ISSN 0012-2661, Differential Equations, 2018, Vol. 54, No. 10, pp. 1338–1353. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © A.S. Makin, 2018, published in Differentsial'nye Uravneniya, 2018, Vol. 54, No. 10, pp. 1367–1382.

ORDINARY DIFFERENTIAL EQUATIONS

Basis Properties of the System of Root Functions of the Sturm–Liouville Operator with Degenerate Boundary Conditions: I

A. S. Makin

Moscow Technological University (MIREA), Moscow, 119454 Russia
e-mail: alexmakin@yandex.ru

Received September 22, 2017

Abstract—The spectral problem for the Sturm–Liouville operator with an arbitrary complex-valued potential $q(x)$ of the class $L_1(0,\pi)$ and degenerate boundary conditions is considered. We prove that the system of root functions of this operator is not a basis in the space $L_2(0,\pi)$.

DOI: 10.1134/S0012266118100051

1. INTRODUCTION

Consider the Sturm–Liouville problem

$$u'' - q(x)u + \lambda u = 0 \tag{1}$$

with the degenerate [1, p. 35] boundary conditions

$$u'(0) + du'(\pi) = 0, \qquad u(0) - du(\pi) = 0, \tag{2}$$

where $d \neq 0$. The completeness of the system of root functions of problem (1), (2) was studied in [2–4]. In particular, it turned out that the set Q of potentials $q(x) \in L_1(0,\pi)$ ensuring the completeness of the system of eigenfunctions and associated functions in the space $L_2(0,\pi)$ is everywhere dense in the space $L_1(0,\pi)$; its complement $\bar{Q}$ has the same property. Since, for a broad class of potentials $q(x)$, the system of root functions of problem (1), (2) is complete in $L_2(0,\pi)$, it is natural to pose the problem of its basis properties.

Let $m(\lambda_n)$ be the multiplicity of the eigenvalue λ_n of problem (1), (2). It follows from [5] that if the inequality

$$m(\lambda_n) < C \tag{3}$$

holds with some constant C for all $n \in \mathbb{N}$, then the system of eigenfunctions and associated functions of problem (1), (2) does not form an unconditional basis in the space $L_2(0,\pi)$. From this and the paper [6], where the constraint (3) is also imposed on the spectrum, we obtain the following stronger assertion: this system is not an ordinary basis in the space $L_2(0,\pi)$. However, later it was proved in [7] that there exists a potential $q(x) \in L_2(0,\pi)$ for which condition (3) is not satisfied, namely, a potential such that

$$c_1 \ln|\mu_n| \le m(\lambda_n) \le c_2 \ln|\mu_n|, \tag{4}$$

where $\mu_n = \sqrt{\lambda_n}$, $\operatorname{Re}\mu_n \ge 0$, $n \in \mathbb{N}$, $c_1 > 0$, and $c_2 > 0$. In the above example, all λ_n are positive. An example of a problem of the form (1), (2) whose spectrum along with condition (4) satisfies the inequality

$$c_3 \ln|\mu_n| \le |\operatorname{Im}\mu_n| \le c_4 \ln|\mu_n|, \tag{5}$$

where $c_3 > 0$ and $c_4 > 0$, was constructed in [8]. Note that the spectrum lies outside the Carleman parabola under condition (5). It was proved that the system of eigenfunctions and associated functions of problem (1), (2) with a specially constructed potential $q(x)$ is complete in the space $L_2(0,\pi)$ in both examples. Obviously, these examples are not covered by the papers cited above, because the multiplicities of the eigenvalues increase unboundedly and hence the system of root functions contains associated functions of arbitrarily large order. This effect is impossible in the case of nondegenerate boundary conditions.

It is well known that if the total number of associated functions is infinite, then the system of root functions may have the basis property for one choice of associated functions and lose it, still remaining complete and minimal, for another choice. Therefore, when speaking about the presence or absence of the basis property, one should have in mind some specific system of eigenfunctions and associated functions. However, note that if in some boundary value problem there exists at least one system of root functions that is a basis, i.e., the choice of associated functions is "correct," then one usually says that the system of eigenfunctions and associated functions of this boundary value problem has the basis property.

The main goal of this paper is to prove the absence of the basis property of an arbitrary system of root functions of problem (1), (2) without any constraints on its spectrum.

2. STATEMENT AND PROOF OF THE MAIN RESULT

Theorem. *No system of eigenfunctions and associated functions of problem* (1), (2) *is a basis in the space* $L_2(0,\pi)$.

Proof. By B_σ^- we denote the class of odd entire functions $f(z)$ of exponential type $\le \sigma$ bounded on the real axis. By $c(x,\mu)$, $s(x,\mu)$ $(\lambda=\mu^2)$ we denote the fundamental system of solutions of Eq. (1) with the initial conditions $c(0,\mu)=s'(0,\mu)=1$, $c'(0,\mu)=s(0,\mu)=0$.

Simple calculations show that the characteristic equation of any problem (1), (2) can be reduced to the form $\Delta(\mu)=0$, where

$$\Delta(\mu)=\frac{d^2-1}{d}+c(\pi,\mu)-s'(\pi,\mu)=\frac{d^2-1}{d}+\int_0^\pi K(t)\frac{\sin\mu t}{\mu}\,dt=\frac{d^2-1}{d}+\frac{f(\mu)}{\mu}, \tag{6}$$

$K(t)\in L_1(0,\pi)$, and $f(\mu)\in B_\pi^-$. It is well known [9] that the spectrum of problem (1), (2) is a countable set of eigenvalues λ_n which does not have a finite limit point if and only if the inequality

$$q(x)-q(\pi-x)\ne 0 \tag{7}$$

holds on a set of positive measure. If $q(x)-q(\pi-x)=0$ almost everywhere on the interval $[0,\pi]$, then the spectrum is absent for $d\ne\pm1$ and fills the entire complex plane for $d=\pm1$. It follows from the representation (6) that under condition (7) the function $\Delta(\mu)$ is a nonconstant function of the class B_π^- and the eigenvalues of problem (1), (2) form an infinite sequence $\lambda_n=\mu_n^2$, where $\operatorname{Re}\mu_n\ge 0$, $n\in\mathbb{N}$. Since the sequence $\{\lambda_n\}$ does not have a finite limit point, we can assume that the numbers λ_n are numbered in nondescending order of absolute values. By $m(\lambda_n)$ we denote the multiplicity of the eigenvalue λ_n. Then the results obtained in [10, p. 130] imply the relation

$$m(\lambda_n)=o(\mu_n),$$

and it follows from [11, pp. 323–324] that all numbers μ_n except possibly for a set of zero density lie inside an arbitrarily small sector $|\arg\mu|<\varepsilon$ $(\varepsilon>0)$. In [12], the above-stated theorem was proved for the case in which $\underline{\lim}_{n\to\infty} m(\lambda_n)<\infty$, and hence, in what follows, we assume that the last condition fails; i.e., $m(\lambda_{n_k})\to\infty$ as $k\to\infty$ for any subsequence $\{\lambda_{n_k}\}$ of eigenvalues. It follows that there exists a subsequence $\{\lambda_{n_k}\}$ for which $\lim_{k\to\infty}|\operatorname{Im}\mu_{n_k}|/|\mu_{n_k}|=0$ and $m(\lambda_{n_k})\to\infty$ as $k\to\infty$. We denote this subsequence by Λ. For convenience, we write $\Omega=(0,\pi)\times(0,\pi)$.

2.1. Estimates of the Green Function of Problem (1), (2)

Let us calculate the Green function of problem (1), (2) by two methods. For the calculations by the first method, we use the fundamental system of solutions of Eq. (1) constructed by Birkhoff.

We write

$$B_1(y) = y'(0) + dy'(\pi), \qquad B_2(y) = y(0) - dy(\pi),$$
$$B_1^*(y) = dy'(\pi) - y'(0), \qquad B_2^*(y) = -dy(\pi) - y(0),$$

and by y_1, y_2 we denote the fundamental system of solutions of Eq. (1) in the domain T_0 [13, pp. 55, 58, 59],

$$y_1(x) = e^{i\mu x}(1 + O(1/\mu)), \qquad y_2(x) = e^{-i\mu x}(1 + O(1/\mu)), \tag{8}$$
$$y_1'(x) = i\mu e^{i\mu x}(1 + O(1/\mu)), \qquad y_2'(x) = -i\mu e^{i\mu x}(1 + O(1/\mu)). \tag{9}$$

Then for the characteristic determinant

$$\Delta_b(\mu) = B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1)$$

we have the relation

$$\Delta_b(\mu) = 2i\mu[1 - d^2 + e^{i\mu\pi}O(1/\mu) + e^{-i\mu\pi}O(1/\mu)],$$

which implies the inequality

$$|\Delta_b(\mu)| < c_5(e^{|\mathrm{Im}\,\mu|\pi} + |\mu|) \tag{10}$$

and, for $\mu > 0$, the inequality

$$|\Delta_b(\mu)| < c_6\mu. \tag{11}$$

It is easily seen that the Wronskian

$$W_b(\xi) = \begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi) \end{vmatrix} \tag{12}$$

satisfies the relation

$$W_b(\mu) = -2i\mu + O(1). \tag{13}$$

By [13, p. 47], the Green function $G(x, \xi, \mu)$ of problem (1), (2) satisfies

$$G(x, \xi, \mu) = H_b(x, \xi, \mu)/\Delta_b(\mu),$$

where

$$H_b(x, \xi, \mu) = \begin{vmatrix} y_1(x) & y_2(x) & g_b(x, \xi) \\ B_1(y_1(x)) & B_1(y_2(x)) & B_1(g) \\ B_2(y_1(x)) & B_2(y_2(x)) & B_2(g) \end{vmatrix}; \tag{14}$$

here

$$g_b(x, \xi) = \mathrm{sgn}(x - \xi)(-y_1(x)y_2(\xi) + y_2(x)y_1(\xi))/(2W_b(\xi)). \tag{15}$$

Since $B_k(g(x, \xi) = [-y_2(\xi)B_k^*(y_1) + y_1(\xi)B_k^*(y_2)]/(2W_b(\xi))$, $k = 1, 2$, we have

$$B_1(g(x, \xi) = [-y_2(\xi)(dy_1'(\pi) - y_1'(0)) + y_1(\xi)(dy_2'(\pi) - y_2'(0))]/(2W_b(\xi)),$$
$$B_2(g(x, \xi) = [-y_2(\xi)(-dy_1(\pi) - y_1(0)) + y_1(\xi)(-dy_2(\pi) - y_2(0))]/(2W_b(\xi)).$$

Following [13, p. 95], to the last column of the determinant (14) we add the first column multiplied by $-y_2(\xi)/(2W_b(\xi))$ and the second column multiplied by $-y_1(\xi)/(2W_b(\xi))$; then the third column becomes

$$\left(\frac{P(x, \xi)}{2W_b(\xi)}, \frac{-2dy_1'(\pi)y_2(\xi) - 2y_2'(0)y_1(\xi)}{2W_b(\xi)}, \frac{2dy_1(\pi)y_2(\xi) - 2y_2(0)y_1(\xi)}{2W_b(\xi)}\right)^{\mathrm{T}},$$

where $P(x, \xi) = -2y_1(x)y_2(\xi)$ if $x > \xi$ and $P(x, \xi) = -2y_2(x)y_1(\xi)$ if $x < \xi$.

Expanding the resulting determinant along the first row, we obtain

$$\begin{aligned}H_b(x,\xi,\mu) = \{&2y_1(x)y_2(\xi)[d(y_2'(0)y_1(\pi)+y_2(0)y_1'(\pi)) + d^2(y_2'(\pi)y_1(\pi)-y_2(\pi)y_1'(\pi))]\\ &- 2y_1(x)y_1(\xi)[d(y_2'(\pi)y_2(0)+y_2(\pi)y_2'(0))] - 2y_2(x)y_2(\xi)[d(y_1'(0)y_1(\pi)+y_1(0)y_1'(\pi))]\\ &+ 2y_2(x)y_1(\xi)[y_1'(0)y_2(0)-y_1(0)y_2'(0)+d(y_1'(\pi)y_2(0)+y_1(\pi)y_2'(0))]\\ &+ P(x,\xi)\Delta_b(\mu)\}/(2W_b(\xi)) = R_b(x,\xi,\mu)+\Delta_b(\mu)g_b(x,\xi),\end{aligned}$$

where

$$\begin{aligned}R_b(x,\xi,\mu) = \{&y_1(x)y_2(\xi)[d(y_2'(0)y_1(\pi)+y_2(0)y_1'(\pi)) + d^2(y_2'(\pi)y_1(\pi)-y_2(\pi)y_1'(\pi))]\\ &- y_1(x)y_1(\xi)[d(y_2'(\pi)y_2(0)+y_2(\pi)y_2'(0))] - y_2(x)y_2(\xi)[d(y_1'(0)y_1(\pi)+y_1(0)y_1'(\pi))]\\ &+ y_2(x)y_1(\xi)[y_1'(0)y_2(0)-y_1(0)y_2'(0)+d(y_1'(\pi)y_2(0)+y_1(\pi)y_2'(0))]\}/W_b(\xi),\\ g_b(x,\xi) = {}&P(x,\xi)/(2W_b(\xi)).\end{aligned}$$

We substitute the asymptotic representations (8) and (9) into the last relation and also use (13) to obtain

$$\begin{aligned}R_b(x,\xi,\mu) = \{&-2i\mu e^{i\mu(x-\xi)}(1+O(1/\mu))d^2 + de^{i\mu(x-\xi+\pi)}O(1)\\ &+ 2i\mu e^{i\mu(x+\xi-\pi)}(1+O(1/\mu))d - 2i\mu d e^{i\mu(\pi-x-\xi)}(1+O(1/\mu))\\ &+ de^{i\mu(\pi-x+\xi)}O(1)\}/W_b(\xi), \qquad (16)\\ g_b(x,\xi) = {}&-e^{i\mu\,\mathrm{sgn}(x-\xi)(x-\xi)}(1+O(1/\mu))\}/(W_b(\xi)). \qquad (17)\end{aligned}$$

From (13) and (16), we obtain the asymptotic representation

$$\begin{aligned}R_b(x,\xi,\mu) = [&d^2e^{i\mu(x-\xi)}(1+O(1/\mu)) - e^{-i\mu(x-\xi)}(1+O(1/\mu))\\ &+ d[e^{i\mu(\pi-x-\xi)}(1+O(1/\mu)) - e^{-i\mu(\pi-x-\xi)}(1+O(1/\mu))\\ &+ d[e^{i\mu(\pi+x-\xi)}O(1/\mu) + e^{i\mu(\pi-x+\xi)}O(1/\mu)]](1+O(1/\mu)). \qquad (18)\end{aligned}$$

In particular, relations (13) and (17) imply the following estimate for $\mu>0$:

$$|g_b(x,\xi)| < c_7/|\mu|. \qquad (19)$$

Let us estimate the function $R_b(x,\xi,\mu)$. For $\mu>0$, relation (18) implies the inequality

$$|R_b(x,\xi,\mu)| < c_8. \qquad (20)$$

Obviously, we have $R_b(x,\xi,\mu) = R_{b,1}(x,\xi,\mu) + R_{b,2}(x,\xi,\mu)$, where

$$\begin{aligned}R_{b,1}(x,\xi,\mu) &= d^2e^{i\mu(x-\xi)} - e^{-i\mu(x-\xi)} + de^{i\mu(\pi-x-\xi)} - de^{-i\mu(\pi-x-\xi)}, \qquad (21)\\ R_{b,2}(x,\xi,\mu) &= d^2e^{i\mu(x-\xi)}O(1/\mu) - e^{-i\mu(x-\xi)}O(1/\mu) + d[e^{i\mu(\pi-x-\xi)}O(1/\mu)\\ &\quad - e^{-i\mu(\pi-x-\xi)}O(1/\mu) + d[e^{i\mu(\pi+x-\xi)}O(1/\mu) + e^{i\mu(\pi-x+\xi)}O(1/\mu)].\end{aligned}$$

One can readily verify the inequalities

$$\frac{c_9}{\sqrt{|\mathrm{Im}\,\mu|+1}} \le \|e^{i\mu t}\|_{L_2(0,\pi)} \le \frac{c_{10}}{\sqrt{|\mathrm{Im}\,\mu|+1}}, \qquad (22)$$

$$\frac{c_9e^{\pi|\mathrm{Im}\,\mu|}}{\sqrt{|\mathrm{Im}\,\mu|+1}} \le \|e^{-i\mu t}\|_{L_2(0,\pi)} \le \frac{c_{10}e^{\pi|\mathrm{Im}\,\mu|}}{\sqrt{|\mathrm{Im}\,\mu|+1}}. \qquad (23)$$

This implies the estimate

$$\|R_{b,2}(x,\xi,\mu))\|_{L_2(\Omega)} \le \frac{c_{11}e^{\pi|\mathrm{Im}\,\mu|}}{|\mu|(|\mathrm{Im}\,\mu|+1)}. \qquad (24)$$

We denote the jth ($j = 1, \ldots, 4$) term on the right-hand side in (21) by $\beta_j(x, \xi, \mu)$. It is easily seen that

$$\|R_{b,1}(x,\xi,\mu)\|^2_{L_2(\Omega))} = \sum_{j=1}^{4} \|\beta_j(x,\xi,\mu)\|^2_{L_2(\Omega)} + \sum_{1\le j,\, k\le 4,\, j\ne k} \int_0^{\pi}\!\!\int_0^{\pi} \beta_j(x,\xi,\mu)\overline{\beta_k(x,\xi,\mu)}\, dx\, d\xi$$
$$= \sum_{j=1}^{4} \|\beta_j(x,\xi,\mu)\|^2_{L_2(\Omega)} + 2\sum_{1\le j<k\le 4} \int_0^{\pi}\!\!\int_0^{\pi} \operatorname{Re}\,[\beta_j(x,\xi,\mu)\overline{\beta_k(x,\xi,\mu)}]\, dx\, d\xi. \quad (25)$$

The estimates (22) and (23) imply the inequality

$$\|\beta_j(x,\xi,\mu)\|^2_{L_2((0,\pi)\times(0,\pi))} \ge \frac{c_{12}e^{2\pi|\operatorname{Im}\mu|}}{(|\operatorname{Im}\mu|+1)^2} \quad (26)$$

for any $j = 1, \ldots, 4$. Moreover, simple calculations for $1 \le j < k \le 4$ lead to the inequality

$$\left|\int_0^{\pi}\!\!\int_0^{\pi} \operatorname{Re}\,[\beta_j(x,\xi,\mu)\overline{\beta_k(x,\xi,\mu)}]\, dx\, d\xi\right| \le c_{13}\max\left(\frac{1}{(\operatorname{Re}\mu)^2}, \frac{e^{\pi|\operatorname{Im}\mu|}}{\operatorname{Re}\mu(|\operatorname{Im}\mu|+1)}\right). \quad (27)$$

It follows from inequalities (24), (26), (27) that for a sufficiently large $\operatorname{Re}\mu$ in the domain T_0 we have the estimate

$$\|R_b(x,\xi,\mu)\|_{L_2(\Omega)} \ge \frac{c_{14}e^{\pi|\operatorname{Im}\mu|}}{|\operatorname{Im}\mu|+1}, \quad (28)$$

where $c_{14} > 0$. By a similar argument, we obtain the estimate (28) for μ lying in the domain T_3 [13, pp. 95, 96].

Let us calculate the Green function of problem (1), (2) by the second method by using the system of fundamental solutions of Eq. (1) related to the transformation operator introduced by Delsarte. Let $e_k(x,\mu)$, $k = 1, 2$, be the solutions of Eq. (1) satisfying the initial conditions $e_k(\pi/2,\mu) = 1$ and $e'_k(\pi/2,\mu) = (-1)^{k+1}i\mu$. Then we have the representations

$$e_1(x,\mu) = e^{i\mu(x-\pi/2)} + \int_{\pi-x}^{x} K_{\pi/2}(x,t)e^{i\mu(t-\pi/2)}\, dt, \quad (29)$$

$$e_2(x,\mu) = e^{-i\mu(x-\pi/2)} + \int_{\pi-x}^{x} K_{\pi/2}(x,t)e^{-i\mu(t-\pi/2)}\, dt, \quad (30)$$

where $K_{\pi/2}(x,t)$ is the transformation operator related to the point $\pi/2$ [1, pp. 18, 19]. We differentiate identities (29) and (30) with respect to x to obtain

$$e'_1(x,\mu) = i\mu e^{i\mu(x-\pi/2)} + \int_{\pi-x}^{x} \frac{\partial K_{\pi/2}(x,t)}{\partial x} e^{i\mu(t-\pi/2)}\, dt$$
$$+ K_{\pi/2}(x,x)e^{i\mu(x-\pi/2)} + K_{\pi/2}(x,\pi-x)e^{i\mu(\pi/2-x)}, \quad (31)$$

$$e'_2(x,\mu) = -i\mu e^{-i\mu(x-\pi/2)} + \int_{\pi-x}^{x} \frac{\partial K_{\pi/2}(x,t)}{\partial x} e^{-i\mu(t-\pi/2)}\, dt$$
$$+ K_{\pi/2}(x,x)e^{-i\mu(x-\pi/2)} + K_{\pi/2}(x,\pi-x)e^{-i\mu(\pi/2-x)}. \quad (32)$$

By the representations (29)–(32), we obtain the following asymptotic formulas, which hold for $|\operatorname{Im}\mu| < M$:

$$e_1(x,\mu) = e^{-i\mu\pi/2}e^{i\mu x}(1+O(1/\mu)), \qquad e_2(x,\mu) = e^{i\mu\pi/2}e^{-i\mu x}(1+O(1/\mu)),$$
$$e_1'(x,\mu) = i\mu e^{-i\mu\pi/2}e^{i\mu x}(1+O(1/\mu)), \qquad e_2'(x,\mu) = -i\mu e^{i\mu\pi/2}e^{i\mu x}(1+O(1/\mu)).$$

It is well known that for $\mu \neq 0$ the functions $e_1(x,\mu)$ and $e_2(x,\mu)$ form a fundamental system of solutions of Eq. (1). Let us calculate the characteristic determinant $\Delta_d(\mu)$ of problem (1), (2). It is easily seen that

$$\Delta_d(\mu) = B_1(e_1)B_2(e_2) - B_1(e_2)B_2(e_1) = 2i\mu[1-d^2] + f(\mu), \tag{33}$$

where $f(\mu) \in B_\sigma^-$, $0 < \sigma \leq \pi$. Calculating the Green function by formulas (12), (14), and (15), where y_1 and y_2 are replaced with e_1 and e_2, respectively, we obtain

$$G(x,\xi,\mu) = \frac{R_d(x,\xi,\mu)}{\Delta_d(\mu)} + g_d(x,\xi), \tag{34}$$

where

$$R_d(x,\xi,\mu) = \{-2i\mu e^{i\mu(x-\xi)}(1+O(1/\mu))d^2 + de^{i\mu(x-\xi+\pi)}O(1) + 2i\mu e^{i\mu(x+\xi-\pi)}(1+O(1/\mu))d$$
$$- 2i\mu de^{i\mu(\pi-x-\xi)}(1+O(1/\mu)) + de^{i\mu(\pi-x+\xi)}O(1)\}/(-2i\mu),$$
$$g_d(x,\xi) = -e^{i\mu\operatorname{sgn}(x-\xi)(x-\xi)}(1+O(1/\mu))/(-2i\mu).$$

Obviously, for $\mu > 0$ we have the relations

$$R_d(x,\xi,\mu) = d^2e^{i\mu(x-\xi)} - e^{-i\mu(x-\xi)} + 2id\sin\mu(\pi-x-\xi) + O(1/\mu), \tag{35}$$
$$|g_d(x,\xi)| < c_{15}/|\mu|. \tag{36}$$

It follows from (35) that for $\mu > 0$ we have the inequality

$$\|R_d(x,\xi,\mu)\|_{L_2(\Omega)} > c_{16}. \tag{37}$$

Relations (14) and (34) imply that

$$G(x,\xi,\mu) = \frac{R_d(x,\xi,\mu)}{\Delta_d(\mu)} + g_d(x,\xi) = \frac{R_b(x,\xi,\mu)}{\Delta_b(\mu)} + g_b(x,\xi). \tag{38}$$

Consider formula (38) for $\mu > 0$. We integrate it to obtain

$$\left|\frac{\Delta_b(\mu)}{\Delta_d(\mu)}\right| \|R_d(x,\xi,\mu)\|_{L_2(\Omega)} = \|R_b(x,\xi,\mu) + (g_b(x,\xi) - g_d(x,\xi))\Delta_b(\mu)\|_{L_2(\Omega)},$$

which, together with the estimates (11), (19), (20), (36), and (37), implies the inequality

$$\left|\frac{\Delta_b(\mu)}{\Delta_d(\mu)}\right| \leq \frac{\|R_b(x,\xi,\mu)\|_{L_2(\Omega)}}{\|R_d(x,\xi,\mu)\|_{L_2(\Omega)}} + \frac{|\Delta_b(\mu)|\|(g_b(x,\xi) - g_d(x,\xi))\|_{L_2(\Omega)}}{\|R_d(x,\xi,\mu)\|_{L_2(\Omega)}} < c_{17}. \tag{39}$$

Since $\Delta_d(\mu)/\mu$ is an entire function of exponential type and $\Delta_d(\mu)/\mu \in B_\sigma$, where $0 < \sigma \leq \pi$, we see by the Krein and Hayman theorems [10, pp. 115, 109] that the following inequality holds everywhere outside the set of disks of finite visibility [10, p. 109]:

$$|\Delta_d(\mu)| \geq e^{\sigma|\operatorname{Im}\mu| - \varepsilon|\mu|} \tag{40}$$

for any $\varepsilon > 0$. Inequality (40) holds on an everywhere dense set of rays issuing from the origin, in particular, on a certain ray $\arg \mu = \alpha_0$, where $0 < \alpha_0 < \pi/2$, $\sin \alpha_0 > 2\varepsilon$, and $\alpha_0 < \sigma$.

Outside the union of exceptional disks, the estimates (10) and (40) imply the inequality

$$|\Delta_b(\mu)/\Delta_d(\mu)| < c_{18} e^{|\operatorname{Im} \mu|(\pi-\sigma)+2\varepsilon|\mu|}. \tag{41}$$

Further, in the sector $\Omega_0 = \{\mu : 0 \le \arg \mu \le \alpha_0\}$ we consider the auxiliary function

$$\chi(\mu) = \Delta_b(\mu) e^{i\mu(\pi-\sigma+\alpha_0)}/\Delta_d(\mu).$$

It follows from inequalities (39) and (41) that the function $\chi(\mu)$ is bounded on the boundary of the sector Ω_0 and satisfies the estimate $|\chi(\mu)| < c_{19} e^{2\varepsilon|\mu|}$ inside it, outside the union of exceptional disks. It follows from this, the definition of exceptional disks, and the maximum principle for an analytic function that the inequality

$$|\chi(\mu)| < c_{20} e^{2\varepsilon(|\mu|+o(\mu))} < c_{19} e^{4\varepsilon|\mu|}$$

holds everywhere insider the sector Ω_0. Then, by a Phragmén–Lindelöf type theorem [14, pp. 186–187 of the Russian translation], the function $\chi(\mu)$ is bounded in Ω_0 and hence the following inequality holds in this sector:

$$|\Delta_b(\mu)/\Delta_d(\mu)| < c_{21} e^{\nu \operatorname{Im} \mu},$$

where $\nu = \pi - \sigma + \alpha_0$, or, which is equivalent,

$$|\Delta_d(\mu)/\Delta_b(\mu)| > c_{22} e^{-\nu \operatorname{Im} \mu}. \tag{42}$$

By a similar argument, we conclude that the inequality

$$|\Delta_d(\mu)/\Delta_b(\mu)| > c_{23} e^{\nu \operatorname{Im} \mu} \tag{42'}$$

holds in the sector $-\alpha_0 \le \arg \mu \le 0$. It follows from the representations (38) that

$$R_d(x,\xi,\mu) = R_b(x,\xi,\mu)\frac{\Delta_d(\mu)}{\Delta_b(\mu)} + \Delta_d(\mu)(g_b(x,\xi) - g_d(x,\xi)).$$

This, together with the estimates (28), (42), and (42′), implies that the inequality

$$\|R_d(x,\xi,\mu_n)\|_{L_2((0,\pi)\times(0,\pi))} \ge \frac{c_{24} e^{|\operatorname{Im} \mu|(\pi-\nu)}}{|\operatorname{Im} \mu| + 1} \tag{43}$$

where $c_{24} > 0$ and $0 < \nu < \pi$, holds for $0 \le |\arg \mu| \le \alpha_0$.

Let us study the function $G(x,\xi,\mu)$ in a neighborhood of the eigenvalues λ_n. It follows from [15] that each root subspace contains one eigenfunction and possibly several associated functions. Assume that the system of functions $\{\overset{h}{u}_n(x)\}$ $(h = 0,\dots,m(\lambda_n))$ is an arbitrary canonical system of eigenfunctions and associated functions of problem (1), (2) and the system of functions $\{\overset{h}{v}_n(x)\}$ is an appropriately normalized canonical system of eigenfunctions and associated functions of the adjoint boundary value problem [16]; i.e., $\overset{0}{u}_n(x)$ and $\overset{0}{v}_n(x)$ are eigenfunctions and $\overset{h}{u}_n(x)$ and $\overset{h}{v}_n(x)$ $(h \ge 1)$ are associated functions of order h, where

$$(\overset{h}{u}_n(x), \overset{g}{v}_k(x))_{L_2(0,\pi)} = \delta_{n,k}\delta_{h,m(\lambda_n)-1-g}.$$

Everywhere below, we consider only the root subspaces corresponding to the above-cited subsequence of eigenvalues Λ. We write

$$\overset{0}{R}_{n_k}(x,\xi) = \overset{0}{u}_{n_k}(x)\overline{\overset{0}{v}_{n_k}(\xi)}, \qquad \overset{m(\lambda_{n_k})-1}{R_{n_k}}(x,\xi) = \sum_{p=0}^{m(\lambda_{n_k})-1} \overset{p}{u}_{n_k}(x)\overline{\overset{m(\lambda_{n_k})-1-p}{v_{n_k}}}(\xi).$$

Since the function $\Delta_d(\mu)$ has a root of multiplicity $m(\lambda_{n_k})$ at the point μ_{n_k}, we have

$$\Delta_d(\mu) = \sum_{l=m(\lambda_{n_k})}^{\infty} C_l(\mu - \mu_{n_k})^l = (\mu - \mu_{n_k})^{m(\lambda_{n_k})} \sum_{l=0}^{\infty} C_{m(\lambda_{n_k})+l}(\mu - \mu_{n_k})^l, \tag{44}$$

where $C_{m(\lambda_{n_k})} = \Delta_d^{(m(\lambda_{n_k}))}(\mu_{n_k})/m(\lambda_{n_k})!$. It follows from (38), (44), (36), and (20) that

$$\overset{0}{R}_{n_k}(x,\xi) = \lim_{\mu\to\mu_{n_k}} (\mu^2 - \mu_{n_k}^2)^{m(\lambda_{n_k})} G(x,\xi,\mu) = \frac{2^{m(\lambda_{n_k})} m(\lambda_{n_k})! \mu_{n_k}^{m(\lambda_{n_k})} R_d(x,\xi,\mu_{n_k})}{\Delta_d^{(m(\lambda_{n_k}))}(\mu_{n_k})}. \tag{45}$$

By the Bernstein inequality [17, p. 115], we have the estimate

$$|\Delta_d^{(m(\lambda_{n_k}))}(\mu_{n_k})| \le c_{25}\pi^{m(\lambda_{n_k})},$$

which, together with the representation (45), implies that

$$|\overset{0}{R}_{n_k}(x,\xi)| \ge c_{26}(2/\pi)^{m(\lambda_{n_k})} m(\lambda_{n_k})! |\mu_{n_k}|^{m(\lambda_{n_k})} |R_d(x,\xi,\mu_{n_k})|,$$

where $c_{26} > 0$, and hence

$$\|\overset{0}{R}_{n_k}(x,\xi)\|^2_{L_2(\Omega)} \ge c_{27}[(2/\pi)^{m(\lambda_{n_k})} m(\lambda_{n_k})!]^2 |\mu_{n_k}|^{2m(\lambda_{n_k})} \|R_d(x,\xi,\mu_{n_k})\|^2_{L_2(\Omega)}, \tag{46}$$

where $c_{27} > 0$. Then the estimates (43) and (46) imply the inequality

$$\|\overset{0}{R}_{n_k}(x,\xi)\|^2_{L_2(\Omega)} \ge c_{28}[(2/\pi)^{m(\lambda_{n_k})} m(\lambda_{n_k})!]^2 |\mu_{n_k}|^{2m(\lambda_{n_k})} (e^{|\operatorname{Im}\mu_{n_k}|(\pi-\nu)}/(|\operatorname{Im}\mu_{n_k}|+1))^2, \tag{47}$$

where $c_{28} > 0$.

2.2. Asymptotic Formulas for the Eigenfunctions of Problem (1), (2) *and the Adjoint Problem*

By [2, p. 34], problem (1), (2) has the solutions

$$u_1(x,\mu) = (e_2'(0,\mu) + de_2'(\pi,\mu))e_1(x,\mu) - (e_1'(\pi,\mu) + de_1'(\pi,\mu))e_2(x,\mu), \tag{48}$$

$$u_2(x,\mu) = (e_1(0,\mu) - de_2(\pi,\mu))e_1(x,\mu) - (e_1(0,\mu) - de_1(\pi,\mu))e_2(x,\mu) \tag{49}$$

if μ is a root of the characteristics determinant (33). The functions $u_1(x,\mu)$ and $u_2(x,\mu)$ are eigenfunctions of problem (1), (2) if they are not identically zero. Replacing the functions the functions e_k, $k = 1,2$, and their derivatives in (48) and (49) with the representations (29)–(32), we obtain

$$\begin{aligned} u_1(\mu,x) = {} & -2i\mu(\cos\mu x + d\cos\mu(\pi - x)) - 2i\mu \int_{\pi-x}^{x} K_{\pi/2}(x,t)[\cos\mu t - d\cos\mu(\pi - t)]\,dt \\ & - 2i \int_0^{\pi} \left[\frac{\partial K_{\pi/2}(0,t)}{\partial x} - d\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right] \sin\mu(x-t)\,dt - 2i \int_{\pi-x}^{x} K_{\pi/2}(x,\tau)\,d\tau \\ & \times \int_0^{\pi} \left[\frac{\partial K_{\pi/2}(0,t)}{\partial x} - d\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right] \sin\mu(\tau - t)\,dt + 2i[K_{\pi/2}(0,0) + dK_{\pi/2}(\pi,0)]\sin\mu x \\ & - 2i[K_{\pi/2}(0,\pi) + dK_{\pi/2}(\pi,\pi)]\sin\mu(\pi - x) + 2i[K_{\pi/2}(0,0) + dK_{\pi/2}(\pi,0)] \\ & \times \int_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu t\,dt - 2i[K_{\pi/2}(0,\pi) + dK_{\pi/2}(\pi,\pi)] \int_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu(\pi - t)\,dt \\ = {} & -2i\mu(\cos\mu x + d\cos\mu(\pi - x)) + \Theta_1(\mu,x), \end{aligned}$$

where

$$\Theta_1(\mu,x) = -2i\mu\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)[\cos\mu t - d\cos\mu(\pi-t)]\,dt - 2i\int\limits_0^{\pi}\left[\frac{\partial K_{\pi/2}(0,t)}{\partial x} - d\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]$$

$$\times\sin\mu(x-t)\,dt - 2i\int\limits_{\pi-x}^{x} K_{\pi/2}(x,\tau)\,d\tau\int\limits_0^{\pi}\left[\frac{\partial K_{\pi/2}(0,t)}{\partial x} - d\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]\sin\mu(\tau-t)\,dt$$

$$+ 2i[K_{\pi/2}(0,0) + dK_{\pi/2}(\pi,0)]\sin\mu x - 2i[K_{\pi/2}(0,\pi) + dK_{\pi/2}(\pi,\pi)]\sin\mu(\pi-x)$$

$$+ 2i[K_{\pi/2}(0,0) + dK_{\pi/2}(\pi,0)]\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu t\,dt - 2i[K_{\pi/2}(0,\pi) + dK_{\pi/2}(\pi,\pi)]$$

$$\times\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu(\pi-t)\,dt = -2i\Bigg[(K_{\pi/2}(x,x) - dK_{\pi/2}(x,\pi-x))\sin\mu x$$

$$-(K_{\pi/2}(x,\pi-x) - dK_{\pi/2}(x,x))\sin\mu(\pi-x) - \int\limits_{\pi-x}^{x}\frac{\partial K_{\pi/2}(x,t)}{\partial t}(\sin\mu t + \sin\mu(\pi-t))\,dt\Bigg]$$

$$-2i\int\limits_0^{\pi}\left[\frac{\partial K_{\pi/2}(0,t)}{\partial x} - d\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]\sin\mu(x-t)\,dt - 2i\int\limits_{\pi-x}^{x} K_{\pi/2}(x,\tau)\,d\tau$$

$$\times\int\limits_0^{\pi}\left[\frac{\partial K_{\pi/2}(0,t)}{\partial x} - d\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]\sin\mu(\tau-t)\,dt + 2i[K_{\pi/2}(0,0) + dK_{\pi/2}(\pi,0)]\sin\mu x$$

$$-2i[K_{\pi/2}(0,\pi) + dK_{\pi/2}(\pi,\pi)]\sin\mu(\pi-x) + 2i[K_{\pi/2}(0,0) + dK_{\pi/2}(\pi,0)]$$

$$\times\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu t\,dt - 2i[K_{\pi/2}(0,\pi) + dK_{\pi/2}(\pi,\pi)]\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu(\pi-t)\,dt,$$

and

$$u_2(\mu,x) = 2i(\sin\mu x + d\sin\mu(\pi-x)) + \int\limits_0^{\pi}\tilde{K}(t)\sin\mu(t-x)\,dt$$

$$+\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt\int\limits_0^{\pi}\tilde{K}(\tau)\sin\mu(\tau-t)\,d\tau + \int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)(\sin\mu t + d\sin\mu(\pi-t))\,dt$$

$$= 2i(\sin\mu x + d\sin\mu(\pi-x)) + \Theta_2(\mu,x), \tag{50}$$

where

$$\tilde{K}(t) = K_{\pi/2}(0,t) + dK_{\pi/2}(\pi,t),$$

$$\Theta_2(\mu,x) = \int\limits_0^{\pi}\tilde{K}(t)\sin\mu(t-x)\,dt + \int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt\int\limits_0^{\pi}\tilde{K}(\tau)\sin\mu(\tau-t)\,d\tau$$

$$+\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)(\sin\mu t + d\sin\mu(\pi-t))\,dt.$$

Integrating by parts in the last relation, we obtain

$$\begin{aligned}\Theta_2(\mu,x)=\mu^{-1}\Bigg[&(\tilde{K}(0)+K_{\pi/2}(x,\pi-x)-dK_{\pi/2}(x,\pi-x))\cos\mu x\\&+(dK_{\pi/2}(x,x)-K_{\pi/2}(x,x)-\tilde{K}(\pi))\cos\mu(\pi-x)\\&+\int\limits_0^\pi \tilde{K}'(t)\cos\mu(t-x)\,dt+\int\limits_{\pi-x}^x\left(\tilde{K}(0)K_{\pi/2}(x,t)+\frac{\partial K_{\pi/2}(x,t)}{\partial t}\right)\cos\mu t\,dt\\&-\int\limits_{\pi-x}^x\left(\tilde{K}(\pi)K_{\pi/2}(x,t)+d\frac{\partial K_{\pi/2}(x,t)}{\partial t}\right)\cos\mu(\pi-t)\,dt\\&+\int\limits_{\pi-x}^x K_{\pi/2}(x,t)\,dt\int\limits_0^\pi \tilde{K}'(\tau)\cos\mu(\tau-t)\,d\tau\Bigg].\end{aligned}\tag{50$'$}$$

Obviously, the problem adjoint to problem (1), (2) has the form

$$\begin{gathered}v''-\bar{q}(x)v+\lambda v=0,\\ \bar{d}v'(0)-v'(\pi)=0,\qquad \bar{d}v(0)+v(\pi)=0.\end{gathered}\tag{51}$$

It follows that if $v_n(x)$ is an eigenfunction of problem (51) corresponding to the eigenvalue $\bar{\lambda}_n$, then the function $w(x)=\bar{v}_n(x)$ is an eigenfunction of the problem

$$\begin{gathered}w''-q(x)w+\lambda w=0,\\ dw'(0)-w'(\pi)=0,\qquad dw(0)+w(\pi)=0\end{gathered}\tag{52}$$

corresponding to the eigenvalue λ_n.

In a similar way, we conclude that for $\mu^2=\lambda_n$ problem (52) has the solutions

$$\begin{aligned}w_1(x,\mu)&=-2i\mu(d\cos\mu x-\cos\mu(\pi-x))-2i\mu\int\limits_{\pi-x}^x K_{\pi/2}(x,t)[d\cos\mu t+\cos\mu(\pi-t)]\,dt\\&\quad-2i\int\limits_0^\pi\left[d\frac{\partial K_{\pi/2}(0,t)}{\partial x}+\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]\sin\mu(x-t)\,dt-2i\int\limits_{\pi-x}^x K_{\pi/2}(x,\tau)\,d\tau\\&\quad\times\int\limits_0^\pi\left[d\frac{\partial K_{\pi/2}(0,t)}{\partial x}+\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]\sin\mu(\tau-t)\,dt+2i[dK_{\pi/2}(0,0)-K_{\pi/2}(\pi,0)]\sin\mu x\\&\quad-2i[dK_{\pi/2}(0,\pi)-K_{\pi/2}(\pi,\pi)]\sin\mu(\pi-x)+2i[dK_{\pi/2}(0,0)-K_{\pi/2}(\pi,0)]\\&\quad\times\int\limits_{\pi-x}^x K_{\pi/2}(x,t)\sin\mu t\,dt-2i[dK_{\pi/2}(0,\pi)-K_{\pi/2}(\pi,\pi)]\int\limits_{\pi-x}^x K_{\pi/2}(x,t)\sin\mu(\pi-t)\,dt\\&=-2i\mu(d\cos\mu x-\cos\mu(\pi-x))+\Theta_1^*(\mu,x),\end{aligned}$$

where

$$\Theta_1^*(\mu,x) = -2i\mu\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)[d\cos\mu t+\cos\mu(\pi-t)]\,dt - 2i\int\limits_0^{\pi}\left[d\frac{\partial K_{\pi/2}(0,t)}{\partial x}+\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]$$
$$\times\sin\mu(x-t)\,dt - 2i\int\limits_{\pi-x}^{x} K_{\pi/2}(x,\tau)\,d\tau\int\limits_0^{\pi}[d\frac{\partial K_{\pi/2}(0,t)}{\partial x}+\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}]\sin\mu(\tau-t)\,dt$$
$$+2i[dK_{\pi/2}(0,0)-K_{\pi/2}(\pi,0)]\sin\mu x - 2i[dK_{\pi/2}(0,\pi)-K_{\pi/2}(\pi,\pi)]\sin\mu(\pi-x)$$
$$+2i[dK_{\pi/2}(0,0)-K_{\pi/2}(\pi,0)]\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu t\,dt - 2i[dK_{\pi/2}(0,\pi)-K_{\pi/2}(\pi,\pi)]$$
$$\times\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu(\pi-t)\,dt = -2i\Bigg[(K_{\pi/2}(x,\pi-x)+dK_{\pi/2}(x,x))\sin\mu x$$
$$-(K_{\pi/2}(x,x)+dK_{\pi/2}(x,\pi-x))\sin\mu(\pi-x)+\int\limits_{\pi-x}^{x}\frac{\partial K_{\pi/2}(x,t)}{\partial t}\sin\mu(\pi-t)\,dt\Bigg]$$
$$-2i\int\limits_0^{\pi}\left[d\frac{\partial K_{\pi/2}(0,t)}{\partial x}+\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]\sin\mu(x-t)\,dt - 2i\int\limits_{\pi-x}^{x} K_{\pi/2}(x,\tau)\,d\tau$$
$$\times\int\limits_0^{\pi}\left[d\frac{\partial K_{\pi/2}(0,t)}{\partial x}+\frac{\partial K_{\pi/2}(\pi,t)}{\partial x}\right]\sin\mu(\tau-t)\,dt + 2i[dK_{\pi/2}(0,0)-K_{\pi/2}(\pi,0)]\sin\mu x$$
$$-2i[dK_{\pi/2}(0,\pi)-K_{\pi/2}(\pi,\pi)]\sin\mu(\pi-x)+2i[dK_{\pi/2}(0,0)-K_{\pi/2}(\pi,0)]$$
$$\times\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu t\,dt - 2i[dK_{\pi/2}(0,\pi)-K_{\pi/2}(\pi,\pi)]\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\sin\mu(\pi-t)\,dt,$$

and

$$w_2(x,\mu) = 2i(d\sin\mu x-\sin\mu(\pi-x))+\int\limits_0^{\pi}\tilde K^*(t)\sin\mu(t-x)\,dt$$
$$+\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt\int\limits_0^{\pi}\tilde K^*(\tau)\sin\mu(\tau-t)\,d\tau+\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)(d\sin\mu t-\sin\mu(\pi-t))\,dt$$
$$= 2i(d\sin\mu x-\sin\mu(\pi-x))+\Theta_2^*(\mu,x), \tag{53}$$

where

$$\tilde K^*(t) = dK_{\pi/2}(0,t)-K_{\pi/2}(\pi,t),$$
$$\Theta_2^*(\mu,x) = \int\limits_0^{\pi}\tilde K^*(t)\sin\mu(t-x)\,dt$$
$$+\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt\int\limits_0^{\pi}\tilde K^*(\tau)\sin\mu(\tau-t)\,d\tau+\int\limits_{\pi-x}^{x} K_{\pi/2}(x,t)(d\sin\mu t-\sin\mu(\pi-t))\,dt.$$

Integrating by parts, we obtain

$$\begin{aligned}\Theta_2^*(\mu,x) = \mu^{-1}\Bigg[&(\tilde{K}^*(0) + dK_{\pi/2}(x,\pi-x) + K_{\pi/2}(x,\pi-x))\cos\mu x \\ &- (\tilde{K}^*(\pi) - K_{\pi/2}(x,x) - dK_{\pi/2}(x,x))\cos\mu(\pi-x) \\ &+ \int\limits_{\pi-x}^{x}\left(\tilde{K}^*(0)K_{\pi/2}(x,t) + d\frac{\partial K_{\pi/2}(x,t)}{\partial t}\right)\cos\mu t\,dt \\ &- \int\limits_{\pi-x}^{x}\left(\tilde{K}^*(\pi)K_{\pi/2}(x,t) - \frac{\partial K_{\pi/2}(x,t)}{\partial t}\right)\cos\mu(\pi-t)\,dt \\ &+ \int\limits_{0}^{\pi}(\tilde{K}^*(t))'\cos\mu(t-x)\,dt + \int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{0}^{\pi}(\tilde{K}^*(\tau))'\cos\mu(\tau-t)\,d\tau\Bigg]. \end{aligned} \qquad (53')$$

One can readily verify that for all sufficiently large $|\mu|$, if at least one of the conditions

$$d \neq \pm 1 \quad \text{or} \quad |\mu - n| > 0,001$$

is satisfied, then $u_2(\mu,x) \not\equiv 0$ and $w_2(\mu,x) \not\equiv 0$ (case 1). If $d=1$, $|\mu-2n| \le 0,001$ or if $d=-1$, $|\mu-2n-1| \le 0,001$, then $u_1(\mu,x) \not\equiv 0$ and $w_2(\mu,x) \not\equiv 0$ (case 2); if $d=1$, $|\mu-2n-1| \le 0,001$ or if $d=-1$, $|\mu-2n| \le 0,001$, then $u_2(\mu,x) \not\equiv 0$ and $w_1(\mu,x) \not\equiv 0$ (case 3). In what follows, we study case 1; cases 2 and 3 can be considered in a similar way.

2.3. Asymptotic Formulas for the Products of Eigenfunctions and Their Derivatives for Problem (1), (2) *and the Adjoint Problem*

For $\mu = \mu_n$, the function $u_2(\mu,x)$ is an eigenfunction of problem (1), (2), and the function $w_2(\mu,\xi)$ is an eigenfunction of problem (52).

The representations (50) and (53) imply the relations

$$u_2(x,\mu)w_2(\xi,\mu) = \sum_{j=0}^{3}\Phi_j(\mu,x,\xi), \qquad (54)$$

where

$$\begin{aligned}\Phi_0(\mu,x,\xi) &= 2i(\sin\mu x + d\sin\mu(\pi-x))2i(d\sin\mu\xi - \sin\mu(\pi-\xi)) \\ &= 2[\cos\mu(\pi-x-\xi) - \cos\mu(\pi+x-\xi) + d(\cos\mu(x+\xi) - \cos\mu(2\pi-x-\xi)) \\ &\quad + d^2(\cos\mu(\pi-x+\xi) - \cos\mu(\pi-x-\xi))] \\ &= 2[(1-d^2)\cos\mu(\pi-x-\xi) - \cos\mu(\pi+x-\xi) + d(\cos\mu(x+\xi) \\ &\quad - \cos\mu(2\pi-x-\xi)) + d^2\cos\mu(\pi-x+\xi)], \qquad (55) \\ \Phi_1(\mu,x,\xi) &= 2i(\sin\mu x + d\sin\mu(\pi-x))\Theta_2^*(\mu,\xi), \\ \Phi_2(\mu,x,\xi) &= 2i(d\sin\mu\xi - \sin\mu(\pi-\xi))\Theta_2(\mu,x), \\ \Phi_3(\mu,x,\xi) &= \Theta_2(\mu,x)\Theta_2^*(\mu,\xi).\end{aligned}$$

This, the representations (53′) and (50′), and the obvious relation $\sin\mu y = \operatorname{sgn} y \sin\mu|y|$, which holds for any real y, imply

$$\Phi_1(\mu,x,\xi) = \mu^{-1}[2i(\sin\mu x + d\sin\mu(\pi-x))]\Bigg[(\tilde{K}^*(0) + dK_{\pi/2}(x,\pi-\xi) + K_{\pi/2}(\xi,\pi-\xi))\cos\mu\xi$$

$$- (\tilde{K}^*(\pi) - K_{\pi/2}(\xi,\xi) - dK_{\pi/2}(\xi,\xi))\cos\mu(\pi-\xi) + \int\limits_{\pi-\xi}^{\xi}\left(\tilde{K}^*(0)K_{\pi/2}(\xi,t) + d\frac{\partial K_{\pi/2}(\xi,t)}{\partial t}\right)\cos\mu t\,dt$$

$$- \int\limits_{\pi-\xi}^{\xi} \left(\tilde{K}^*(\pi)K_{\pi/2}(\xi,t) - \frac{\partial K_{\pi/2}(\xi,t)}{\partial t}\right)\cos\mu(\pi-t)\,dt + \int\limits_0^{\pi}(\tilde{K}^*(t))'\cos\mu(t-\xi)\,dt$$

$$+ \int\limits_{\pi-\xi}^{\xi} K_{\pi/2}(\xi,t)\,dt\int\limits_0^{\pi}(\tilde{K}^*(\tau))'\cos\mu(\tau-t)\,d\tau\Bigg] = i\mu^{-1}\Bigg[(\tilde{K}^*(0) + dK_{\pi/2}(x,\pi-\xi) + K_{\pi/2}(\xi,\pi-\xi))$$

$$\times (\sin\mu(x+\xi) + \operatorname{sgn}(x-\xi)\sin\mu|x-\xi| + d(\sin\mu(\pi-x+\xi) + \operatorname{sgn}(\pi-x-\xi)\sin\mu|\pi-x-\xi|))$$

$$- (\tilde{K}^*(\pi) - K_{\pi/2}(\xi,\xi) - dK_{\pi/2}(\xi,\xi))(\sin\mu(x+\pi-\xi) + \operatorname{sgn}(x-\pi+\xi)\sin\mu|x-\pi+\xi|$$

$$+ d(\sin\mu(2\pi-x-\xi) - \sin\mu(x+\xi))) + \int\limits_{\pi-\xi}^{\xi}\left(\tilde{K}^*(0)K_{\pi/2}(\xi,t) + d\frac{\partial K_{\pi/2}(\xi,t)}{\partial t}\right)$$

$$\times (\sin\mu(x+t) + \operatorname{sgn}(x-t)\sin\mu|x-t| + d(\sin\mu(\pi-x+t) + \operatorname{sgn}(\pi-x-t)\sin\mu|\pi-x-t|))\,dt$$

$$- \int\limits_{\pi-\xi}^{\xi}\left(\tilde{K}^*(\pi)K_{\pi/2}(\xi,t) - \frac{\partial K_{\pi/2}(\xi,t)}{\partial t}\right)(\sin\mu(x+\pi-t) + \operatorname{sgn}(x-\pi+t)\sin\mu|x-\pi+t|$$

$$+ d(\sin\mu(2\pi-x-t) - \sin\mu(x+t)))\,dt + \int\limits_0^{\pi}(\tilde{K}^*(t))'(\operatorname{sgn}(x+t-\xi)\sin\mu|x+t-\xi| + \operatorname{sgn}(x-t+\xi)$$

$$\times \sin\mu|x-t+\xi| + d(\operatorname{sgn}(\pi-x+t-\xi)\sin\mu|\pi-x+t-\xi| + \operatorname{sgn}(\pi-x-t+\xi)\sin\mu|\pi-x-t+\xi|))\,dt$$

$$+ \int\limits_{\pi-\xi}^{\xi} K_{\pi/2}(\xi,t)\,dt\int\limits_0^{\pi}(\tilde{K}^*(\tau))'(\operatorname{sgn}(x+t-\tau)\sin\mu|x+t-\tau| + \operatorname{sgn}(x-t+\tau)\sin\mu|x-t+\tau|$$

$$+ d(\operatorname{sgn}(\pi-x+t-\tau)\sin\mu|\pi-x+t-\tau| + \operatorname{sgn}(\pi-x-t+\tau)\sin\mu|\pi-x-t+\tau|))\,d\tau\Bigg], \tag{56}$$

$$\Phi_2(\mu,x,\xi) = [2i(d\sin\mu\xi - \sin\mu(\pi-\xi))]\Theta_2(\mu,x) = \mu^{-1}[2i(d\sin\mu\xi - \sin\mu(\pi-\xi))]$$

$$\times\Bigg[(\tilde{K}(0) + K_{\pi/2}(x,\pi-x) - dK_{\pi/2}(x,\pi-x))\cos\mu x + (dK_{\pi/2}(x,x) - K_{\pi/2}(x,x) - \tilde{K}(\pi))\cos\mu(\pi-x)$$

$$+ \int\limits_0^{\pi}\tilde{K}'(t)\cos\mu(t-x)\,dt + \int\limits_{\pi-x}^{x}\left(\tilde{K}(0)K_{\pi/2}(x,t) + \frac{\partial K_{\pi/2}(x,t)}{\partial t}\right)\cos\mu t\,dt - \int\limits_{\pi-x}^{x}\Bigg(\tilde{K}(\pi)K_{\pi/2}(x,t)$$

$$+ d\frac{\partial K_{\pi/2}(x,t)}{\partial t}\Bigg)\cos\mu(\pi-t)\,dt + \int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_0^{\pi}\tilde{K}'(\tau)\cos\mu(\tau-t)\,d\tau\Bigg]$$

$$= i\mu^{-1}\Bigg[(\tilde{K}(0) + K_{\pi/2}(x,\pi-x) - dK_{\pi/2}(x,\pi-x))(d\sin\mu(\xi+x) + \operatorname{sgn}(\xi-x)d\sin\mu|\xi-x|$$

$$- \sin\mu(\pi-\xi+x) - \operatorname{sgn}(\pi-\xi-x)\sin\mu|\pi-\xi-x|) + (dK_{\pi/2}(x,x) - K_{\pi/2}(x,x) - \tilde{K}(\pi))(d\sin\mu(\xi+\pi-x)$$

$$+ d\,\operatorname{sgn}(\xi-\pi+x)\sin\mu|\xi-\pi+x| - \sin\mu(2\pi-\xi-x) - \operatorname{sgn}(x-\xi)\sin\mu|x-\xi|)$$

$$+ \int\limits_0^{\pi}\tilde{K}'(t)(d(\operatorname{sgn}(\xi+t-x)\sin\mu|\xi+t-x| + \operatorname{sgn}(\xi-t+x)\sin\mu|\xi-t+x|) - \operatorname{sgn}(\pi-\xi+t-x)$$

$$\times \sin\mu|\pi-\xi+t-x| - \operatorname{sgn}(\pi-\xi-t+x)\sin\mu|\pi-\xi-t+x|)\,dt + \int\limits_{\pi-x}^{x}\left(\tilde{K}(0)K_{\pi/2}(x,t) + \frac{\partial K_{\pi/2}(x,t)}{\partial t}\right)$$

$$\times (d(\sin\mu(\xi+t) + \operatorname{sgn}(\xi-t)\sin\mu|\xi-t| - \sin\mu(\pi-\xi+t)) - \operatorname{sgn}(\pi-\xi-t)\sin\mu|\pi-\xi-t|)\,dt$$

$$- \int\limits_{\pi-x}^{x}\left(\tilde{K}(\pi)K_{\pi/2}(x,t) + d\frac{\partial K_{\pi/2}(x,t)}{\partial t}\right)(d(\sin\mu(\xi+\pi-t) + \operatorname{sgn}(\xi-\pi+t)\sin\mu|\xi-\pi+t|)$$

$$- \sin\mu(2\pi - \xi - t) - \operatorname{sgn}(t-\xi)\sin\mu|t-\xi|)\,dt + \int_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt \int_0^\pi \tilde{K}'(\tau)$$

$$\times (d(\operatorname{sgn}(\xi+\tau-t)\sin\mu|\xi+\tau-t| + \operatorname{sgn}(\xi-\tau+t)\sin\mu|\xi-\tau+t|$$

$$- \operatorname{sgn}(\pi-\xi+\tau-t)\sin\mu|\pi-\xi+\tau-t| + \operatorname{sgn}(\pi-\xi-\tau+t)\sin\mu|\pi-\xi-\tau+t|)\,d\tau\Big], \tag{57}$$

$$\Phi_3(\mu,x,\xi) = \Theta_2(\mu,x)\Theta_2^*(\mu,\xi) = \mu^{-1}\Big[\int_0^\pi \tilde{K}(t)\sin\mu(t-x)\,dt + \int_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt \int_0^\pi \tilde{K}(\tau)\sin\mu(\tau-t)\,d\tau$$

$$+ \int_{\pi-x}^{x} K_{\pi/2}(x,t)(\sin\mu t + d\sin\mu(\pi-t))\,dt\Big]\Big[(\tilde{K}^*(0) + dK_{\pi/2}(\xi,\pi-\xi) + K_{\pi/2}(\xi,\pi-\xi))\cos\mu\xi$$

$$- (\tilde{K}^*(\pi) - K_{\pi/2}(\xi,\xi) - dK_{\pi/2}(\xi,\xi))\cos\mu(\pi-\xi) + \int_{\pi-\xi}^{\xi} \left(\tilde{K}^*(0)K_{\pi/2}(\xi,\alpha) + d\frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\right)\cos\mu\alpha\,d\alpha$$

$$- \int_{\pi-\xi}^{\xi} \left(\tilde{K}^*(\pi)K_{\pi/2}(\xi,\alpha) - \frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\right)\cos\mu(\pi-\alpha)\,d\alpha + \int_0^\pi (\tilde{K}^*(\alpha))'\cos\mu(\alpha-\xi)\,d\alpha$$

$$+ \int_{\pi-\xi}^{\xi} K_{\pi/2}(\xi,\alpha)\,d\alpha \int_0^\pi (\tilde{K}^*(\beta))'\cos\mu(\beta-\alpha)\,d\beta\Big]$$

$$= (2\mu)^{-1}\Big[(\tilde{K}^*(0) + dK_{\pi/2}(\xi,\pi-\xi) + K_{\pi/2}(\xi,\pi-\xi)) \int_0^\pi \tilde{K}(t)(\operatorname{sgn}(t-x+\xi)\sin\mu|t-x+\xi|$$

$$+ \operatorname{sgn}(t-x-\xi)\sin\mu|t-x-\xi|)\,dt - (\tilde{K}^*(\pi) - K_{\pi/2}(\xi,\xi) - dK_{\pi/2}(\xi,\xi))$$

$$\times \int_0^\pi \tilde{K}(t)(\operatorname{sgn}(t-x+\pi-\xi)\sin\mu|t-x+\pi-\xi| + \operatorname{sgn}(t-x-\pi+\xi)\sin\mu|t-x-\pi+\xi|)\,dt$$

$$+ \int_0^\pi \tilde{K}(t)\,dt \int_{\pi-\xi}^{\xi} \left(\tilde{K}^*(0)K_{\pi/2}(\xi,\alpha) + d\frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\right)(\operatorname{sgn}(t-x+\alpha)\sin\mu|t-x+\alpha| + \operatorname{sgn}(t-x-\alpha)$$

$$\times \sin\mu|t-x-\alpha|)\,d\alpha - \int_0^\pi \tilde{K}(t)\,dt \int_{\pi-\xi}^{\xi} \left(\tilde{K}^*(\pi)K_{\pi/2}(\xi,\alpha) - \frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\right)$$

$$\times (\operatorname{sgn}(t-x+\pi-\alpha)\sin\mu|t-x+\pi-\alpha| + \operatorname{sgn}(t-x-\pi+\alpha)\sin\mu|t-x-\pi+\alpha|)\,d\alpha$$

$$+ \int_0^\pi \tilde{K}(t)\,dt \int_0^\pi (\tilde{K}^*(\alpha))'(\operatorname{sgn}(t-x+\alpha-\xi)\sin\mu|t-x+\alpha-\xi| + \operatorname{sgn}(t-x-\alpha+\xi)\sin\mu|t-x-\alpha+\xi|)\,d\alpha$$

$$+ \int_0^\pi \tilde{K}(t)\,dt \int_{\pi-\xi}^{\xi} K_{\pi/2}(\xi,\alpha)\,d\alpha \int_0^\pi (\tilde{K}^*(\beta))'(\operatorname{sgn}(t-x+\beta-\alpha)\sin\mu|t-x+\beta-\alpha| + \operatorname{sgn}(t-x-\beta+\alpha)$$

$$\times \sin\mu|t-x-\beta+\alpha|)\,d\beta + (\tilde{K}^*(0) + dK_{\pi/2}(\xi,\pi-\xi) + K_{\pi/2}(\xi,\pi-\xi)) \int_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt \int_0^\pi \tilde{K}(\tau)$$

$$\times (\operatorname{sgn}(\tau-t+\xi)\sin\mu|\tau-t+\xi| + \operatorname{sgn}(\tau-t-\xi)\sin\mu|\tau-t-\xi|)\,d\tau - (\tilde{K}^*(\pi) - K_{\pi/2}(\xi,\xi)$$

$$- dK_{\pi/2}(\xi,\xi)) \int_{\pi-x}^{x} K_{\pi/2}(x,t)\,dt \int_0^\pi \tilde{K}(\tau)\,d\tau(\operatorname{sgn}(\tau-t+\pi-\xi)\sin\mu|\tau-t+\pi-\xi|$$

$$+\operatorname{sgn}(\tau-t-\pi+\xi)\sin\mu|\tau-t-\pi+\xi|)+\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{0}^{\pi}\tilde{K}(\tau)\,d\tau\int\limits_{\pi-\xi}^{\xi}\Big(\tilde{K}^*(0)K_{\pi/2}(\xi,\alpha)$$

$$+d\frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\Big)(\operatorname{sgn}(\tau-t+\alpha)\sin\mu|\tau-t+\alpha|+\operatorname{sgn}(\tau-t-\alpha)\sin\mu|\tau-t-\alpha|)\,d\alpha$$

$$-\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{0}^{\pi}\tilde{K}(\tau)\,d\tau\int\limits_{\pi-\xi}^{\xi}\Big(\tilde{K}^*(\pi)K_{\pi/2}(\xi,\alpha)-\frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\Big)(\operatorname{sgn}(\tau-t+\pi-\alpha)$$

$$\times\sin\mu|\tau-t+\pi-\alpha|+\operatorname{sgn}(\tau-t+\pi-\alpha)\sin\mu|\tau-t+\pi-\alpha|)\,d\alpha+\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{0}^{\pi}\tilde{K}(\tau)\,d\tau$$

$$\times\int\limits_{0}^{\pi}(\tilde{K}^*(\alpha))'(\operatorname{sgn}(\tau-t+\alpha-\xi)\sin\mu|\tau-t+\alpha-\xi|+\operatorname{sgn}(\tau-t-\alpha+\xi)\sin\mu|\tau-t-\alpha+\xi|)\,d\alpha$$

$$+\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{0}^{\pi}\tilde{K}(\tau)\,d\tau\int\limits_{\pi-\xi}^{\xi}K_{\pi/2}(\xi,\alpha)\,d\alpha\int\limits_{0}^{\pi}(\tilde{K}^*(\beta))'(\operatorname{sgn}(\tau-t+\beta-\alpha)\sin\mu|\tau-t+\beta-\alpha|$$

$$+\operatorname{sgn}(\tau-t-\beta+\alpha)\sin\mu|\tau-t-\beta+\alpha|)\,d\beta+(\tilde{K}^*(0)+dK_{\pi/2}(\xi,\pi-\xi)+K_{\pi/2}(\xi,\pi-\xi))\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)$$

$$\times(\sin\mu(t+\xi)+\operatorname{sgn}(t-\xi)\sin\mu|t-\xi|+d(\sin\mu(\pi-t+\xi)+\operatorname{sgn}(\pi-t-\xi)\sin\mu|\pi-t-\xi|))\,dt$$

$$-(\tilde{K}^*(\pi)-K_{\pi/2}(\xi,\xi)-dK_{\pi/2}(\xi,\xi))\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)(\sin\mu(t+\pi-\xi)+\operatorname{sgn}(t-\pi+\xi)\sin\mu|t-\pi+\xi|$$

$$+d(\sin\mu(2\pi-t-\xi)+\operatorname{sgn}(\xi-t)\sin\mu|\xi-t|))\,dt+\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{\pi-\xi}^{\xi}\Big(\tilde{K}^*(0)K_{\pi/2}(\xi,\alpha)+d\frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\Big)$$

$$\times(\sin\mu(t+\alpha)+\operatorname{sgn}(t-\alpha)\sin\mu|t-\alpha|+d(\sin\mu(\pi-t+\alpha)+\operatorname{sgn}(\pi-t-\alpha)\sin\mu|\pi-t-\alpha|))\,d\alpha$$

$$-\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{\pi-\xi}^{\xi}\Big(\tilde{K}^*(\pi)K_{\pi/2}(\xi,\alpha)-\frac{\partial K_{\pi/2}(\xi,\alpha)}{\partial\alpha}\Big)(\sin\mu(t+\pi-\alpha)+\operatorname{sgn}(t-\pi+\alpha)$$

$$\times\sin\mu|t-\pi+\alpha|+d(\sin\mu(2\pi-t-\alpha)+\operatorname{sgn}(\alpha-t)\sin\mu|\alpha-t|))\,d\alpha+\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{0}^{\pi}(\tilde{K}^*(\alpha))'$$

$$\times(\operatorname{sgn}(t+\alpha-\xi)\sin\mu|t+\alpha-\xi|+\operatorname{sgn}(t-\alpha+\xi)\sin\mu|t-\alpha+\xi|$$

$$+d(\operatorname{sgn}(\pi-t+\alpha-\xi)\sin\mu|\pi-t+\alpha-\xi|$$

$$+\operatorname{sgn}(\pi-t-\alpha+\xi)\sin\mu|\pi-t-\alpha+\xi|))\,d\alpha+\int\limits_{\pi-x}^{x}K_{\pi/2}(x,t)\,dt\int\limits_{\pi-\xi}^{\xi}K_{\pi/2}(\xi,\alpha)\,d\alpha\int\limits_{0}^{\pi}(\tilde{K}^*(\beta))'$$

$$\times(\operatorname{sgn}(t+\beta-\alpha)\sin\mu|t+\beta-\alpha|+\operatorname{sgn}(t-\beta+\alpha)\sin\mu|t-\beta+\alpha|$$

$$+d(\operatorname{sgn}(\pi-t+\beta-\alpha)\sin\mu|\pi-t+\beta-\alpha|+|\operatorname{sgn}(\pi-t-\beta+\alpha)\sin\mu|\pi-t-\beta+\alpha|))\,d\beta\Big]. \tag{58}$$

REFERENCES

1. Marchenko, V.A., *Operatory Shturma–Liuvillya i ikh prilozheniya* (Sturm–Liouville Operators and Their Applications), Kiev: Naukova Dumka, 1977.
2. Biyarov, B.N., Spectral properties of correct restrictions and extensions of the Sturm–Liouville operator, *Differ. Equations*, 1994, vol. 30, no. 12, pp. 1863–1868.

3. Malamud, M.M., On the completeness of the system of root vectors of the Sturm–Liouville operator with general boundary conditions, *Funct. Anal. Appl.*, 2008, vol. 42, no. 3, pp. 198–204.
4. Makin, A.S., On the completeness of the system of root functions of the Sturm–Liouville operator with degenerate boundary conditions, *Differ. Equations*, 2014, vol. 50, no. 6, pp. 835–839.
5. Minkin, A., Resolvent growth and Birkhoff regularity, *J. Math. Anal. Appl.*, 2006, vol. 323, no. 1, pp. 387–402.
6. Kerimov, N.B., Necessary and sufficient conditions for a system of root functions of a differential operator to be a basis, *Differ. Equations*, 1996, vol. 32, no. 1, pp. 38–45.
7. Makin, A.S., On a two-point boundary value problem for the Sturm–Liouville operator with a nonclassical asymptotics of the spectrum, *Differ. Equations*, 2013, vol. 49, no. 5, pp. 536–544.
8. Makin, A.S., Problem with nonclassical eigenvalue asymptotics, *Differ. Equations*, 2015, vol. 51, no. 3, pp. 318–324.
9. Makin, A.S., Characterization of the spectrum of the Sturm–Liouville operator with irregular boundary conditions, *Differ. Equations*, 2010, vol. 46, no. 10, pp. 1427–1437.
10. Levin, B.Ya., *Tselye funktsii* (Entire Functions), Moscow: Mosk. Gos. Univ., 1971.
11. Levin, B.Ya., *Raspredelenie kornei tselykh funktsii* (Distribution of Roots of Entire Functions), Moscow: Gos. Izd. Fiz. Mat. Lit., 1956.
12. Makin, A.S., On the divergence of expansions in systems of root functions of the Sturm–Liouville operator with degenerate boundary conditions, *Differ. Equations*, 2012, vol. 48, no. 8, pp. 1183–1187.
13. Naimark, M.A., *Lineinye differentsial'nye operatory* (Linear Differential Operators), Moscow: Nauka, 1969.
14. Titchmarsh, E.C., *The Theory of Functions*, Oxford: Oxford Univ., 1939. Translated under the title *Teoriya funktsii*, Moscow: Nauka, 1980.
15. Makin, A.S., On a new class of boundary value problems for the Sturm–Liouville operator, *Differ. Equations*, 2013, vol. 49, no. 2, pp. 262–266.
16. Keldysh, M.V., On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators, *Uspekhi Mat. Nauk*, 1971, vol. 26, no. 4, pp. 15–41.
17. Nikol'skii, S.M., *Priblizhenie funktsii mnogikh peremennykh i teoremy vlozheniya* (Approximation of Functions of Many Variables and Embedding Theorems), Moscow: Nauka, 1977.