ISSN 0012-2661, Differential Equations, 2018, Vol. 54, No. 10, pp. 1285–1309. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © S.D. Glyzin, A.Yu. Kolesov, N.Kh. Rozov, 2018, published in Differentsial'nye Uravneniya, 2018, Vol. 54, No. 10, pp. 1313–1337.

ORDINARY DIFFERENTIAL EQUATIONS

Stable Relaxation Cycle in a Bilocal Neuron Model

S. D. Glyzin[1,2]*, A. Yu. Kolesov[1], and N. Kh. Rozov[3]*****

[1] *Demidov Yaroslavl State University, Yaroslavl, 150003 Russia*
[2] *Scientific Center of the Russian Academy of Sciences in Chernogolovka, Moscow oblast, 142432 Russia*
[3] *Lomonosov Moscow State University, Moscow, 119991 Russia*
*e-mail: *glyzin@uniyar.ac.ru, *glyzin.s@gmail.com, **andkolesov@mail.ru, ***nicolas_rozov@mail.ru*

Received December 22, 2017

Abstract—We consider the so-called bilocal neuron model, which is a special system of two nonlinear delay differential equations coupled by linear diffusion terms. The system is invariant under the interchange of phase variables. We prove that, under an appropriate choice of parameters, the system under study has a stable relaxation cycle whose components turn into each other under a certain phase shift.

DOI: 10.1134/S0012266118100026

1. STATEMENT OF THE PROBLEM AND DESCRIPTION OF THE RESULTS

Following [1, 2], we assume that the operation of an individual neuron is modeled by the equation

$$\dot u = \lambda f(u(t-1))u \tag{1.1}$$

for the membrane potential $u = u(t) > 0$. Here the parameter $\lambda > 0$ characterizing the rate of electric processes in the neuron is assumed to be large, the dot over a function stands for differentiation with respect to t, and the function $f(u) \in C^2(\mathbb{R}_+)$, $\mathbb{R}_+ = \{u \in \mathbb{R} : u \ge 0\}$, has the properties

$$f(0) = 1, \qquad f(u) = -a + O(1/u), \qquad uf'(u) = O(1/u), \qquad u^2 f''(u) = O(1/u) \tag{1.2}$$

as $u \to +\infty$, where $a =$ const > 0. An example of such a function is given by

$$f(u) = \frac{1-u}{1+u/a}. \tag{1.3}$$

Equation (1.1), which is a modification of the well-known Hutchinson equation [3], was proposed in [4], where it was shown that for $\lambda \gg 1$ it admits an exponentially orbitally stable cycle $u_*(t,\lambda) > 0$, $u_*(0,\lambda) \equiv 1$, of period $T_*(\lambda)$ satisfying the limit relations

$$\lim_{\lambda\to+\infty} T_*(\lambda) = T_*, \qquad \max_{0\le t\le T_*(\lambda)} |\omega_*(t,\lambda) - \omega_*(t)| = O(1/\lambda), \qquad \lambda \to +\infty, \tag{1.4}$$

where $T_* = (1+a)t_0$, $t_0 = 1 + 1/a$, $\omega_*(t,\lambda) = (1/\lambda)\ln u_*(t,\lambda)$, and the T_*-periodic function $\omega_*(t)$ is given by the relations

$$\omega_*(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ 1 - a(t-1) & \text{for } 1 \le t \le t_0 + 1, \\ t - T_* & \text{for } t_0 + 1 \le t \le T_*, \end{cases} \qquad \omega_*(t + T_*) \equiv \omega_*(t). \tag{1.5}$$

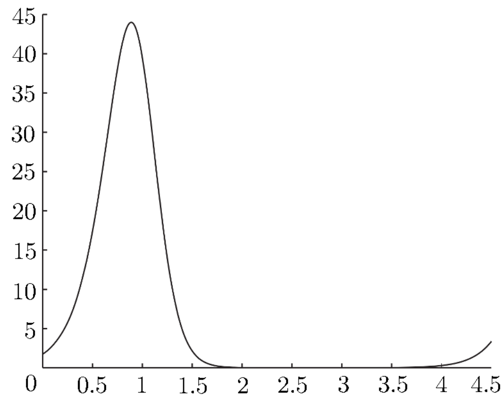

Fig. 1.

The relaxation properties of this cycle are visualized by its graph on the plane (t, u) numerically constructed for the case of Eq. (1.1) with $f(u)$ given by (1.3), $\lambda = 5$, and $a = 2$ (see Fig. 1).

Now consider a system of two electrically coupled neurons. Their membrane potentials $u_1(t)$ and $u_2(t)$ satisfy a system of differential-difference equations of the form

$$\begin{aligned} \dot{u}_1 &= d(u_2 - u_1) + \lambda f(u_1(t-1))u_1, \\ \dot{u}_2 &= d(u_1 - u_2) + \lambda f(u_2(t-1))u_2, \end{aligned} \tag{1.6}$$

where the parameter $d > 0$ characterizes the neuron coupling strength.

It is of interest to note that system (1.6) also admits an ecological interpretation; namely, one can assume that Eq. (1.1) (recall that it is a generalization of the Hutchinson equation in ecology) describes the dynamics of the mammal population density in a homogeneous natural habitat. We also assume that the food potential is stably recovered to a certain fixed level and migration is strong enough to damp spatial perturbations. Under these biological hypotheses, we consider two local habitats connected by a narrow passage. As a result, we obtain system (1.6) for the population densities u_1 and u_2 in these habitats. Following the biological terminology, we refer to this system as the bilocal neuron model or simply the bilocal model.

One possible stationary mode of system (1.6) is the so-called self-symmetric cycle, which is preserved under the change of variables $(u_1, u_2) \mapsto (u_2, u_1)$ and does not coincide with the homogeneous cycle

$$(u_1, u_2) = (u_*(t, \lambda), u_*(t, \lambda)), \tag{1.7}$$

where $u_*(t, \lambda)$ is a periodic solution of Eq. (1.1). By the above-cited properties, this cycle admits the representation

$$(u_1, u_2) = (u_{**}(t, \lambda), u_{**}(t - h(\lambda), \lambda)), \tag{1.8}$$

where $h = h(\lambda) > 0$ is a phase shift, and has the period $T = 2h$. The function $u_{**}(t, \lambda)$, $u_{**}(0, \lambda) \equiv 1$, in the representation (1.8) is a $2h$-periodic solution of the scalar equation

$$\dot{u} = d(u(t-h) - u) + \lambda f(u(t-1))u. \tag{1.9}$$

In the present paper, we show that, under the conditions

$$d = \lambda \exp(-b\lambda), \qquad b = \text{ const } > 0, \qquad \lambda \gg 1, \tag{1.10}$$

$$a > 1, \qquad 1 + \frac{1}{a} < b < 1 + \frac{1}{2}\left(a + \frac{1}{a}\right), \tag{1.11}$$

there exists a self-symmetric cycle (1.8) of the bilocal model (1.6) and that this cycle is stable. Its asymptotic properties are also studied.

To state the corresponding rigorous result, we introduce some notation. Namely, consider the $2b$-periodic function $\omega_{**}(t)$ given by the expression

$$\omega_{**}(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ 1 - a(t-1) & \text{for } 1 \le t \le t_1, \\ t - 2b & \text{for } t_1 \le t \le 2b, \end{cases} \qquad \omega_{**}(t+2b) \equiv \omega_{**}(t), \tag{1.12}$$

where $t_1 = 1 + 2b/(a+1)$. The graph of this function is shown in Fig. 2, where the dashed graph of the function $\omega_*(t)$ (see (1.5)) is given for comparison. By the inequality $t_1 < t_0 + 1$, which follows from inequalities (1.11), the minimum of the function $\omega_{**}(t)$ is greater than the minimum of the function $\omega_*(t)$, and its period $2b$ is less than T_*.

It turns out that the periodic function (1.12) is the zero-order approximation as $\lambda \to +\infty$ to the function $\omega_{**}(t,\lambda) = (1/\lambda)\ln u_{**}(t,\lambda)$, where $u_{**}(t,\lambda)$ is the function given in (1.8). More precisely, the following assertions hold.

Theorem 1.1. *Under conditions* (1.10) *and* (1.11), *there exists a sufficiently large* $\lambda_0 > 0$ *such that system* (1.6) *admits a self-symmetric cycle* (1.8) *for all* $\lambda \ge \lambda_0$. *As* $\lambda \to +\infty$, *the following asymptotic representations hold for this cycle*:

$$h(\lambda) = b - \frac{\ln\lambda}{\lambda} - \frac{\ln(b-t_0)}{\lambda} + O\left(\frac{\ln\lambda}{\lambda^2}\right),$$
$$\max_{0 \le t \le 2h(\lambda)} |\omega_{**}(t,\lambda) - \omega_{**}(t)| = O\left(\frac{\ln\lambda}{\lambda}\right). \tag{1.13}$$

Theorem 1.2. *The cycle* (1.8) *in Theorem* 1.1 *is exponentially orbitally stable.*

The proof of Theorem 1.1 is based on the search of a $2h$-periodic solution $u = u_{**}(t,\lambda)$ of the auxiliary equation (1.9) with the properties (1.13). The stability of the cycle (1.8) is established separately by an asymptotic analysis of the corresponding linear variational system.

Comparing formulas (1.4), (1.5) with formulas (1.12), (1.13), we see that the self-symmetric cycle (1.8) of system (1.6) has significantly better biological characteristics than the homogeneous cycle (1.7). Indeed, by the mutual location of the graphs of the functions (1.5) and (1.12) in Fig. 2, the period of the cycle (1.8) is less than the period of the homogeneous cycle, and conversely, the minima of its components are greater than the corresponding minima of the cycle (1.7). This fact allows us to speak about the self-organization phenomenon observed in the framework of the

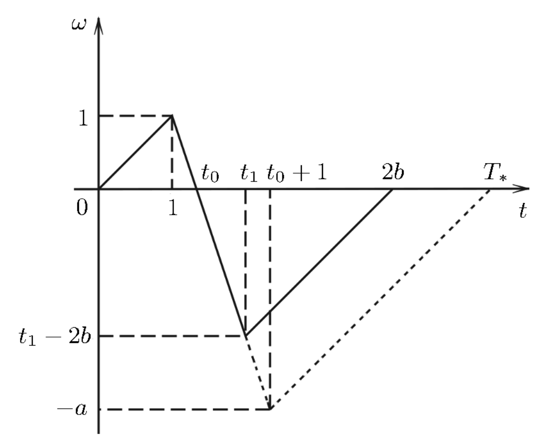

Fig. 2.

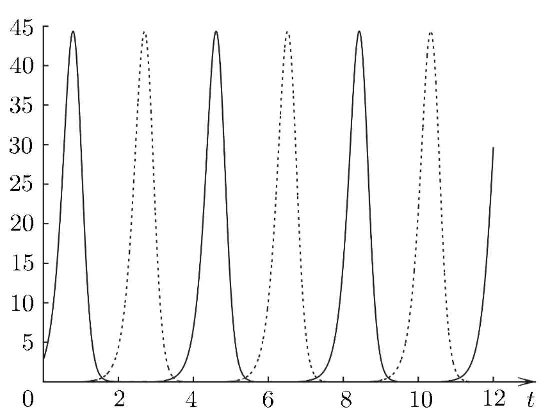

Fig. 3.

biological model (1.6). In the ecological interpretation, the essence of this phenomenon is that the species under study artificially creates an inhomogeneous habitat by migration between the two local habitats and thus improves its own biological characteristics. Therefore, it is expedient to say that the periodic mode (1.8) is a self-organization mode. It is visualized in Fig. 3, where the graphs of its components $u_1(t)$ and $u_2(t)$ are drawn on the plane (t, u) in case (1.3), (1.10) for $\lambda = 5$, $a = 2$, and $b = 1.87$. (The solid line shows the graph of the function $u_1(t)$; the dashed line, the graph of the function $u_2(t)$.)

2. PROOF OF THEOREM 1.1.

2.1. General Scheme of the Study

As was already said, Theorem 1.1 is justified by analyzing the auxiliary equation (1.9), and more precisely, by seeking its nonconstant $2h$-periodic solution.

Under conditions (1.10) and (1.11), in Eq. (1.9) we make the change the variable $u = \exp(\lambda\,\omega)$ and set $\varepsilon = 1/\lambda \ll 1$. As a result, for the new variable $\omega = \omega(t)$ we obtain the equation

$$\dot\omega = \exp\left(\frac{\omega(t-h)-\omega-b}{\varepsilon}\right) - \exp\left(-\frac{b}{\varepsilon}\right) + f\left(\exp\left(\frac{\omega(t-1)}{\varepsilon}\right)\right). \tag{2.1}$$

Throughout the following, we assume that the delay h in this equation ranges in the set

$$\Omega \overset{\text{def}}{=} [b-\delta_0, b+\delta_0], \tag{2.2}$$

where the value of the constant $\delta_0 > 0$ will be specified below. Now we assume that the following inequality holds:

$$\delta_0 < b - 1. \tag{2.3}$$

Now let us describe the class of initial conditions for Eq. (2.1). To this end, we fix a sufficiently small constant $\sigma_0 > 0$. Just as in the case of constant δ_0, we further impose several restrictions on it. In particular, we assume everywhere below that

$$\sigma_0 < (h-1)/2, \qquad h \in \Omega. \tag{2.4}$$

(This inequality is possible, because $h > 1$ for all $h \in \Omega$ by inequality (2.3).)

Condition (2.4) permits representing the interval $I = [-h-\sigma_0, -\sigma_0]$ in the form $I = I_1 \bigcup I_2$, where $I_1 = [-h-\sigma_0, -h+1+\sigma_0]$ and $I_2 = [-h+1+\sigma_0, -\sigma_0]$. We use this representation to define the desired set S of initial functions $\varphi(t)$ continuous for $t \in I$ by the conditions

$$S = \{\varphi(t) : -q_1 \le \varphi(t) \le -q_2 \text{ for } t \in I_1,\ \varphi(-\sigma_0) = -\sigma_0,\ |\varphi(t) - t| \le \exp(-1/\sqrt{\varepsilon}) \text{ for } t \in I_2\}, \tag{2.5}$$

where $q_1 > q_2 > 0$ are some universal constants (constants independent of t, ε, h, and φ) which we choose below.

Consider the solution $\omega = \omega_\varphi(t, \varepsilon, h)$, $t \ge -\sigma_0$, of Eq. (2.1) with an arbitrary initial function $\varphi(t) \in S$ for $t \in I$. By $t = T_\varphi(\varepsilon, h)$ we denote the second positive root of the equation

$$\omega_\varphi(t-\sigma_0, \varepsilon, h) = -\sigma_0 \tag{2.6}$$

(if it exists) and define an operator Π acting from the set S to the space $C(I)$ of functions continuous for $t \in I$ by the rule

$$\Pi(\varphi) = \omega_\varphi(t + T_\varphi(\varepsilon, h), \varepsilon, h), \qquad t \in I. \tag{2.7}$$

As will be shown below, for an appropriate choice of the parameters q_1, q_2, δ_0, and σ_0, the operator (2.7) is defined on the set (2.5), and one has $\Pi(S) \subset S$ and $T_\varphi(\varepsilon, h) > h$ for all $h \in \Omega$ and $\varphi \in S$. Since the set S is closed, bounded, and convex and the operator Π is compact by the inequality $T_\varphi > h$, it has at least one fixed point $\varphi = \widetilde\varphi(t, \varepsilon, h)$ in the set S by the Schauder principle. It is also obvious that the solution $\widetilde{\omega}(t,\varepsilon,h)=\omega_\varphi|_{\varphi=\widetilde{\varphi}}$ of Eq. (2.1) is periodic with period $\widetilde{T}(\varepsilon,h)=T_\varphi|_{\varphi=\widetilde{\varphi}}$. As to the additional parameter h ranging in the set (2.2), it can be determined from the equation

$$\widetilde{T}(\varepsilon,h)=2h. \tag{2.8}$$

It turns out that Eq. (2.8) admits a solution $h=h(\varepsilon)$ bounded in ε and such that $h(\varepsilon)\to b$ as $\varepsilon\to 0$. In turn, this implies that for $h=h(\varepsilon)$ the auxiliary equation (2.1) has the desired $2h(\varepsilon)$-periodic solution $\omega(t,\varepsilon)=\widetilde{\omega}(t,\varepsilon,h)|_{h=h(\varepsilon)}$.

2.2. Asymptotic Integration of an Auxiliary Scalar Equation

To implement the scheme described in the preceding section, it is necessary to know the asymptotics as $\varepsilon\to 0$ (uniform in $h\in\Omega$ and $\varphi\in S$) of the solution $\omega_\varphi(t,\varepsilon,h)$ on various intervals of time t. The process of constructing this asymptotics is divided into nine stages, each of which has its own lemma. Further, the same letter q is used to denote some universal positive constants (constants independent of t, ε, h, and φ) whose exact values are of no importance.

At the first of these stages, consider the interval

$$-\sigma_0\le t\le 1-\sigma_0. \tag{2.9}$$

We have the following assertion.

Lemma 2.1. *The asymptotic representation*

$$\omega_\varphi(t,\varepsilon,h)=t+O(\exp(-q/\varepsilon)) \tag{2.10}$$

holds as $\varepsilon\to 0$ *on the interval* (2.9) *uniformly in* t, h, *and* φ.

Proof. Since the delay h is greater than unity by inequality (2.3) and the interval (2.9) has unit length, it follows that the functions $\omega(t-1)$ and $\omega(t-h)$ coincide with the functions $\varphi(t-1)$ and $\varphi(t-h)$, respectively, for t in this interval. Thus, we obtain the Cauchy problem

$$\dot{\omega}=\exp((\varphi(t-h)-\omega-b)/\varepsilon)-\exp(-b/\varepsilon)+f(\exp(\varphi(t-1)/\varepsilon)),\qquad \omega|_{t=-\sigma_0}=-\sigma_0. \tag{2.11}$$

The analysis of problem (2.11) is based on estimates of the form

$$\varphi(t-1)\le -M_1,\qquad \varphi(t-h)-\omega_\varphi(t,\varepsilon,h)-b\le -M_2,\qquad M_1,M_2=\text{ const }>0. \tag{2.12}$$

Just as with the letter q, from now on by const we denote various positive constants independent of t, ε, h, and φ. Note that, by the definition of the set S (see (2.5)), the first inequality in (2.12) is satisfied automatically, while the second inequality is proved at the end of the lemma.

In the equation in (2.11), we take into account relations (2.12) together with the representation

$$f(\exp(\varphi(t-1)/\varepsilon))=1+O(\exp(-q/\varepsilon))\quad\text{as}\quad \varepsilon\to 0,$$

which follows from properties (1.2), and conclude that the Cauchy problem in question can be written as

$$\dot{\omega}=1+O(\exp(-q/\varepsilon)),\qquad \omega|_{t=-\sigma_0}=-\sigma_0,$$

which obviously implies the asymptotic formula (2.10).

To complete the justification of the lemma, it remains to verify the second inequality in (2.12). Based on relation (2.10) (yet only a priori one), we have

$$\varphi(t-h)-\omega_\varphi(t,\varepsilon,h)-b<-\omega_\varphi(t,\varepsilon,h)-b=-t-b+O(\exp(-q/\varepsilon)).$$

This means that this inequality is indeed satisfied with any constant M_2 in the interval $(0,b-\sigma_0)$. The proof of Lemma 2.1 is complete.

At the second stage, consider the interval

$$1 - \sigma_0 \le t \le 1 + \sigma_0 \tag{2.13}$$

under the condition that

$$\sigma_0 < \min(2^{-1}, (b+1)/a). \tag{2.14}$$

Inequality (2.14) guarantees the inclusions

$$t - 1 \in [-\sigma_0, \sigma_0] \subset [-\sigma_0, 1 - \sigma_0], \qquad t - h \in [-h + 1 - \sigma_0, -h + 1 + \sigma_0] \subset I_1.$$

This, definition (2.5) and the representation (2.10) imply that

$$\omega_\varphi(t-1, \varepsilon, h) = t - 1 + O(\exp(-q/\varepsilon)), \qquad \omega_\varphi(t-h, \varepsilon, h) = \varphi(t-h). \tag{2.15}$$

We have the following assertion for the function $\omega_\varphi(t, \varepsilon, h)$ on the interval (2.13).

Lemma 2.2. *The asymptotic representation*

$$\omega_\varphi(t, \varepsilon, h) = 1 + \varepsilon v_0(\tau)|_{\tau=(t-1)/\varepsilon} + O(\exp(-q/\varepsilon)) \tag{2.16}$$

as $\varepsilon \to 0$, *where*

$$v_0(\tau) = \tau + \int\limits_{-\infty}^{\tau} [f(\exp s) - 1]\, ds, \tag{2.17}$$

holds uniformly in $t \in [1 - \sigma_0, 1 + \sigma_0]$, $h \in \Omega$, *and* $\varphi \in S$.

Proof. As in the proof of Lemma 2.1, formulas (2.16) and (2.17) are justified first under the a priori assumption

$$\varphi(t-h) - \omega_\varphi(t, \varepsilon, h) - b \le -M, \qquad M = \text{ const } > 0, \tag{2.18}$$

and then we verify whether condition (2.18) itself is satisfied.

On the right-hand side in Eq. (2.1), we take into account relations (2.15) and (2.18) and make the change of time $\tau = (t-1)/\varepsilon$ in this equation. As a result, we obtain the Cauchy problem

$$\begin{gathered} \frac{d\omega}{d\tau} = \varepsilon f[\exp(\tau + O(\exp(-q/\varepsilon)))] + O(\exp(-q/\varepsilon)), \\ \omega|_{\tau=-\sigma_0/\varepsilon} = \omega_\varphi(t, \varepsilon, h)|_{t=1-\sigma_0} = 1 - \sigma_0 + O(\exp(-q/\varepsilon)). \end{gathered} \tag{2.19}$$

To analyze problem (2.19), we use the estimate

$$|f(u_1) - f(u_2)| \le \frac{M}{1 + \min(u_1^2, u_2^2)} |u_1 - u_2|, \quad M = \sup_{u \ge 0} (1 + u^2)|f'(u)| < \infty, \quad u_1, u_2 \ge 0, \tag{2.20}$$

and the asymptotic representation

$$v_0(\tau) = \tau + O(\exp \tau) \quad \text{as} \quad \tau \to -\infty, \tag{2.21}$$

which follows from properties (1.2). By relations (2.20) and (2.21), it is easily seen that the asymptotic relations

$$\begin{gathered} f[\exp(\tau + O(\exp(-q/\varepsilon)))] = f(\exp \tau) + O(\exp(-q/\varepsilon)), \\ 1 + \varepsilon v_0(\tau)|_{\tau=-\sigma_0/\varepsilon} = 1 - \sigma_0 + O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0 \end{gathered}$$

hold uniformly in $\tau \in [-\sigma_0/\varepsilon, \sigma_0/\varepsilon]$. This and problem (2.19) automatically imply the desired formulas (2.16) and (2.17).

To prove the estimate (2.18), along with (2.21), we use the asymptotic representation (which holds due to properties (1.2))

$$v_0(\tau) = -a\tau + c_0 + O(\exp(-\tau)) \quad \text{as} \quad \tau \to +\infty, \tag{2.22}$$

where

$$c_0 = \int_0^1 \frac{f(u)-1}{u}\,du + \int_1^{+\infty} \frac{f(u)+a}{u}\,du. \tag{2.23}$$

As a result, in view of the inequalities $a > 1$ and $a\sigma_0 < b+1$ (see (1.11) and (2.14)), we see that

$$\varphi(t-h) - \omega_\varphi(t,\varepsilon,h) - b < -\omega_\varphi(t,\varepsilon,h) - b = -b - 1 - \varepsilon v_0(\tau)|_{\tau=(t-1)/\varepsilon} + O(\exp(-q/\varepsilon)) \le -M,$$

where $M = \text{const} \in (0, b+1-a\sigma_0)$. The proof of Lemma 2.2 is complete.

At the third stage, consider the values of t in the interval

$$1 + \sigma_0 \le t \le t_1(h) - \sigma_0, \tag{2.24}$$

where $t_1(h) = 1 + (b+h)/(a+1)$, under the assumptions that

$$t_0 < t_1(h) < t_0 + 1, \qquad t_0 < h, \qquad \sigma_0 < \min(a^{-1}, (t_1(h)-1)/2). \tag{2.25}$$

Note that for $\delta_0 = 0$, i.e., for $h = b$, the first two of these inequalities are satisfied (see (1.11)). Therefore, they are also satisfied for a sufficiently small $\delta_0 > 0$. Throughout the following, we assume that δ_0 in (2.2) is chosen precisely in this way.

Lemma 2.3. *The asymptotic relation*

$$\omega_\varphi(t,\varepsilon,h) = 1 - a(t-1) + \varepsilon c_0 + O(\exp(-q/\varepsilon)), \tag{2.26}$$

where c_0 is the constant (2.23), *holds as $\varepsilon \to 0$ on the interval* (2.24) *uniformly in t, h, and φ.*

Proof. The lemma is proved by the step method using the a priori estimates

$$\omega_\varphi(t-1,\varepsilon,h) \ge M_1, \quad \omega_\varphi(t-h,\varepsilon,h) - \omega_\varphi(t,\varepsilon,h) - b \le -M_2, \quad M_1, M_2 = \text{const} > 0. \tag{2.27}$$

In this method, the interval (2.24) is divided into intervals of length at most unity and then, considering these intervals as they are successively located, we first derive (2.26) on the current interval under assumptions (2.27) and then use the result to prove the estimates (2.27) themselves.

At the first stage, consider the interval

$$1 + \sigma_0 \le t \le \min(2 + \sigma_0,\ t_1(h) - \sigma_0), \tag{2.28}$$

on which, by the inclusions

$$\begin{aligned} &t - 1 \in [\sigma_0, \min(1+\sigma_0,\ t_1(h) - 1 - \sigma_0)] \subset [\sigma_0, 1+\sigma_0], \\ &t - h \in [-h + 1 + \sigma_0, \min(2 - h + \sigma_0,\ t_1(h) - h - \sigma_0)] \subset [-h+1+\sigma_0, 1-\sigma_0] \end{aligned}$$

following from inequalities (2.25), the function $\omega_\varphi(t-1,\varepsilon,h)$ is defined by relations (2.10), (2.16), and (2.17), and for the function $\omega_\varphi(t-h,\varepsilon,h)$, by formulas (2.5), (2.10), we have

$$\omega_\varphi(t-h,\varepsilon,h) = t - h + O(\exp(-1/\sqrt{\varepsilon}\,)). \tag{2.29}$$

Thus, in this case, the first inequality in (2.27) holds with any constant $M_1 \in (0, \min(\sigma_0, 1 - a\sigma_0))$, and the second inequality will be proved later.

Taking into account relations (2.27) on the right-hand side in Eq. (2.1) and using properties (1.2) of the function $f(u)$, we conclude that, on the interval (2.28), this equation becomes

$$\dot{\omega} = -a + O(\exp(-q/\varepsilon)). \tag{2.30}$$

Further, by relations (2.16), (2.17), and (2.22), Eq. (2.30) must be supplemented with the initial condition
$$\omega|_{t=1+\sigma_0} = \omega_\varphi(t,\varepsilon,h)|_{t=1+\sigma_0} = 1 - a\sigma_0 + \varepsilon c_0 + O(\exp(-q/\varepsilon)). \tag{2.31}$$
Solving the resulting Cauchy problem, we obtain the desired relation (2.26).

Now let us verify the second estimate in (2.27). By the representation (2.29) and the asymptotic formula (2.26) (which has already been obtained for the values of t under study), we have
$$\omega_\varphi(t-h,\varepsilon,h) - \omega_\varphi(t,\varepsilon,h) - b = (a+1)(t - t_1(h)) - \varepsilon c_0 + O(\exp(-q/\varepsilon)).$$
Since the function $(a+1)(t-t_1(h))$ is negative on the interval (2.28), it follows that the second inequality in (2.27) holds with an arbitrary constant M_2 in the interval $(0,(a+1)\sigma_0)$.

At the subsequent stages, by the estimates $1 - a(t-2) > 0$, $(a+1)(t-t_1(h)) < 0$, the whole above argument is repeated practically word by word. Namely, at the current stage, first, under conditions (2.27), we derive the asymptotic representation (2.26) from the Cauchy problem of the form (2.30), (2.31), and then we verify that the estimates (2.27) are indeed satisfied with constants $M_1 \in (0, \min(1-a\sigma_0,\, 1-a(t_1(h)-2-\sigma_0)))$ and $M_2 \in (0,(a+1)\sigma_0)$. The proof of Lemma 2.3 is complete.

At the fourth stage, consider the time interval
$$t_1(h) - \sigma_0 \le t \le t_1(h) + \sigma_0 \tag{2.32}$$
and the next additional restriction
$$\sigma_0 < 2^{-1}(t_0 + 1 - t_1(h)), \qquad h \in \Omega, \tag{2.33}$$
on the parameter σ_0. Inequalities (2.25) and (2.33) guarantee the inclusions
$$\begin{aligned} &t-1 \in [t_1(h)-1-\sigma_0, t_1(h)-1+\sigma_0] \subset [\sigma_0, t_0-\sigma_0],\\ &t-h \in [-h+t_1(h)-\sigma_0, -h+t_1(h)+\sigma_0] \subset [-h+1+\sigma_0, 1-\sigma_0]. \end{aligned}$$
Thus, for the above-listed t, formula (2.29) holds for the function $\omega_\varphi(t-h,\varepsilon,h)$, and the function $\omega_\varphi(t-1,\varepsilon,h)$ is defined on the corresponding time intervals by the relations obtained at the three preceding stages (see (2.10), (2.16), (2.17), and (2.26)). In this case, these formulas imply that
$$\omega_\varphi(t-1,\varepsilon,h) \ge M, \qquad M = \text{const} \in (0,\sigma_0). \tag{2.34}$$

In Eq. (2.1), we change the variables
$$\omega = t_1(h) - h - b + \varepsilon w, \qquad \tau = (t - t_1(h))/\varepsilon \tag{2.35}$$
and take into account the above-obtained information about the functions $\omega_\varphi(t-1,\varepsilon,h)$ and $\omega_\varphi(t-h,\varepsilon,h)$ on the right-hand side in (2.1); we see that the function w satisfies an equation of the form
$$\frac{dw}{d\tau} = \exp[\tau - w + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,))] - a + O(\exp(-q/\varepsilon)). \tag{2.36}$$
By formulas (2.26) and (2.35), it should be supplemented with the initial condition
$$w|_{\tau=-\sigma_0/\varepsilon} = \varepsilon^{-1}(\omega_\varphi(t_1(h)-\sigma_0,\varepsilon,h) - t_1(h) + h + b) = a\sigma_0/\varepsilon + c_0 + O(\exp(-q/\varepsilon)). \tag{2.37}$$
We denote the solution of the Cauchy problem (2.36), (2.37) by $w_\varphi(\tau,\varepsilon,h)$. The following assertion holds.

Lemma 2.4. *The asymptotic representation*
$$w_\varphi(\tau,\varepsilon,h) = w_0(\tau) + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,)), \tag{2.38}$$
where
$$w_0(\tau) = \ln(\exp(-a\tau + c_0) + (a+1)^{-1}\exp\tau), \tag{2.39}$$
holds as $\varepsilon \to 0$ *uniformly in* $\tau \in [-\sigma_0/\varepsilon, \sigma_0/\varepsilon]$, $h \in \Omega$, *and* $\varphi \in S$.

Proof. We directly verify that the function $w = w_0(\tau)$ is a solution of the Cauchy problem

$$\frac{dw}{d\tau} = \exp(\tau - w) - a, \qquad w|_{\tau=-\sigma_0/\varepsilon} = \varepsilon^{-1}a\sigma_0 + c_0 + O(\exp(-q/\varepsilon)).$$

Therefore, we have the representation $w_\varphi(\tau,\varepsilon,h) = w_0(\tau) + \Delta$, where the remainder term Δ is asymptotically small. More precisely, the principal order of this remainder is determined from the linear Cauchy problem

$$\begin{aligned}&\frac{d\Delta}{d\tau} = -\exp(\tau - w_0(\tau))\Delta + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,))\exp(\tau - w_0(\tau)) + O(\exp(-q/\varepsilon)),\\ &\Delta|_{\tau=-\sigma_0/\varepsilon} = O(\exp(-q/\varepsilon)).\end{aligned} \tag{2.40}$$

In turn, from relations (2.40) we derive the estimate

$$\begin{aligned}|\Delta| \le M_1 \exp\left(-\frac{q}{\varepsilon}\right)\frac{w_*(-\sigma_0/\varepsilon)}{w_*(\tau)} + \frac{M_2}{\varepsilon w_*(\tau)}\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)\int\limits_{-\sigma_0/\varepsilon}^{\tau}\exp[(a+1)s]\,ds\\ + \frac{M_3}{w_*(\tau)}\exp\left(-\frac{q}{\varepsilon}\right)\int\limits_{-\sigma_0/\varepsilon}^{\tau} w_*(s)\,ds, \qquad M_1, M_2, M_3 = \text{ const } > 0,\end{aligned}$$

where

$$w_*(\tau) = \exp c_0 + (a+1)^{-1}\exp[(a+1)\tau]. \tag{2.41}$$

This and the explicit form of the functions $w_0(\tau)$ and $w_*(\tau)$ (see (2.39) and (2.41)) imply

$$|\Delta| \le M\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,), \qquad M = \text{ const } > 0.$$

The proof of Lemma 2.4 is complete.

Summarizing the above, we substitute the functions defined by relations (2.38) and (2.39) into (2.35). As a result, we see that, at the fourth stage, the following asymptotic representation holds for the solution $\omega_\varphi(t,\varepsilon,h)$ as $\varepsilon \to 0$ uniformly in t in the interval (2.32) and in $h \in \Omega$ and $\varphi \in S$:

$$\omega_\varphi(t,\varepsilon,h) = t_1(h) - h - b + \varepsilon w_0(\tau)|_{\tau=(t-t_1(h))/\varepsilon} + O(\exp(-1/\sqrt{\varepsilon}\,)). \tag{2.42}$$

At the fifth stage, consider the values of t in the interval

$$t_1(h) + \sigma_0 \le t \le t_0 + 1 - \sigma_0. \tag{2.43}$$

In this case, by conditions (2.25) and (2.33), we have the inclusions

$$\begin{aligned}&t - 1 \in [t_1(h) - 1 + \sigma_0, t_0 - \sigma_0] \subset [\sigma_0, t_0 - \sigma_0],\\ &t - h \in [t_1(h) - h + \sigma_0, t_0 + 1 - h - \sigma_0] \subset [-h + 1 + \sigma_0, 1 - \sigma_0].\end{aligned}$$

Therefore, formulas (2.29) and the estimate (2.34) hold on the interval (2.43) for the functions $\omega_\varphi(t-h,\varepsilon,h)$ and $\omega_\varphi(t-1,\varepsilon,h)$, respectively.

With regard to the above information, Eq. (2.1) is transformed to the form

$$\dot\omega = [1 + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,))]\exp((t - h - b - \omega)/\varepsilon) - a + O(\exp(-q/\varepsilon)).$$

Further, by the representation (2.42), we supplement this equation with the initial condition

$$\omega|_{t=t_1(h)+\sigma_0} = \omega_\varphi(t_1(h)+\sigma_0,\varepsilon,h) = t_1(h) - h - b + \sigma_0 - \varepsilon\ln(a+1) + O(\exp(-1/\sqrt{\varepsilon}\,))$$

and then set

$$\omega = t - h - b - \varepsilon\ln(a+1) + \Delta$$

in the resulting Cauchy problem. As a result, to seek the remainder Δ, in the linear approximation we have the following problem:

$$\dot{\Delta} = -\varepsilon^{-1}(a+1)\Delta + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon})), \qquad \Delta|_{t=t_1(h)+\sigma_0} = O(\exp(-1/\sqrt{\varepsilon})).$$

A simple analysis of the problem leads to the estimates

$$|\Delta| \le M_1 \exp\left[-\frac{1}{\sqrt{\varepsilon}} - \frac{a+1}{\varepsilon}(t - t_1(h) - \sigma_0)\right] + \frac{M_2}{\varepsilon}\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)\int\limits_{t_1(h)+\sigma_0}^{t} \exp\left[-\frac{a+1}{\varepsilon}(t-s)\right]ds \le M_3 \exp\left(-\frac{1}{\sqrt{\varepsilon}}\right),$$

where $M_1, M_2, M_3 =$ const > 0.

Thus, we obtain the following assertion.

Lemma 2.5. *As $\varepsilon \to 0$, the asymptotic relation*

$$\omega_\varphi(t,\varepsilon,h) = t - h - b - \varepsilon\ln(a+1) + O(\exp(-1/\sqrt{\varepsilon})) \tag{2.44}$$

holds for the solution $\omega_\varphi(t,\varepsilon,h)$ uniformly in t in the interval (2.43) *and in $h \in \Omega$ and $\varphi \in S$.*

At the sixth stage, consider the time interval

$$t_0 + 1 - \sigma_0 \le t \le t_0 + 1 + \sigma_0 \tag{2.45}$$

under the next additional condition

$$\sigma_0 < 2^{-1}\min(t_0 - 1,\, t_1(h) - t_0,\, h - t_0), \qquad h \in \Omega, \tag{2.46}$$

on the parameter σ_0. This condition ensures the inclusions

$$\begin{aligned} &t - 1 \in [t_0 - \sigma_0, t_0 + \sigma_0] \subset [1 + \sigma_0, t_1(h) - \sigma_0], \\ &t - h \in [t_0 + 1 - h - \sigma_0, t_0 + 1 - h + \sigma_0] \subset [-h + 1 + \sigma_0, 1 - \sigma_0], \end{aligned}$$

which imply that, first, formula (2.29) holds for the functions $\omega_\varphi(t-h,\varepsilon,h)$ and second, in this case, by the representation (2.26), the following asymptotic relation holds for the function $\omega_\varphi(t-1,\varepsilon,h)$:

$$\omega_\varphi(t-1,\varepsilon,h) = 1 - a(t-2) + \varepsilon c_0 + O(\exp(-q/\varepsilon)). \tag{2.47}$$

Taking into account relations (2.29) and (2.47) and making the change of variables

$$\omega = t_0 + 1 - h - b + \varepsilon z(\tau), \qquad \tau = (t - t_0 - 1)/\varepsilon \tag{2.48}$$

in Eq. (2.1), we obtain an equation of the form

$$\frac{dz}{d\tau} = [1 + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}))]\exp(\tau - z) + f(\exp(-a\tau + c_0)) + O(\exp(-q/\varepsilon)) \tag{2.49}$$

for determining z. Further, omitting the exponentially small additional terms on the right-hand side in Eq. (2.49), we obtain the model equation

$$\frac{dz}{d\tau} = \exp(\tau - z) + f(\exp(-a\tau + c_0)).$$

We are interested in its special solution $z = z_0(\tau)$ given by the formulas

$$z_0(\tau) = \ln\left(K(\tau)\int\limits_{-\infty}^{\tau}\frac{\exp s}{K(s)}ds\right), \qquad K(\tau) = \exp\left(\int\limits_0^{\tau} f(\exp(-as + c_0))\,ds\right), \qquad \tau \in \mathbb{R}. \tag{2.50}$$

We separately consider the asymptotic behavior of the function $z_0(\tau)$ as $\tau \to \pm\infty$. It follows from properties (1.2) of the function $f(u)$ that

$$\int_0^\tau f(\exp(-as+c_0))ds = \int_0^\tau [f(\exp(-as+c_0))+a]ds - a\tau = c_1 + \int_{-\infty}^\tau [f(\exp(-as+c_0))+a]\,ds - a\tau$$

$$= c_1 - a\tau + O(\exp a\tau) \quad \text{as} \quad \tau \to -\infty, \quad c_1 = -\int_{-\infty}^0 [f(\exp(-as+c_0))+a]\,ds.$$

Taking into account these relations in the formula for $K(\tau)$ (see (2.50)), we successively find that

$$K(\tau) = \exp[c_1 - a\tau + O(\exp a\tau)],$$
$$\int_{-\infty}^\tau \frac{\exp s}{K(s)}\,ds = \frac{\exp(-c_1+(a+1)\tau)}{a+1} + O(\exp(2a+1)\tau) \quad \text{as} \quad \tau \to -\infty. \tag{2.51}$$

From this and the definition of the function $z_0(\tau)$, we finally obtain

$$z_0(\tau) = \tau - \ln(a+1) + O(\exp a\tau) \quad \text{as} \quad \tau \to -\infty. \tag{2.52}$$

As $\tau \to +\infty$, properties (1.2) imply the relations

$$\int_0^\tau f(\exp(-as+c_0))\,ds = \int_0^\tau [f(\exp(-as+c_0))-1]\,ds + \tau = \tau + c_2 - \int_\tau^{+\infty} [f(\exp(-as+c_0))-1]\,ds$$

$$= \tau + c_2 + O(\exp(-a\tau)), \quad c_2 = \int_0^{+\infty} [f(\exp(-as+c_0))-1]\,ds,$$

$$\int_{-\infty}^\tau \frac{\exp s}{K(s)}\,ds = \int_{-\infty}^0 \frac{\exp s}{K(s)}\,ds + \int_0^\tau \left(\frac{\exp s}{K(s)} - \exp(-c_2)\right) ds + \tau\exp(-c_2)$$

$$= c_3 + \tau\exp(-c_2) + O(\exp(-a\tau)), \quad c_3 = \int_{-\infty}^0 \frac{\exp s}{K(s)}\,ds + \int_0^{+\infty} \left(\frac{\exp s}{K(s)} - \exp(-c_2)\right) ds. \tag{2.53}$$

Taking them into account in the formula for the function $z_0(\tau)$ in (2.50), we conclude that

$$z_0(\tau) = \tau + \ln(c_* + \tau) + O(\exp(-a\tau)) \quad \text{as} \quad \tau \to +\infty, \tag{2.54}$$

where $c_* = c_3 \exp c_2$.

It turns out that the function $z_0(\tau)$ plays a key role in the construction of the asymptotics of the solution $\omega_\varphi(t,\varepsilon,h)$ on the interval (2.45). Namely, we have the following lemma.

Lemma 2.6. *The function $\omega_\varphi(t,\varepsilon,h)$ admits the asymptotic representation*

$$\omega_\varphi(t,\varepsilon,h) = t_0 + 1 - h - b + \varepsilon z_0(\tau)|_{\tau=(t-t_0-1)/\varepsilon} + O(\exp(-1/\sqrt{\varepsilon}\,)) \tag{2.55}$$

as $\varepsilon \to 0$ on the interval (2.45) *uniformly in t, h, and φ.*

Proof. By (2.44), we supplement Eq. (2.49) with the initial condition

$$z|_{\tau=-\sigma_0/\varepsilon} = \varepsilon^{-1}(\omega_\varphi(t_0+1-\sigma_0,\varepsilon,h) - t_0 - 1 + h + b)$$
$$= -\varepsilon^{-1}\sigma_0 - \ln(a+1) + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,)). \tag{2.56}$$

Further, by setting $z = z_0(\tau) + \Delta$ in the Cauchy problem (2.49), (2.56) and by taking into account property (2.52) of the function $z_0(\tau)$, for the remainder Δ we obtain the linear inhomogeneous first approximation problem

$$\begin{aligned} \frac{d\Delta}{d\tau} &= -\exp(\tau - z_0(\tau))\Delta + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,))\exp(\tau - z_0(\tau)) + O(\exp(-q/\varepsilon)), \\ \Delta|_{\tau=-\sigma_0/\varepsilon} &= O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,)). \end{aligned} \tag{2.57}$$

The analysis of problem (2.57) is based on the obvious estimate

$$\begin{aligned} |\Delta| \le \frac{M_1}{\varepsilon}\,\frac{z_*(-\sigma_0/\varepsilon)}{z_*(\tau)}\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right) + \frac{M_2}{\varepsilon z_*(\tau)}\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)\int\limits_{-\sigma_0/\varepsilon}^{\tau}\frac{\exp s}{K(s)}\,ds \\ + \frac{M_3}{z_*(\tau)}\exp\left(-\frac{q}{\varepsilon}\right)\int\limits_{-\sigma_0/\varepsilon}^{\tau} z_*(s)\,ds, \qquad M_1, M_2, M_3 = \text{ const } > 0, \end{aligned}$$

where

$$z_*(\tau) = \int\limits_{-\infty}^{\tau}\frac{\exp s}{K(s)}\,ds.$$

Combining this estimate with properties (2.51) and (2.53) of the function $z_*(\tau)$, we finally obtain

$$|\Delta| \le M\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,), \qquad M = \text{ const } > 0.$$

This and the change of variables (2.48) automatically imply the desired asymptotic relation (2.55). The proof of Lemma 2.6 is complete.

At the seventh stage, consider the values of t in the interval

$$t_0 + 1 + \sigma_0 \le t \le h + 1 - \sigma_0. \tag{2.58}$$

In this case, we have the inclusions

$$t - 1 \in [t_0 + \sigma_0, h - \sigma_0], \qquad t - h \in [t_0 + 1 - h + \sigma_0, 1 - \sigma_0] \subset [-h + 1 + \sigma_0, 1 - \sigma_0],$$

and hence formula (2.29) still holds for the function $\omega_\varphi(t-h,\varepsilon,h)$. For the function $\omega_\varphi(t,\varepsilon,h)$, for the values of t in question, we have the following assertion.

Lemma 2.7. *The function $\omega_\varphi(t,\varepsilon,h)$ admits the following asymptotic relation as $\varepsilon \to 0$ uniformly in t in the interval* (2.58) *and in $h \in \Omega$ and $\varphi \in S$:*

$$\omega_\varphi(t,\varepsilon,h) = t - h - b + \varepsilon\ln(c_* + \varepsilon^{-1}(t - t_0 - 1)) + O(\exp(-1/\sqrt{\varepsilon}\,)), \tag{2.59}$$

where c_* is the constant in (2.54).

Proof. Just as in Lemma 2.3, the proof of the above-stated lemma is based on the method of a priori estimates and the step method. Namely, in this situation, we assume that the following inequality holds:

$$\omega_\varphi(t-1,\varepsilon,h) \le -M, \qquad M = \text{ const } > 0. \tag{2.60}$$

As usual, we divide the interval (2.58) into parts of length at most equal to unity and successively consider the obtained time intervals. We immediately note that, at the first of them, i.e., for

$$t_0 + 1 + \sigma_0 \le t \le \min(t_0 + 2 + \sigma_0,\, h + 1 - \sigma_0), \tag{2.61}$$

by the inclusion $t-1 \in [t_0+\sigma_0, t_0+1+\sigma_0]$ and the already obtained formulas (2.26), (2.42), (2.44), (2.55), the estimate (2.60) is satisfied with the constant M in the interval

$$(0, \min(a\sigma_0,\, h+b-t_0-1-\sigma_0)).$$

Taking into account relations (2.29) and (2.60) on the right-hand side of Eq. (2.1) and substituting the expression

$$\omega = t-h-b+\Delta \tag{2.62}$$

into it, we obtain an equation of the form

$$\dot{\Delta} = [1+O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,))]\exp(-\Delta/\varepsilon)+O(\exp(-q/\varepsilon)) \tag{2.63}$$

for the remainder Δ. By formulas (2.54), (2.55), and (2.62), we supplement Eq. (2.63) with the initial condition

$$\Delta|_{t=t_0+1+\sigma_0} = \varepsilon\ln(c_* + \varepsilon^{-1}\sigma_0) + O(\exp(-1/\sqrt{\varepsilon}\,)). \tag{2.64}$$

A simple analysis of the Cauchy problem (2.63), (2.64) show that the asymptotic representation

$$\Delta = \varepsilon\ln(c_*+\varepsilon^{-1}(t-t_0-1)) + O(\exp(-1/\sqrt{\varepsilon}\,)) \quad \text{as} \quad \varepsilon\to 0$$

holds on the interval (2.61) uniformly in t, h, and φ. This and (2.62) imply the desired formula (2.59).

When considering the subsequent intervals of this division, the desired a priori condition (2.60) is necessarily satisfied by the estimate $t-1-h-b<0$. This means that we can generalize the asymptotic formula (2.59) to the entire interval (2.58). The proof of Lemma 2.7 is complete.

At the eighth stage, consider the time interval

$$h+1-\sigma_0 \le t \le h+1+\widetilde{\sigma}_0, \tag{2.65}$$

where $\widetilde{\sigma}_0$ is an arbitrarily fixed constant in the interval $(0,\sigma_0)$. For the above-distinguished t, by the inclusions

$$t-1 \in [h-\sigma_0, h+\widetilde{\sigma}_0] \subset [t_0+\sigma_0, h+1-\sigma_0], \qquad t-h \in [1-\sigma_0, 1+\widetilde{\sigma}_0]$$

and formulas (2.16), (2.17), (2.26), (2.42), (2.44), (2.55), and (2.59), we have

$$\begin{aligned} &\omega_\varphi(t-h,\varepsilon,h) = 1+\varepsilon v_0(\tau)|_{\tau=(t-h-1)/\varepsilon} + O(\exp(-q/\varepsilon)),\\ &\omega_\varphi(t-1,\varepsilon,h) \le -M, \qquad M = \text{ const} \in (0,\min(a\sigma_0, b-1+\sigma_0)). \end{aligned} \tag{2.66}$$

Further, we take into account relations (2.66) on the right-hand side in Eq. (2.1) and set

$$\omega = 1-b+\varepsilon\gamma(\tau), \qquad \tau = (t-h-1)/\varepsilon \tag{2.67}$$

in this equation. As a result, to determine the function γ, we obtain an equation of the form

$$\frac{d\gamma}{d\tau} = (1+O(\exp(-q/\varepsilon)))\exp(v_0(\tau)-\gamma)+1+O(\exp(-q/\varepsilon)). \tag{2.68}$$

It follows from (2.59) and (2.67) that it is necessary to supplement this equation with the initial condition

$$\begin{aligned} \gamma|_{\tau=-\sigma_0/\varepsilon} &= \varepsilon^{-1}(\omega_\varphi(h+1-\sigma_0,\varepsilon,h)-1+b)\\ &= -\varepsilon^{-1}\sigma_0 + \ln(c_*+\varepsilon^{-1}(h-t_0-\sigma_0)) + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,)). \end{aligned} \tag{2.69}$$

Just as in the case of Eq. (2.49), neglecting the exponentially small additional terms on the right-hand side in Eq. (2.68), we obtain the model equation

$$\frac{d\gamma}{d\tau} = \exp(v_0(\tau)-\gamma)+1. \tag{2.70}$$

It is easily seen that the solution of the Cauchy problem (2.69), (2.70) is given by the explicit expressions

$$\gamma = \tau + \ln\left(c + \int\limits_{-\sigma_0/\varepsilon}^{\tau} \exp(v_0(s) - s)\,ds\right), \qquad c = \exp(\gamma|_{\tau=-\sigma_0/\varepsilon} + \varepsilon^{-1}\sigma_0). \tag{2.71}$$

To analyze formulas (2.71), we need the asymptotic representation

$$\int\limits_{\tau}^{0} \exp(v_0(s) - s)\,ds = -\tau + \int\limits_{-\infty}^{0} [\exp(v_0(s) - s) - 1]\,ds + O(\exp\tau) \quad \text{as} \quad \tau \to -\infty,$$

which follows from property (2.21) of the function (2.17). It follows from this relation that

$$c + \int\limits_{-\sigma_0/\varepsilon}^{0} \exp(v_0(s) - s)\,ds = c_{**} + \varepsilon^{-1}(h - t_0) + O(\varepsilon^{-2}\exp(-1/\sqrt{\varepsilon}\,)) \quad \text{as} \quad \varepsilon \to 0, \tag{2.72}$$

where

$$c_{**} = c_* + \int\limits_{-\infty}^{0} [\exp(v_0(s) - s) - 1]\,ds. \tag{2.73}$$

Further, taking into account relations (2.72) and (2.73) in the expression (2.71) for the function γ, we conclude that the asymptotic representation

$$\gamma = \tau + \ln\left(c_{**} + \varepsilon^{-1}(h - t_0) + \int\limits_{0}^{\tau} \exp(v_0(s) - s)\,ds\right) + O(\varepsilon^{-1}\exp(-1/\sqrt{\varepsilon}\,)) \tag{2.74}$$

as $\varepsilon \to 0$ holds uniformly in $\tau \in [-\sigma_0/\varepsilon, \sigma_0/\varepsilon]$, $h \in \Omega$, and $\varphi \in S$. Obviously, the asymptotic representation (2.74) is preserved when we pass backward from Eq. (2.70) to the original equation (2.68). Thus, we have the following assertion.

Lemma 2.8. *The asymptotic relation*

$$\omega_\varphi(t, \varepsilon, h) = 1 - b + \varepsilon\gamma(\tau, \varepsilon)|_{\tau=(t-h-1)/\varepsilon} + O(\exp(-1/\sqrt{\varepsilon}\,)), \tag{2.75}$$

where

$$\gamma(\tau, \varepsilon) = \tau + \ln\left(c_{**} + \frac{h - t_0}{\varepsilon} + \int\limits_{0}^{\tau} \exp(v_0(s) - s)\,ds\right), \tag{2.76}$$

holds as $\varepsilon \to 0$ *for the solution* $\omega_\varphi(t, \varepsilon, h)$ *uniformly in* t *in the interval* (2.65), $h \in \Omega$, *and* $\varphi \in S$.

The ninth, last stage is related to the consideration of the time interval

$$h + 1 + \widetilde{\sigma}_0 \le t \le h + b - \sigma_0/2. \tag{2.77}$$

Just as at the first stage, we assume that the a priori estimates

$$\omega_\varphi(t - h, \varepsilon, h) - \omega_\varphi(t, \varepsilon, h) - b \le -M_1, \quad \omega_\varphi(t - 1, \varepsilon, h) \le -M_2, \quad M_1, M_2 = \text{const} > 0, \tag{2.78}$$

similar to (2.12), are satisfied on this interval.

By the step method, using inequalities (2.78), we see that, on each part of the interval (2.77) of length at most equal to unity, we must consider an equation of the form

$$\dot{\omega} = 1 + O(\exp(-q/\varepsilon)). \tag{2.79}$$

This and relations (2.75) and (2.76) imply that, uniformly in $t \in [h + 1 + \widetilde{\sigma}_0, h + b - \sigma_0/2]$, $h \in \Omega$, and $\varphi \in S$, we have the relation

$$\omega_\varphi(t, \varepsilon, h) = \omega_\varphi(h + 1 + \widetilde{\sigma}_0, \varepsilon, h) + t - h - 1 - \widetilde{\sigma}_0 + O(\exp(-q/\varepsilon)) \tag{2.80}$$

as $\varepsilon \to 0$, where, in turn,

$$\omega_\varphi(h + 1 + \widetilde{\sigma}_0, \varepsilon, h) = 1 - b + \widetilde{\sigma}_0 + \varepsilon \ln(c_{***} + \varepsilon^{-1}(h - t_0)) + O(\exp(-1/\sqrt{\varepsilon})),$$
$$c_{***} = c_{**} + \int\limits_0^{+\infty} \exp(v_0(s) - s)\, ds. \tag{2.81}$$

Thus, under conditions (2.78), relation (2.80) holds on the entire interval (2.77). Now let us verify conditions (2.78). To this end, we combine formulas (2.80) and (2.81) with the asymptotic representations already obtained for the function $\omega_\varphi(t, \varepsilon, h)$ (starting from the second stage). As a result, we have

$$\omega_\varphi(t - h, \varepsilon, h) \le 1 - a\widetilde{\sigma}_0 + O(\exp(-q/\varepsilon)),$$
$$\omega_\varphi(t, \varepsilon, h) = t - h - b + O(\varepsilon \ln \varepsilon^{-1}),$$
$$\omega_\varphi(t - 1, \varepsilon, h) \le -M,$$

where $M = \text{const} \in (0, a\widetilde{\sigma}_0)$, whence the desired estimates (2.78) obviously follows.

Therefore, at the last stage we have the following assertion.

Lemma 2.9. *The solution $\omega_\varphi(t, \varepsilon, h)$ admits the asymptotic representation* (2.80) *as $\varepsilon \to 0$ on the interval* (2.77) *uniformly in the variables t, h, and φ.*

Let us summarize the results. To this end, consider the function

$$\omega_0(t, h) = \begin{cases} t & \text{for } -\sigma_0 \le t \le 1, \\ 1 - a(t - 1) & \text{for } 1 \le t \le t_1(h), \\ t - h - b & \text{for } t_1(h) \le t \le h + b - \sigma_0/2. \end{cases} \tag{2.82}$$

Combining Lemmas 2.1–2.9, we conclude that the asymptotic representation

$$\omega_\varphi(t, \varepsilon, h) = \omega_0(t, h) + O(\varepsilon \ln \varepsilon^{-1}) \tag{2.83}$$

holds as $\varepsilon \to 0$ uniformly in $t \in [-\sigma_0, h + b - \sigma_0/2]$, $h \in \Omega$, and $\varphi \in S$.

Relations (2.82) and (2.83) permit localizing the desired second positive root $t = T_\varphi(\varepsilon, h)$ of Eq. (2.6). Indeed, it follows from these relations that the value $t = T_\varphi(\varepsilon, h) - \sigma_0$ belongs to the interval (2.77). By relations (2.79)–(2.81), this automatically implies that the root $T_\varphi(\varepsilon, h)$ is determined uniquely and admits the following asymptotics as $\varepsilon \to 0$ uniformly in $h \in \Omega$ and $\varphi \in S$:

$$T_\varphi(\varepsilon, h) = h + 1 + \widetilde{\sigma}_0 - \omega_\varphi(h + 1 + \widetilde{\sigma}_0, \varepsilon, h) + O(\exp(-q/\varepsilon)), \tag{2.84}$$

where, as we recall, the function $\omega_\varphi(h + 1 + \widetilde{\sigma}_0, \varepsilon, h)$ satisfies formulas (2.81).

2.3. Completion of the Proof of Theorem 1.1

Now we implement the scheme for studying Eq. (1.9) described in Section 2.1. Consider the operator (2.7), which, by our constructions, is well defined on the set (2.5), and show that the inclusion $\Pi(S) \subset S$ holds under an appropriate choice of the constants q_1 and q_2 in definition (2.5) and under some additional condition on the parameter δ_0 in (2.2).

Indeed, by the asymptotic relations (2.81)–(2.84), we can readily see that the condition

$$-q_1 \le \omega_\varphi(t + T_\varphi(\varepsilon, h), \varepsilon, h) \le -q_2, \qquad t \in I_1,$$

is necessarily satisfied with any fixed constants

$$q_1 > -\min_{b-\sigma_0 \le t \le b+1+\sigma_0} \omega_0(t, h), \qquad q_2 \in \Bigl(0, -\max_{b-\sigma_0 \le t \le b+1+\sigma_0} \omega_0(t, h)\Bigr). \tag{2.85}$$

Further, under the next additional assumption

$$\delta_0 < \sigma_0 - \widetilde{\sigma}_0, \tag{2.86}$$

which guarantees the inclusions

$$[T_\varphi(\varepsilon, h) - h + 1 + \sigma_0, T_\varphi(\varepsilon, h) - \sigma_0] \subset [h + 1 + \widetilde{\sigma}_0, h + b - \sigma_0/2],$$

formulas (2.80) and (2.84) imply the relation

$$\omega_\varphi(t + T_\varphi(\varepsilon, h), \varepsilon, h) = t + O(\exp(-q/\varepsilon)), \qquad t \in I_2. \tag{2.87}$$

Thus, the inequality

$$|\omega_\varphi(t + T_\varphi(\varepsilon, h), \varepsilon, h) - t| \le \exp(-1/\sqrt{\varepsilon}\,), \qquad t \in I_2,$$

required in definition (2.5), is satisfied automatically.

Thus, we have shown that the inclusion $\Pi(S) \subset S$ holds under conditions (2.3), (2.4), (2.14), (2.25), (2.33), (2.46), (2.85), and (2.86) on the parameters σ_0, q_1, q_2, and δ_0. Further, by the inequality $T_\varphi(\varepsilon, h) > h$, which follows from (2.81) and (2.84), the operator Π is compact. Thus, by the Schauder principle, it has at least one fixed point $\varphi = \widetilde{\varphi}(t, \varepsilon, h)$ in the set S. As was shown in Section 2.1, the corresponding solution $\widetilde{\omega}(t, \varepsilon, h) = \omega_\varphi|_{\varphi=\widetilde{\varphi}}$ of Eq. (2.1) turns out to be periodic with period

$$\widetilde{T}(\varepsilon, h) = T_\varphi|_{\varphi=\widetilde{\varphi}}.$$

Moreover, it follows from (2.87) that, in the case of $\varphi = \widetilde{\varphi}$, in all formulas for the solution $\omega_\varphi(t, \varepsilon, h)$, the remainders of the form $O(\exp(-1/\sqrt{\varepsilon}\,))$ (where they are encountered) can be replaced with $O(\exp(-q/\varepsilon))$. Taking this fact into account, from (2.81) and (2.84) with $\varphi = \widetilde{\varphi}$ we derive

$$\widetilde{T}(\varepsilon, h) = h + b - \varepsilon \ln(c_{***} + \varepsilon^{-1}(h - t_0) + O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0. \tag{2.88}$$

Further, consider Eq. (2.8) for the free parameter $h \in \Omega$. By the asymptotic representation (2.88), we conclude that Eq. (2.8) admits the solution $h = h(\varepsilon)$ with the asymptotics

$$h(\varepsilon) = b + h_0(\varepsilon) + O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0, \tag{2.89}$$

where $h_0 = h_0(\varepsilon)$ is determined from the equation

$$h_0 = -\varepsilon \ln(c_{***} + \varepsilon^{-1}(b - t_0 + h_0)). \tag{2.90}$$

In turn, the solution $h_0(\varepsilon)$ of Eq. (2.90) has the asymptotics

$$h_0(\varepsilon) = \varepsilon \ln \varepsilon - \varepsilon \ln(b - t_0) + O(\varepsilon^2 \ln \varepsilon^{-1}) \quad \text{as} \quad \varepsilon \to 0. \tag{2.91}$$

The constructions made in this section, together with the asymptotic analysis in Section 2.2, allow us to prove Theorem 1.1. To this end, consider the function

$$\omega(t, \varepsilon) = \widetilde{\omega}(t, \varepsilon, h)|_{h=h(\varepsilon)}, \tag{2.92}$$

which, by construction, is periodic with period $2h(\varepsilon)$. Further, consider the root $t = t_0(\varepsilon)$ of the equation $\omega(t, \varepsilon) = 0$, which is asymptotically close to zero. The representation (2.10) and the formula

$$\dot{\omega}_\varphi = 1 + O(\exp(-q/\varepsilon)),$$

which holds on the interval (2.9), imply that this root is simple and admits the asymptotics

$$t_0(\varepsilon) = O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0. \tag{2.93}$$

Further, we assume that

$$u_{**}(t,\lambda) = \exp(\lambda\,\omega_{**}(t,\lambda)), \qquad \omega_{**}(t,\lambda) = \omega(t + t_0(\varepsilon), \varepsilon)|_{\varepsilon=1/\lambda}, \quad h(\lambda) = h(\varepsilon)|_{\varepsilon=1/\lambda} \tag{2.94}$$

and obtain the desired cycle (1.8) of system (1.6), (1.10).

In conclusion, note that our relations imply the asymptotic representations (1.13) for the functions $\omega_{**}(t,\lambda)$ and $h(\lambda)$. Indeed, the desired formula for $h(\lambda)$ obviously follows from (2.89), (2.91), and (2.94), and the asymptotic formula for $\omega_{**}(t,\lambda)$ is a consequence of relations (2.82), (2.83), (2.89), (2.91), (2.93), and (2.94). The proof of Theorem 1.1 is complete.

3. PROOF OF THEOREM 1.2

3.1. General Plan of the Study

To analyze the stability properties of the cycle (1.8) in system (1.6) under conditions (1.10) and (1.11), we change the variables $u_1 = \exp(\lambda\omega_1)$, $u_2 = \exp(\lambda\omega_2)$ and the parameter $\lambda = 1/\varepsilon$.

As a result, we obtain a system of the form

$$\begin{aligned} \dot{\omega}_1 &= \exp((\omega_2 - \omega_1 - b)/\varepsilon) - \exp(-b/\varepsilon) + f(\exp(\omega_1(t-1)/\varepsilon)), \\ \dot{\omega}_2 &= \exp((\omega_1 - \omega_2 - b)/\varepsilon) - \exp(-b/\varepsilon) + f(\exp(\omega_2(t-1)/\varepsilon)). \end{aligned} \tag{3.1}$$

Note that in system (3.1) the self-symmetric cycle (1.8) is associated with the periodic solution

$$(\omega_1, \omega_2) = (\omega(t,\varepsilon), \omega(t-h,\varepsilon)), \tag{3.2}$$

where $h = h(\varepsilon)$ and $\omega(t,\varepsilon)$ are the functions (2.89) and (2.92). In turn, the stability of the cycle (3.2) is determined by the multipliers of the linear variational system

$$\begin{aligned} \dot{g}_1 &= a(t,\varepsilon)(g_2 - g_1) + b(t,\varepsilon)g_1(t-1), \\ \dot{g}_2 &= a(t-h,\varepsilon)(g_1 - g_2) + b(t-h,\varepsilon)g_2(t-1) \end{aligned} \tag{3.3}$$

with the coefficients

$$\begin{aligned} a(t,\varepsilon) &= \frac{1}{\varepsilon}\exp\left(\frac{\omega(t-h,\varepsilon) - \omega(t,\varepsilon) - b}{\varepsilon}\right), \\ b(t,\varepsilon) &= \frac{1}{\varepsilon}f'\left(\exp\left(\frac{\omega(t-1,\varepsilon)}{\varepsilon}\right)\right)\exp\left(\frac{\omega(t-1,\varepsilon)}{\varepsilon}\right). \end{aligned} \tag{3.4}$$

Let us explain the meaning of the term "multiplier" as applied to system (3.3). In this connection, consider the Banach space E of vector functions $g_0(t) = (g_{1,0}(t), g_{2,0}(t))$ continuous with respect to t on the interval $[-1, 0]$ with the norm

$$\|g_0\|_E = \max_{j=1,2}\ \max_{-1\le t\le 0} |g_{j,0}(t)|. \tag{3.5}$$

Further, the monodromy operator system (3.3) is a bounded linear operator $V(\varepsilon)$ from E to E acting on an arbitrary function $g_0(t) \in E$ by the rule

$$V(\varepsilon)g_0 = g(t + 2h, \varepsilon), \qquad -1 \le t \le 0, \tag{3.6}$$

where $g(t,\varepsilon)=(g_1(t,\varepsilon),g_2(t,\varepsilon))$ is a solution of system (3.3) on the interval $0\le t\le 2h$ with the initial function $g_0(t)$, $t\in[-1,0]$. Note that, by the obvious inequality $2h>1$, the operator (3.6) is compact. By analogy with the case of ordinary differential equations, the multipliers of system (3.3) are defined to be the eigenvalues of the operator $V(\varepsilon)$. We number them in descending order of absolute values and denote by $\nu_s(\varepsilon)\in\mathbb{C}$, $s\in\mathbb{N}$.

Note that system (3.1) is a special case of a ring-like system of unidirectionally coupled equations, and the cycle (3.2) is a periodic mode of traveling wave type. For such periodic solutions, a special method for analyzing the stability properties was developed in [5–7]. The essence of this method is described below.

Along with system (3.3), consider the auxiliary scalar linear equation

$$\dot g=a(t,\varepsilon)(\varkappa g(t-h)-g)+b(t,\varepsilon)g(t-1), \tag{3.7}$$

where h is the phase shift in (3.2), $g(t)$ is a complex-valued function, and $\varkappa\in\mathbb{C}$ is an arbitrary parameter. Just as in the case of system (3.3), let $\nu_s(\varkappa,\varepsilon)$, $s\in\mathbb{N}$, denote the multipliers of this equation numbered in descending order of absolute values. Note that the multipliers for the scalar equation (3.7) are the eigenvalues of the monodromy operator $W(\varkappa,\varepsilon):C(G)\to C(G)$, similar to (3.6), where $G=[1+\sigma_0,h+1+\sigma_0]$ and $C(G)$ is the space of complex-valued functions continuous in $t\in G$. The norm on the space $C(G)$ is defined as usual, i.e., by a relation similar to (3.5). The operator $W(\varkappa,\varepsilon)$ satisfies the formulas

$$W(\varkappa,\varepsilon)g_0=g(t+2h,\varkappa,\varepsilon),\qquad t\in G, \tag{3.8}$$

where $g_0(t)$ is an arbitrary element of the space $C(G)$ and $g(t,\varkappa,\varepsilon)$ is the solution of Eq. (3.7) with the initial function $g_0(t)$, $t\in G$.

Consider the problem of the relationship between the multipliers of system (3.3) and of Eq. (3.7). We have the following assertion (see [5–7]).

Lemma 3.1. *Each multiplier $\nu\ne 0$ of system* (3.3) *admits the representation*

$$\nu=\varkappa^2,$$

where $\varkappa$ is a root of one of the equations

$$\nu_s(\varkappa,\varepsilon)=\varkappa^2,\qquad s\in\mathbb{N}. \tag{3.9}$$

Conversely, if for some $s=s_0$ Eq. (3.9) *has a nonzero root $\varkappa=\varkappa_0$, then the original system* (3.3) *has the multiplier $\nu=\varkappa_0^2$.*

In the next two sections, we asymptotically calculate the multipliers $\nu_s(\varkappa,\varepsilon)$ and analyze Eqs. (3.9). In this way, we obtain the relation

$$\nu_1(\varepsilon)\equiv 1,\qquad |\nu_2(\varepsilon)|\le M\varepsilon^2,\qquad \sup_{s\ge 3}|\nu_s(\varepsilon)|\le\exp(-q/\varepsilon),\qquad M=\text{const}>0, \tag{3.10}$$

which means that the cycle (1.8) is exponentially orbitally stable (in the metric of the phase space E).

3.2. Analysis of the Auxiliary Linear Equation

Consider the set of initial functions

$$B=\{g_0(t)\in C(G):g_0(h+1+\sigma_0)=0,\quad \|g_0\|\le 2\}, \tag{3.11}$$

where $\|\cdot\|$ is the norm on the space $C(G)$. By $g_1(t,g_0,\varkappa,\varepsilon)$ we denote the solution of Eq. (3.7) with an arbitrary initial condition $g_0(t)$ in the set (3.11); by $g_2(t,\varkappa,\varepsilon)$, the solution of this equation with the initial function $g_2\equiv 1$, $t\in G$. It follows from Eq. (3.7) that, on the interval $t\in[h+1+\sigma_0,3h+1+\sigma_0]$ of length $2h$, the dependence of the functions g_1 and g_2 on $\varkappa$ is quadratic; i.e.,

$$\begin{gathered} g_1(t,g_0,\varkappa,\varepsilon)=g_{1,1}(t,g_0,\varepsilon)+\varkappa\, g_{1,2}(t,g_0,\varepsilon)+\varkappa^2 g_{1,3}(t,g_0,\varepsilon),\\ g_2(t,\varkappa,\varepsilon)=g_{2,1}(t,\varepsilon)+\varkappa\, g_{2,2}(t,\varepsilon)+\varkappa^2 g_{2,3}(t,\varepsilon). \end{gathered} \tag{3.12}$$

Moreover, $g_{1,3}(t,g_0,\varepsilon)\equiv g_{2,3}(t,\varepsilon)\equiv 0$ for $t\in[h+1+\sigma_0,2h+1+\sigma_0]$.

For the asymptotic analysis of multipliers of Eq. (3.7), we need the following assertion.

Lemma 3.2. *There exists a sufficiently small* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon \le \varepsilon_0$ $\varkappa \in \Lambda \overset{\text{def}}{=} \{\varkappa \in \mathbb{C} : |\varkappa| \le 1\}$, $g_0 \in B$, *on the interval* $h + 1 + \sigma_0 \le t \le 3h + 1 + \sigma_0$, *the estimates*

$$\sum_{k=0}^{2} \left| \frac{\partial^k g_1}{\partial \varkappa^k}(t, g_0, \varkappa, \varepsilon) \right| \le \exp\left(-\frac{q}{\varepsilon} \right), \qquad \sum_{k=0}^{2} \left| \frac{\partial^k g_2}{\partial \varkappa^k}(t, \varkappa, \varepsilon) \right| \le M \tag{3.13}$$

hold with constants $q, M > 0$ *independent of* t, ε, $\varkappa$, *and* g_0. *Moreover, as* $\varepsilon \to 0$, *the asymptotic representations*

$$\frac{\partial^k g_2}{\partial \varkappa^k}(3h+1+\sigma_0, \varkappa, \varepsilon) = \left[1 + (\varkappa - 1)\left(1 - \frac{\varepsilon}{h - t_0 + \varepsilon c_{***}} \right) \right]^{(k)} + O\left(\exp\left(-\frac{q}{\varepsilon} \right) \right), \quad k = 0, 1, 2, \tag{3.14}$$

hold uniformly in $\varkappa \in \Lambda$. *(Here and below,* $[\cdot]^{(k)}$ *denotes the kth derivative with respect to* $\varkappa$.*)*

Proof. By relations (3.4) and well-known asymptotic properties of the function $\omega(t, \varepsilon)$, we have the inequalities

$$\max_{h+1+\sigma_0 \le t \le 2h+1+\sigma_0} |a(t, \varepsilon)| \le \exp(-q/\varepsilon), \qquad \max_{h+1+\sigma_0 \le t \le 2h+1-\sigma_0} |b(t, \varepsilon)| \le \exp(-q/\varepsilon), \tag{3.15}$$

where $q = \text{const} > 0$. Combining the estimates (3.15) with $g_0(h + 1 + \sigma_0) = 0$ and integrating Eq. (3.7) with an arbitrary initial condition $g_0 \in B$ by the step method (i.e., successively considering time intervals of length at most unity), we obtain

$$\sum_{k=0}^{2} \left| \frac{\partial^k g_1}{\partial \varkappa^k}(t, g_0, \varkappa, \varepsilon) \right| \le \exp(-q/\varepsilon), \qquad t \in [h + 1 + \sigma_0, 2h + 1 - \sigma_0]. \tag{3.16}$$

Further, for $2h + 1 - \sigma_0 \le t \le 2h + 1 + \sigma_0$, by (3.4) and the estimate (3.16), we have

$$a(t, \varepsilon) = O(\exp(-q/\varepsilon)), \qquad |b(t, \varepsilon)| = O(1/\varepsilon), \qquad \sum_{k=0}^{2} \left| \frac{\partial^k g_1}{\partial \varkappa^k}(t - 1, g_0, \varkappa, \varepsilon) \right| = O(\exp(-q/\varepsilon)).$$

In turn, it follows from this and Eq. (3.7) that the first estimate in (3.13) also holds on the interval $h + 1 + \sigma_0 \le t \le 2h + 1 + \sigma_0$.

On the remaining interval $2h + 1 + \sigma_0 \le t \le 3h + 1 + \sigma_0$, inequality (3.16) is proved by induction. To this end, we divide the interval $[2h + 1 + \sigma_0, 3h + 1 + \sigma_0]$ under study into the intervals $[2h + \sigma_0 + n, 2h + \sigma_0 + n + 1]$, $n = 1, \dots, n_0 - 1$, and $[2h + \sigma_0 + n_0, 3h + 1 + \sigma_0]$, where

$$n_0 = \begin{cases} h & \text{for integer } h, \\ \lfloor h \rfloor + 1 & \text{otherwise,} \end{cases}$$

and $\lfloor \cdot \rfloor$ is the integer part of a number.

At the first stage, i.e., for $2h + 1 + \sigma_0 \le t \le 2h + 2 + \sigma_0$, the solution $g_1(t, g_0, \varkappa, \varepsilon)$ satisfies the explicit formula

$$g_1(t, g_0, \varkappa, \varepsilon) = g_1(2h + 1 + \sigma_0, g_0, \varkappa, \varepsilon) \exp\left(-\int_{2h+1+\sigma_0}^{t} a(\sigma, \varepsilon)\, d\sigma \right)$$

$$+ \int_{2h+1+\sigma_0}^{t} \exp\left(-\int_{s}^{t} a(\sigma, \varepsilon)\, d\sigma \right) [\varkappa a(s, \varepsilon) g_1(s - h, g_0, \varkappa, \varepsilon) + b(s, \varepsilon) g_1(s - 1, g_0, \varkappa, \varepsilon)]\, ds. \tag{3.17}$$

In this formula, we take into account inequality (3.16) for $t \le 2h + 1 + \sigma_0$ and the properties

$$\exp\left(-\int\limits_s^t a(\sigma,\varepsilon)\,d\sigma\right) \le 1, \quad a(t,\varepsilon) + |b(t,\varepsilon)| = O(1/\varepsilon), \quad 2h + 1 + \sigma_0 \le t \le 3h + 1 + \sigma_0, \quad (3.18)$$

which follow from (3.4), and see that the estimates (3.16) hold for $2h + 1 + \sigma_0 \le t \le 2h + 2 + \sigma_0$.

The reasoning at the subsequent stages is quite similar. Namely, at the nth stage, i.e., for $t \in [2h + \sigma_0 + n, 2h + \sigma_0 + n + 1]$, we first write a formula of the form (3.17) for the initial time $t = 2h + \sigma_0 + n$ and then use the already known estimate (3.16) for $t \le 2h + \sigma_0 + n$ and relations (3.18). As a result, inequality (3.16) is generalized to a stage ahead, and after n_0 stages, it is established on the whole interval $2h + 1 + \sigma_0 \le t \le 3h + 1 + \sigma_0$. Thus, the first estimate in (3.13) has been proved.

The second inequality in (3.13) and formulas (3.14) are proved simultaneously by asymptotic integration of Eq. (3.7) with the initial condition $g \equiv 1$ for $t \in G$. The corresponding analysis is divided into nine stages similar to the already considered in the case of Eq. (2.1). Therefore, omitting the technical details, we only present the final results here.

At the first stage, consider the values $t \in [h + 1 + \sigma_0, 2h + 1 - \sigma_0]$. In this case, as was noted above (see (3.15)),

$$a(t,\varepsilon) + |b(t,\varepsilon)| = O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0, \quad g_2(t-h,\varkappa,\varepsilon) \equiv 1.$$

From this and Eq. (3.7), using the step method, we obtain the asymptotic representations

$$g_2(t,\varkappa,\varepsilon) = 1 + O(\exp(-q/\varepsilon)), \quad \frac{\partial g_2}{\partial \varkappa}(t,\varkappa,\varepsilon) = O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0,$$
$$\frac{\partial^2 g_2}{\partial \varkappa^2}(t,\varkappa,\varepsilon) \equiv 0 \qquad (3.19)$$

uniform in t and $\varkappa$.

At the second stage, consider the interval $t \in [2h + 1 - \sigma_0, 2h + 1 + \sigma_0]$ on which, as $\varepsilon \to 0$, we have

$$a(t,\varepsilon) = O(\exp(-q/\varepsilon)), \qquad \omega(t-1,\varepsilon) = \varepsilon\tau|_{\tau=(t-2h-1)/\varepsilon} + O(\exp(-q/\varepsilon)).$$

From this, with regard to the inequality

$$|u_1 f'(u_1) - u_2 f'(u_2)| \le \frac{M}{1 + \min(u_1^2,\, u_2^2)}|u_1 - u_2|, \quad M = \sup_{u \ge 0}(1 + u^2)|f'(u) + uf''(u)|, \quad u_1, u_2 \ge 0,$$

which follows from properties (1.2), we conclude that

$$b(t,\varepsilon) = \varepsilon^{-1} f'(\exp\tau)\exp\tau|_{\tau=(t-2h-1)/\varepsilon} + O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0.$$

Moreover, here we still have $g_2(t-h,\varkappa,\varepsilon) \equiv 1$, and for $g_2(t-1,\varkappa,\varepsilon)$ we have formulas of the form (3.19). It follows from the above representations that, as $\varepsilon \to 0$, the asymptotic relations

$$g_2(t,\varkappa,\varepsilon) = v_0'(\tau)|_{\tau=(t-2h-1)/\varepsilon} + O(\exp(-q/\varepsilon)),$$
$$\frac{\partial g_2}{\partial \varkappa}(t,\varkappa,\varepsilon) = O(\exp(-q/\varepsilon)), \qquad \frac{\partial^2 g_2}{\partial \varkappa^2}(t,\varkappa,\varepsilon) \equiv 0 \qquad (3.20)$$

hold uniformly in $t \in [2h+1-\sigma_0, 2h+1+\sigma_0]$ and $\varkappa \in \Lambda$, where $v_0(\tau)$ is the function in (2.17) and the prime stands for differentiation with respect to τ.

At the third stage, consider the interval $2h+1+\sigma_0 \le t \le t_1(h)+2h-\sigma_0$. For the t listed above, we have

$$a(t,\varepsilon) = O(\exp(-q/\varepsilon)), \quad |b(t,\varepsilon)| = O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0.$$

From this and Eq. (3.7), we conclude that, as $\varepsilon \to 0$, the asymptotic representations

$$g_2(t, \varkappa, \varepsilon) = -a + O(\exp(-q/\varepsilon)), \qquad \frac{\partial^k g_2}{\partial \varkappa^k}(t, \varkappa, \varepsilon) = O(\exp(-q/\varepsilon)), \qquad k = 1, 2, \tag{3.21}$$

hold uniformly in $t \in [2h + 1 + \sigma_0, t_1(h) + 2h - \sigma_0]$ and $\varkappa \in \Lambda$.

At the fourth stage, i.e., for $t_1(h) + 2h - \sigma_0 \le t \le t_1(h) + 2h + \sigma_0$, by (3.19), the function $g_2(t - h, \varkappa, \varepsilon)$ satisfies the relations

$$g_2(t - h, \varkappa, \varepsilon) = 1 + O(\exp(-q/\varepsilon)), \qquad \frac{\partial^k g_2}{\partial \varkappa^k}(t - h, \varkappa, \varepsilon) = O(\exp(-q/\varepsilon)), \qquad k = 1, 2, \tag{3.22}$$

as $\varepsilon \to 0$, and the functions $a(t, \varepsilon)$ and $b(t, \varepsilon)$ satisfy the relations

$$a(t, \varepsilon) = \varepsilon^{-1} \exp(\tau - w_0(\tau))|_{\tau = (t - t_1(h) - 2h)/\varepsilon} + O(\exp(-q/\varepsilon)), \qquad |b(t, \varepsilon)| = O(\exp(-q/\varepsilon)),$$

where $w_0(\tau)$ is the function (2.39). Substituting these asymptotic relations into Eq. (3.7) and then integrating the resulting equation, we see that, as $\varepsilon \to 0$, the asymptotic representations

$$\begin{aligned} \frac{\partial^k g_2}{\partial \varkappa^k}(t, \varkappa, \varepsilon) = {} & \left[w_0'(\tau) + (\varkappa - 1) \frac{\exp(a + 1)\tau}{\exp(a + 1)\tau + (a + 1)\exp c_0} \right]^{(k)} \Bigg|_{\tau = (t - t_1(h) - 2h)/\varepsilon} \\ & + O(\exp(-q/\varepsilon)), \qquad k = 0, 1, 2, \end{aligned} \tag{3.23}$$

hold uniformly in $t \in [t_1(h) + 2h - \sigma_0, t_1(h) + 2h + \sigma_0]$ and $\varkappa \in \Lambda$.

At the fifth stage, consider the interval $t_1(h) + 2h + \sigma_0 \le t \le 2h + t_0 + 1 - \sigma_0$. In this case, we have the asymptotic relations (3.22) and the formulas

$$a(t, \varepsilon) = \varepsilon^{-1}(a + 1) + O(\exp(-q/\varepsilon)), \quad |b(t, \varepsilon)| = O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0,$$

which, in turn, imply that, as $\varepsilon \to 0$, the asymptotic representations

$$\frac{\partial^k g_2}{\partial \varkappa^k}(t, \varkappa, \varepsilon) = [\varkappa]^{(k)} + O(\exp(-q/\varepsilon)), \qquad k = 0, 1, 2, \tag{3.24}$$

hold uniformly in t and $\varkappa$.

At the sixth stage, consider the interval $2h + t_0 + 1 - \sigma_0 \le t \le 2h + t_0 + 1 + \sigma_0$, on which

$$\begin{aligned} a(t, \varepsilon) &= \varepsilon^{-1} \exp(\tau - z_0(\tau))|_{\tau = (t - t_0 - 1 - 2h)/\varepsilon} + O(\exp(-q/\varepsilon)), \\ b(t, \varepsilon) &= \varepsilon^{-1} f'(\exp(-a\tau + c_0)) \exp(-a\tau + c_0)|_{\tau = (t - t_0 - 1 - 2h)/\varepsilon} + O(\exp(-q/\varepsilon)), \end{aligned}$$

where $z_0(\tau)$ is the function in (2.50). Moreover, for the function $g_2(t - h, \varkappa, \varepsilon)$, by the representations (3.19), (3.21), formulas (3.22) hold as $\varepsilon \to 0$, and for the function $g_2(t - 1, \varkappa, \varepsilon)$, we have

$$g_2(t - 1, \varkappa, \varepsilon) = -a + O(\exp(-q/\varepsilon)), \qquad \frac{\partial^k g_2}{\partial \varkappa^k}(t - 1, \varkappa, \varepsilon) = O(\exp(-q/\varepsilon)), \qquad k = 1, 2.$$

With regard to the above relations, at the sixth stage, we obtain the representations

$$\frac{\partial^k g_2}{\partial \varkappa^k}(t, \varkappa, \varepsilon) = [z_0'(\tau) + \varkappa - 1]^{(k)}|_{\tau = (t - t_0 - 1 - 2h)/\varepsilon} + O(\exp(-q/\varepsilon)) \;\; \text{as} \;\; \varepsilon \to 0, \quad k = 0, 1, 2, \tag{3.25}$$

which hold uniformly in t and $\varkappa$.

At the seventh stage, consider the interval $2h + t_0 + 1 + \sigma_0 \le t \le 3h + 1 - \sigma_0$, on which, as $\varepsilon \to 0$, the asymptotic relations

$$a(t, \varepsilon) = \frac{1}{t - t_0 - 1 - 2h + \varepsilon c_*} + O(\exp(-q/\varepsilon)), \qquad |b(t, \varepsilon)| = O(\exp(-q/\varepsilon))$$

hold, and for the function $g_2(t-h,\varkappa,\varepsilon)$ we still have formulas (3.22). By this, we conclude that, as $\varepsilon\to 0$, the representations

$$\frac{\partial^k g_2}{\partial\varkappa^k}(t,\varkappa,\varepsilon)=\left[\varkappa+\frac{1}{c_*+(t-t_0-1-2h)/\varepsilon}\right]^{(k)}+O(\exp(-q/\varepsilon)),\qquad k=0,1,2, \tag{3.26}$$

hold uniformly in t and $\varkappa$.

At the eighth stage, consider the interval $3h+1-\sigma_0\le t\le 3h+1+\widetilde{\sigma}_0$. In this case,

$$a(t,\varepsilon)=\varepsilon^{-1}\exp(v_0(\tau)-\gamma(\tau,\varepsilon))|_{\tau=(t-3h-1)/\varepsilon}+O(\exp(-q/\varepsilon)),\quad |b(t,\varepsilon)|=O(\exp(-q/\varepsilon))\ \text{ as }\ \varepsilon\to 0,$$

where $\gamma(\tau,\varepsilon)$ is the function (2.76), and for the function $g_2(t-h,\varkappa,\varepsilon)$, as $\varepsilon\to 0$, the asymptotic representations

$$g_2(t-h,\varkappa,\varepsilon)=v_0'(\tau)|_{\tau=(t-3h-1)/\varepsilon}+O(\exp(-q/\varepsilon)),\quad \frac{\partial^k g_2}{\partial\varkappa^k}(t-h,\varkappa,\varepsilon)=O(\exp(-q/\varepsilon)),\quad k=1,2,$$

hold by relations (3.20). Taking into account the above relations in Eq. (3.7), we see that

$$\frac{\partial^k g_2}{\partial\varkappa^k}(t,\varkappa,\varepsilon)=\left[\varkappa\gamma_\tau'(\tau,\varepsilon)-(\varkappa-1)\left(c_{**}+\frac{h-t_0}{\varepsilon}+\int_0^\tau\exp(v_0(s)-s)\,ds\right)^{-1}\right]^{(k)}\Bigg|_{\tau=(t-3h-1)/\varepsilon}$$
$$+O(\exp(-q/\varepsilon))\quad\text{as}\quad\varepsilon\to 0,\quad k=0,1,2. \tag{3.27}$$

At the ninth, last stage, consider the time interval $3h+1+\widetilde{\sigma}_0\le t\le 3h+1+\sigma_0$. It is easy to see that, by the relations

$$a(t,\varepsilon)=O(\exp(-q/\varepsilon)),\quad |b(t,\varepsilon)|=O(\exp(-q/\varepsilon))\quad\text{as}\quad\varepsilon\to 0,$$

in this case, as $\varepsilon\to 0$, the asymptotic relations

$$\frac{\partial^k g_2}{\partial\varkappa^k}(t,\varkappa,\varepsilon)=\left[1+(\varkappa-1)\left(1-\frac{\varepsilon}{h-t_0+\varepsilon c_{***}}\right)\right]^{(k)}+O\left(\exp\left(-\frac{q}{\varepsilon}\right)\right),\qquad k=0,1,2, \tag{3.28}$$

hold uniformly in t and $\varkappa$.

Combining the above-obtained formulas (3.19)–(3.21), (3.23)–(3.28), we conclude that properties (3.13), (3.14) of the function g_2 are indeed satisfied. The proof of Lemma 3.2 is complete.

3.3. Completion of the Proof of Theorem 1.2

First, we localize the values of the parameter $\varkappa$ for which Eqs. (3.9) must be considered. In this connection, we arbitrarily fix the initial condition $g_0(t)\in C(G)$, $\|g_0\|\le 1$, and note that the operator (3.8) acts on the function g_0 by the rule

$$W(\varkappa,\varepsilon)g_0=g_1(t+2h,\widetilde{g}_0,\varkappa,\varepsilon)+g_0(h+1+\sigma_0)g_2(t+2h,\varkappa,\varepsilon),\qquad t\in G, \tag{3.29}$$

where $\widetilde{g}_0(t)=g_0(t)-g_0(h+1+\sigma_0)\in B$ (see (3.11)) and g_1 and g_2 are the solutions of Eq. (3.7) studied above. In turn, relations (3.12) and (3.29) imply the representation

$$W(\varkappa,\varepsilon)=W_0(\varepsilon)+\varkappa W_1(\varepsilon)+\varkappa^2W_2(\varepsilon), \tag{3.30}$$

where the $W_j(\varepsilon):C(G)\to C(G)$, $j=0,1,2$, are bounded linear operators. Moreover, the above-established asymptotic properties of the functions g_1 and g_2 ensure the estimates

$$\|W_j(\varepsilon)\|_{C(G)\to C(G)}\le M_{j+1},\qquad j=0,1,\qquad \|W_2(\varepsilon)\|_{C(G)\to C(G)}\le\exp(-q/\varepsilon), \tag{3.31}$$

where $M_1,M_2=$ const >0.

Further, we show that, for a fixed sufficiently large $R > 0$, Eqs. (3.9) do not have roots in the set $\{\varkappa \in \mathbb{C} : |\varkappa| > R\}$. Indeed, the representation (3.30) and the estimates (3.31) imply the inequality

$$\begin{aligned}\sup_{s\geq 1} |\nu_s(\varkappa,\varepsilon)| &\leq \|W_0(\varepsilon)\|_{C(G)\to C(G)} + |\varkappa|\|W_1(\varepsilon)\|_{C(G)\to C(G)} + |\varkappa|^2\|W_2(\varepsilon)\|_{C(G)\to C(G)} \\ &\leq M_1 + |\varkappa|M_2 + |\varkappa|^2\exp(-q/\varepsilon).\end{aligned}$$

Since, for sufficiently large fixed constant R and for all sufficiently small ε, we have the estimate

$$M_1 + M_2R + R^2\exp(-q/\varepsilon) < R^2, \tag{3.32}$$

we see that $\sup_{s\geq 1}|\nu_s(\varkappa,\varepsilon)| < |\varkappa|^2$ necessarily for all $\varkappa \in \mathbb{C}$, $|\varkappa| > R$. Thus, Eqs. (3.9) indeed do not have roots $\varkappa$ satisfying the inequality $|\varkappa| > R$.

Now we directly asymptotically calculate the spectrum of the operator (3.8). We have the following assertion.

Lemma 3.3. *For any $R > 0$, there exist $\varepsilon_0 = \varepsilon_0(R) > 0$, $q = q(R) > 0$, and $\delta = \delta(R) > 0$ such that the following inequality holds for all $0 < \varepsilon \leq \varepsilon_0$, $\varkappa \in \Lambda_{\delta,R} \stackrel{\text{def}}{=} \{\varkappa \in \mathbb{C} : \exp(-\delta/\varepsilon) \leq |\varkappa| \leq R\}$:*

$$\sup_{s\geq 2}|\nu_s(\varkappa,\varepsilon)| \leq \exp(-q/\varepsilon). \tag{3.33}$$

The multiplier $\nu_1(\varkappa,\varepsilon)$ is simple, analytically depends on $\varkappa \in \Lambda_{\delta,R}$, and admits the asymptotic representations

$$\frac{\partial^k\nu_1}{\partial\varkappa^k}(\varkappa,\varepsilon) = \left[1 + (\varkappa-1)\left(1 - \frac{\varepsilon}{h - t_0 + \varepsilon c_{***}}\right)\right]^{(k)} + O(\exp(-q/\varepsilon)), \qquad k = 0,1, \tag{3.34}$$

uniform in $\varkappa$ as $\varepsilon \to 0$.

Proof. We arbitrarily fix $R > 0$ and fix that the parameter $\varkappa$ in Eq. (3.7) belongs to the set $\Lambda_{\delta,R}$ for some $\delta > 0$ (we choose δ somewhat later). It follows from formula (3.29) that $W(\varkappa,\varepsilon) = U_1(\varkappa,\varepsilon) + U_2(\varkappa,\varepsilon)$, where the operators U_1 and U_2 are given by the relations

$$U_1(\varkappa,\varepsilon)g_0 = g_1(t+2h,\widetilde{g}_0,\varkappa,\varepsilon), \qquad U_2(\varkappa,\varepsilon)g_0 = g_0(h+1+\sigma_0)g_2(t+2h,\varkappa,\varepsilon), \qquad t\in G, \tag{3.35}$$

and, by inequalities (3.13), admit the estimates

$$\begin{aligned}&\|U_1(\varkappa,\varepsilon)\|_{C(G)\to C(G)} + \left\|\frac{\partial U_1}{\partial\varkappa}(\varkappa,\varepsilon)\right\|_{C(G)\to C(G)} \leq \exp(-q/\varepsilon), \\ &\|U_2(\varkappa,\varepsilon)\|_{C(G)\to C(G)} + \left\|\frac{\partial U_2}{\partial\varkappa}(\varkappa,\varepsilon)\right\|_{C(G)\to C(G)} \leq M, \qquad M = \text{const} > 0.\end{aligned} \tag{3.36}$$

First, we study the spectral properties of the operator $U_2(\varkappa,\varepsilon)$. By the second relation in (3.35), it remains finite-dimensional, and its spectrum consists of the following two points, the simple eigenvalue $\nu = \nu_*(\varkappa,\varepsilon)$, where $\nu_*(\varkappa,\varepsilon) = g_2(3h+1+\sigma_0,\varkappa,\varepsilon)$, and the eigenvalue $\nu = 0$ of infinite multiplicity. For the eigenvalue $\nu_*(\varkappa,\varepsilon)$, the representations (3.14) imply the asymptotic relations

$$\frac{\partial^k\nu_*}{\partial\varkappa^k}(\varkappa,\varepsilon) = \left[1 + (\varkappa-1)\left(1 - \frac{\varepsilon}{h - t_0 + \varepsilon c_{***}}\right)\right]^{(k)} + O(\exp(-q/\varepsilon)), \qquad k = 0,1, \tag{3.37}$$

uniform in $\varkappa \in \Lambda_{\delta,R}$.

Now consider the initial operator $W(\varkappa,\varepsilon)$ and note that, by the relations

$$W = U_1 + U_2, \qquad (\nu\mathrm{I} - W)^{-1} = (\mathrm{I} - (\nu\mathrm{I} - U_2)^{-1}U_1)^{-1}(\nu\mathrm{I} - U_2)^{-1},$$

where I is the identity operator, any value $\nu \in \mathbb{C}$ such that

$$\|(\nu\mathrm{I} - U_2(\varkappa,\varepsilon))^{-1} U_1(\varkappa,\varepsilon)\|_{C(G)\to C(G)} < 1 \tag{3.38}$$

belongs to the resolvent set of this operator. Recall that the operator U_1 admits the estimate in (3.36). But for the case of the operator $(\nu\mathrm{I} - U_2)^{-1}$, taking into account its implicit form

$$(\nu\mathrm{I} - U_2)^{-1} g_0 = \frac{g_0(t)}{\nu} + \frac{g_0(h+1+\sigma_0)}{\nu(\nu - \nu_*(\varkappa,\varepsilon))}\, g_2(t+2h,\varkappa,\varepsilon), \qquad t \in G,$$

and the second estimate in (3.13), we obtain the inequality

$$\|(\nu\mathrm{I} - U_2(\varkappa,\varepsilon))^{-1}\|_{C(G)\to C(G)} \le \frac{M(1+|\nu|)}{|\nu||\nu - \nu_*(\varkappa,\varepsilon)|}, \qquad \nu \in \mathbb{C}, \qquad \nu \neq 0, \qquad \nu_*(\varkappa,\varepsilon), \tag{3.39}$$

where $M = \mathrm{const} > 0$.

At the final stage of the proof of Lemma 3.3, we combine the estimates (3.36) and (3.39) with the asymptotic representations (3.37). As a result, we see that condition (3.38) is satisfied for any $\varkappa \in \Lambda_{\delta,R}$ and $\nu \in \mathbb{C} \setminus \{O_1 \bigcup O_2\}$, where

$$O_1 = \{\nu : |\nu| < \exp(-\delta_1/\varepsilon)\}, \qquad O_2 = \{\nu : |\nu - \nu_*(\varkappa,\varepsilon)| < \exp(-\delta_2/\varepsilon)\}, \tag{3.40}$$

and the constants $\delta, \delta_1, \delta_2 > 0$ are sufficiently small. Thus, the spectrum of the operator (3.29) necessarily belongs to the balls (3.40), which obviously implies inequality (3.33).

To justify relations (3.34), note that, under the perturbation of the operator $U_2(\varkappa,\varepsilon)$ with respect to $\varkappa$ by the additional term $U_1(\varkappa,\varepsilon)$ of order $O(\exp(-q/\varepsilon))$, the eigenvalue $\nu = \nu_*(\varkappa,\varepsilon)$ turns into the simple eigenvalue $\nu = \nu_1(\varkappa,\varepsilon)$ analytically depending on $\varkappa \in \Lambda_{\delta,R}$, and

$$\nu_1(\varkappa,\varepsilon) - \nu_*(\varkappa,\varepsilon) = O(\exp(-q/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0 \tag{3.41}$$

(in the C^1-metric with respect to the variable $\varkappa$). Then we combine relations (3.37) and (3.41) to conclude that, for $\varkappa \in \Lambda_{\delta,R}$, the multiplier $\nu_1(\varkappa,\varepsilon)$ has all the desired properties. The proof of Lemma 3.3 is complete.

At the final stage of the proof of Theorem 1.2, we analyze Eqs. (3.9). We assume that the constant $R > 0$ satisfies condition (3.32). Then, when considering these equations, we can necessarily restrict ourselves to the values $\varkappa \in \mathbb{C}$, $|\varkappa| \le R$. Further, we choose the constant $\delta = \delta(R) > 0$ by Lemma 3.3 and consider the set $\Lambda_{\delta,R}$. By the asymptotic relations (2.89), (2.91), and (3.34), Eq. (3.9) has two simple roots in the set $\Lambda_{\delta,R}$ for $s = 1$,

$$\varkappa = 1, \quad \varkappa = \varkappa_0(\varepsilon), \quad \varkappa_0(\varepsilon) = -\frac{\varepsilon}{b - t_0} + O(\varepsilon^2 \ln \varepsilon^{-1}) \quad \text{as} \quad \varepsilon \to 0. \tag{3.42}$$

We point out that the existence of the root $\varkappa = 1$ is guaranteed by the identity $\nu_1(1,\varepsilon) \equiv 1$. This identity follows from the fact that Eq. (3.7) has the unit multiplier for $\varkappa = 1$. (In this case, Eq. (3.7) is the linearization of Eq. (2.1) on the cycle (2.92).)

By Lemma 3.1, the roots (3.42) are associated with the multipliers $\nu_1(\varepsilon) \equiv 1$ and $\nu_2(\varepsilon) = \varkappa_0^2(\varepsilon)$ of system (3.3). For the remaining multipliers of this system, the desired exponential estimate in (3.10) holds by inequality (3.33). The proof of Theorem 1.2 is complete.

CONCLUSION

Thus, we have found the range of the parameter d for which the bilocal model (1.6) has an exponentially orbitally stable self-symmetric cycle. Then it is natural to pose the question of what occurs outside this range.

First, we assume that

$$d = \lambda \exp(-b\lambda), \qquad b = \text{ const } > 1 + \frac{1}{2}\left(a + \frac{1}{a}\right), \qquad a = \text{ const } > 1, \qquad \lambda \gg 1. \tag{3.43}$$

Then the function (1.5) satisfies the inequality

$$\omega_*(t - T_*/2) - \omega_*(t) - b < 0, \qquad 0 \le t \le T_*. \tag{3.44}$$

Further, using relations (3.43) and (3.44) and analyzing the auxiliary equation (2.1) for $\varepsilon = 1/\lambda$, we conclude that, for some $h = h(\varepsilon)$, it admits a $2h(\varepsilon)$-periodic solution $\omega = \omega(t, \varepsilon)$, where

$$h(\varepsilon) = T_*/2 + O(\exp(-q/\varepsilon)), \qquad \max_{0 \le t \le 2h(\varepsilon)} |\omega(t,\varepsilon) - \omega_*(t)| = O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0. \tag{3.45}$$

Under conditions (3.43), the original system (1.6) has a self-symmetric cycle determined by relations (1.8), (2.94), and (3.45). Based on the methods developed above, we can also readily show that this cycle is quasi-stable; i.e., its multipliers $\nu_s(\varepsilon)$, $s \in \mathbb{N}$, satisfy the conditions

$$\nu_1(\varepsilon) \equiv 1, \qquad \nu_2(\varepsilon) = -1 + O(\exp(-q/\varepsilon)), \qquad \sup_{s \ge 3} |\nu_s(\varepsilon)| \le \exp(-q/\varepsilon).$$

As numerical analysis shows, for the parameter values

$$d = \lambda \exp(-b\lambda), \qquad b = \text{ const } \in (0, 1 + 1/a), \qquad a = \text{ const } > 1, \qquad \lambda \gg 1, \tag{3.46}$$

system (1.6) admits two stable cycles which pass into each other under the permutation of the coordinates u_1 and u_2. In the case of $d \sim 1$ and $\lambda \gg 1$, the existence of such cycles can be proved analytically (see [8]), and in the case of (3.46), this problem is open yet.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation (grant no. 14-21-00158).

REFERENCES

1. Glyzin, S.D., Kolesov, A.Yu., and Rozov, N.Kh., On a method for mathematical modeling of chemical synapses, *Differ. Equations*, 2013, vol. 49, no. 10, pp. 1193–1210.
2. Glyzin, S.D., Kolesov, A.Yu., and Rozov, N.Kh., Self-excited relaxation oscillations in networks of impulse neurons, *Russian Math. Surveys*, 2015, vol. 70, no. 3, pp. 383–452.
3. Hutchinson, G.E., Circular causal systems in ecology, *Ann. N.Y. Acad. Sci.*, 1948, vol. 50, pp. 221–246.
4. Kolesov, A.Yu., Mishchenko, E.F., and Rozov, N.Kh., A modification of Hutchinson's equation, *Comput. Math. Math. Phys.*, 2010, vol. 50, no. 12, pp. 1990–2002.
5. Glyzin, S.D., Kolesov, A.Yu., and Rozov, N.Kh., Relaxation self-oscillations in Hopfield networks with delay, *Izv. Math.*, 2013, vol. 77, no. 2, pp. 271–312.
6. Glyzin, S.D., Kolesov, A.Yu., and Rozov, N.Kh., Periodic traveling-wave-type solutions in circular chains of unidirectionally coupled equations, *Theor. Math. Phys.*, 2013, vol. 175, no. 1, pp. 499–517.
7. Glyzin, S.D., Kolesov, A.Yu., and Rozov, N.Kh., The buffer phenomenon in ring-like chains of unidirectionally connected generators, *Izv. Math.*, 2014, vol. 78, no. 4, pp. 708–743.
8. Kolesov, A.Yu. and Rozov, N.Kh., Self-excited wave processes in chains of diffusion-linked delay equations, *Russian Math. Surveys*, 2012, vol. 67, no. 2, pp. 297–343.