot), \dots, x^n(\cdot))$ is normal, and the exponents of its solutions are the exponents of system (8). Thus, $\Lambda(\mu; C) = F(\mu)$ for any $\mu \in M$.

Note that, by construction, the matrix $B[1, 6m^*(\mu); q](t)$ satisfies the condition of uniform convergence in the parameter on the time semiaxis: if $\mu_k \to \mu_0$ as $k \to +\infty$, then $B[1, 6m^*(\mu_k); q](t) \to B[1, 6m^*(\mu_0); q](t)$ as $k \to +\infty$ uniformly in $t \in \mathbb{R}_+$. Indeed, the matrix $B[1, 6m^*(\mu); q](t)$ is constant on each interval Δ_k, $k \in \mathbb{N}_0$, and its coefficients take at most five values on each of such intervals, namely, either 0, or 1, or -1, or $6m^*(\mu)$, or $-6m^*(\mu)$. Since the function $m^*(\cdot)$ is continuous, the convergence $\mu_k \to \mu_0$ implies the convergence $m^*(\mu_k) \to m^*(\mu_0)$ as $k \to +\infty$, and hence if $|m^*(\mu_k) - m^*(\mu_0)| \le \varepsilon$ for $k \ge N_\varepsilon$, then $\|B[1, 6m^*(\mu_k); q](t) - B[1, 6m^*(\mu_0); q](t)\| \le \varepsilon$ for all $t \in \mathbb{R}_+$, or, which is equivalent, $\|C_0(t, \mu_k) - C_0(t, \mu_0)\| \le \varepsilon$ for all $t \in \mathbb{R}_+$.

Since $C(t, \mu) = C_0(t, \mu) + Q(t, \mu)$ and $\|Q(t, \mu)\| \le \exp(-t/7)$ by inequality (10), we have the convergence $C(t, \mu_k) \to C(t, \mu_0)$ uniformly on $\mathbb{R}_+$ as $\mu_k \to \mu_0$.

4. Let us show that there exists a family

$$\dot{x} = B(t, \mu)x, \qquad x \in \mathbb{R}^n, \qquad t \ge 0, \tag{11}$$

of the class $\mathcal{U}^n(M)$ whose Lyapunov spectrum $\Lambda(\cdot\,; B)$ coincides with the vector function $F(\cdot)$. As follows from the Bogdanov–Grobman theorem [15, 16], the Lyapunov exponents of the systems

$$\dot{x} = C(t)x \quad \text{and} \quad \dot{x} = (C(t) + R(t))x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+,$$

coincide if the integral $\int_0^{+\infty} \|R(\tau)\| \exp(\tau^2)\, d\tau$ converges.

As follows from the construction of the matrix $C(t, \mu)$, the points of discontinuity of this matrix are contained in the set $\{T_m : m \in \mathbb{N}\}$. Let $\delta_m = [\delta_{m,1}, \delta_{m,2}]$ be an interval centered at the point T_m of length $|\delta_m| = 2^{-m} \exp(-\delta_{m,2}^2)$. We set the matrix $R(t)$ to be equal to O_n for $t \in \mathbb{R}_+ \setminus \bigcup_{m\in\mathbb{N}} \delta_m$ and to $-C(t, \mu) + |\delta_m|^{-1}(C(\delta_{m,1}, \mu)(\delta_{m,2} - t) + C(\delta_{m,2}, \mu)(t - \delta_{m,1}))$ for $t \in \delta_m$, $m \in \mathbb{N}$. Then the matrix $B(t, \mu) \stackrel{\text{def}}{=} C(t, \mu) + R(t, \mu)$ is obviously jointly continuous with respect to its arguments, is bounded for all $t \ge 0$ for each $\mu \in M$, and hence belongs to the class $\mathcal{U}^n(M)$. Since the inequality

$$\int_0^{+\infty} \|R(\tau, \mu)\| \exp(\tau^2)\, d\tau \le \sum_{m=1}^{\infty} 18m^*(\mu)|\delta_m| \exp(\delta_{m,2}^2) = 18m^*(\mu)$$

holds for each $\mu \in M$, it follows that the Lyapunov spectra of the families (8) and (11) coincide.

Thus, family (11) belongs to the class $\mathcal{U}^n(M)$, and its Lyapunov spectrum coincides with the vector function $F(\cdot)$.

5. Finally, we construct a family $A \in \mathcal{U}^n(M)$ whose Lyapunov spectrum coincides with the vector function $f(\cdot)$. We use the following well-known assertion: the Lyapunov exponents $\lambda_k(B)$ and $\lambda_k(B + aE_n)$, $k = 1, \dots, n$, of the respective systems

$$\dot{x} = B(t)x \quad \text{and} \quad \dot{y} = (B(t) + aE_n)y, \quad x, y \in \mathbb{R}^n, \quad t \in \mathbb{R}_+,$$

where $a \in \mathbb{R}$ is fixed, are related as $\lambda_k(B + aE_n) = \lambda_k(B) + a$ which follows from the fact that the identity $x(t) \equiv y(t)\exp(-at)$, $t \in \mathbb{R}_+$, holds for the solutions $x(t)$ and $y(t)$ of these systems with the same initial vector $(x(0) = y(0))$.

Set $A(t,\mu) = B(t,\mu) + (\underline{m}(\mu) - 1)E_n$, $t \in \mathbb{R}_+$, $\mu \in M$. Then it follows from the above and the already proved identity $\Lambda(\mu; B) = F(\mu)$ for all $\mu \in M$ that the Lyapunov spectrum $\Lambda(\mu; A)$ of the family (1) with the matrix $A(\cdot\,,\cdot)$ thus defined is identically equal to the vector function $f(\mu)$ for all $\mu \in M$. The proof of the theorem is complete.

4. COROLLARIES

In the case of a bounded vector function $f(\cdot)$ ($\|f(\mu)\| \le \text{const}$, for all $\mu \in M$), the theorem proved above implies the following assertion announced in the report [17], which can be considered as a sufficiently wide generalization of Perron's example [3].

Corollary 1. *For any $n \ge 2$, any metric space M, and any vector function $(f_1, \dots, f_n)^{\mathrm{T}} : M \to \mathbb{R}^n$ whose components belong to the class $({}^*, G_\delta)$, are bounded, and satisfy the inequalities $f_1(\mu) \le \dots \le f_n(\mu)$ for any $\mu \in M$, there exists a continuous bounded $n \times n$ matrix $A(t)$, $t \in \mathbb{R}_+$, and a continuous $n \times n$ matrix $Q(t,\mu)$, $t \in \mathbb{R}_+$, $\mu \in M$, exponentially decaying to zero as $t \to +\infty$ uniformly with respect to μ such that the Lyapunov exponents $\lambda_k(\cdot)$ of the family*

$$\dot{x} = (A(t) + Q(t,\mu))x, \qquad x \in \mathbb{R}^n, \qquad \mu \in M,$$

satisfy the relations $\lambda_k(\mu) = f_k(\mu)$ for all $k = 1, \dots, n$ and $\mu \in M$.

There is another application of the theorem proved above. Millionshchikov [7] proved that if M is a complete metric space and $A \in \mathcal{C}^n(M)$, then for each $i = 1, \dots, n$ the set $US_i(A)$ of points of upper semicontinuity of the function $\lambda_i(\cdot\,; A)$ contains a dense G_δ-set. In other words, the upper semicontinuity of these functions is typical in the sense of Baire in the space M. In the case of lower semicontinuity, this it not true: an example of a family $A \in \mathcal{C}^n([0,1])$ for which the set $LS_i(A)$ of points of lower semicontinuity of the function $\lambda_i(\cdot\,; A)$, $i = 1, \dots, n$, is empty was constructed in [18] for each $n \ge 1$. A complete description of the n-tuples $(LS_1(A), \dots, LS_n(A))$, where $A \in \mathcal{C}^n(M)$, for any metric space M and a complete description of the n-tuples $(US_1(A), \dots, US_n(A))$ for a complete space M were obtained in [19].

A family $A \in \mathcal{U}^n([0,1])$ for which the set $LS_i(A)$ is empty was constructed for any $n \ge 2$ and $i = 1, \dots, n$ in [20]. Then the ideas of this paper and the results obtained in [19] were used in [21] to obtain a complete description of the sets $LS_i(A)$ and, in the case of a complete space M, also of the sets $US_i(A)$, $i = 1, \dots, n$, for families $A \in \mathcal{U}^n(M)$.

Using the theorem proved above, we can completely describe the n-tuples $(LS_1(A), \dots, LS_n(A))$ for each metric space M, and in the case of a complete space M we can also describe the n-tuples $(US_1(A), \dots, US_n(A))$ for families $A \in \mathcal{U}^n(M)$ and thus obtain an answer to the question posed in [21].

Corollary 2. *For any $n \ge 2$ and any metric space M, an n-tuple $(M_1, \dots, M_n)$ of subsets of M is the n-tuple of the sets of lower semicontinuity of the Lyapunov exponents of a family $A \in \mathcal{U}^n(M)$, i.e., $M_i = LS_i(A)$, $i = 1, \dots, n$, if and only if each of the sets M_i, $i = 1, \dots, n$, is an $F_{\sigma\delta}$-set containing all isolated points of the space M. If the space M is complete, then an n-tuple $(M_1, \dots, M_n)$ is the n-tuple of the sets of upper semicontinuity of the Lyapunov exponents of a family $A \in \mathcal{U}^n(M)$, i.e., $M_i = US_i(A)$, $i = 1, \dots, n$, if and only if each of the sets M_i, $i = 1, \dots, n$, is a dense G_δ-set in M.*

Proof. **Necessity** of the conditions of the corollary follows from [19], because $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$.

Sufficiency. We prove the assertion for the points of lower semicontinuity. Assume that each of the sets M_i, $i = 1, \dots, n$, is an $F_{\sigma\delta}$-set containing all isolated points of the space M. By [19, Lemma 5], for each $i = 1, \dots, n$, there exists a function $g_i \colon M \to [0,1]$ of the class $({}^*, G_\delta)$ such that the set of points of its lower semicontinuity coincides with the set M_i. We use the theorem proved above to construct a family $A \in \mathcal{U}^n(M)$ with the Lyapunov spectrum $\Lambda(\cdot\,; A) = (g_1, g_2 + 1, \dots, g_n + n - 1)^{\mathrm{T}}$. Then $LS_i(A) = M_i$ for all $i = 1, \dots, n$. The assertion of the corollary for the points of upper semicontinuity can be proved in a similar way. The proof of the corollary is complete.

ACKNOWLEDGMENTS

The work of E.A. Barabanov and M.V. Karpuk was supported by the Belarusian Republican Foundation for Fundamental Research under grant no. F17-102.

REFERENCES

1. Lyapunov, A.M., *Sobranie sochinenii* (Collected Papers in 6 Volumes), Moscow–Leningrad: Akad. Nauk SSSR, 1956, vol. 2.
2. Bylov, B.F., Vinograd, R.E., Grobman, D.M., and Nemytskii, V.V., *Teoriya pokazatelei Lyapunova i ee prilozheniya k voprosam ustoichivosti* (Theory of Lyapunov Exponents and Its Applications to Stability Problems), Moscow: Nauka, 1966.
3. Perron O., Die Stabilitätsfrage bei Differentialgleichungen, *Math. Zeitschr.*, 1930, vol. 32, no. 1, pp. 703–728.
4. Izobov, N.A., *Lyapunov Exponents and Stability*, Cambridge: Cambridge Scientific Publishers, 2013.
5. Millionshchikov, V.M., Lyapunov exponents as functions of a parameter, *Math. USSR Sb.*, 1990, vol. 65, no. 2, pp. 369–384.
6. Hausdorff, F., *Set Theory*, New York: Chelsea Publishing Company, 1962.
7. Millionshchikov, V.M., Baire function classes and Lyapunov indices. I, *Differ. Equations*, 1981, vol. 16, pp. 902–907.
8. Rakhimberdiev, M.I., Baire class of the Lyapunov indices, *Math. Notes*, 1982, vol. 31, no. 6, pp. 467–470.
9. Karpuk, M.V., Lyapunov exponents of families of morphisms of metrized vector bundles as functions on the base of the bundle, *Differ. Equations*, 2014, vol. 50, no. 10, pp. 1322–1328.
10. Bykov, V.V., Functions determined by the Lyapunov exponents of families of linear differential systems continuously depending on the parameter uniformly on the half-line, *Differ. Equations*, 2017, vol. 53, no. 12, pp. 1529–1542.
11. Barabanov, E.A., Bykov, V.V., and Karpuk, M.V., Functions defined by n-tuples of the Lyapunov exponents of linear differential systems continuously depending on the parameter uniformly on the semiaxis, in *Int. Workshop on the Qualitative Theory of Differ. Equations (December, 24–26, 2017)*, Tbilisi, 2017, pp. 16–19.
12. Barabanov, E.A. and Konyukh, A.V., Uniform exponents of linear systems of differential equations, *Differ. Equations*, 1994, vol. 30, no. 10, pp. 1536–1545.
13. Barabanov, E.A., The structure of the ranges of the lower and the upper Bohl exponents of a linear differential system, *Differ. Equations*, 1996, vol. 32, no. 3, pp. 295–303.
14. Kuratowski, K., *Topology*, New York: Academic, 1966, Vol. 1.
15. Bogdanov, Yu.S., To the theory of systems of linear differential equations, *Dokl. Akad. Nauk SSSR*, 1955, vol. 104, no. 6, pp. 813–814.
16. Grobman, D.M., Characteristic exponents of almost linear systems, *Mat. Sb.*, 1952, vol. 30, no. 1, pp. 121–166.
17. Barabanov, E.A., Bykov, V.V., and Karpuk, M.V., On uniformly bounded Lyapunov exponents of families of linear differential systems, in *Theses Reports XVIII Intern. Sci. Conf. on Differ. Equations "Erugin Readings—2018" (May 15–18, 2018)*, Grodno, 2018, pp. 32–34.
18. Vetokhin, A.N., On the set of lower semicontinuity points of Lyapunov exponents of linear systems with a continuous dependence on a real parameter, *Differ. Equations*, 2014, vol. 50, no. 12, pp. 1673–1676.
19. Karpuk, M.V., Structure of the semicontinuity sets of the Lyapunov exponents of linear differential systems continuously depending on a parameter, *Differ. Equations*, 2015, vol. 51, no. 10, pp. 1397–1401.
20. Vetokhin, A.N., Emptiness of the set of points of lower semicontinuity of Lyapunov exponents, *Differ. Equations*, 2016, vol. 52, no. 3, pp. 272–281.
21. Bykov, V.V., Structure of the sets of points of semicontinuity for the Lyapunov exponents of linear systems continuously depending on a parameter in the uniform norm on the half-line, *Differ. Equations*, 2017, vol. 53, no. 4, pp. 433–438.